# Chapter 3
# A Non-Linear Heat Equation

In this chapter, we study the nonlinear equation of the form

$$\frac{\partial}{\partial t}u(x,t) - c^2(-\otimes)^k u(x,t) = f(x,t,u(x,t))$$

where $\otimes^k$ is the operator iterated $k$-times, defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k$$

where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $k$ is a positive integer and $c$ is a positive constant, $f$ is the given function in nonlinear form depending on $x, t$ and $u(x,t)$. On suitable conditions for $f$, $p$, $q$, $k$ and the spectrum, we obtain the unique solution $u(x,t)$ of such equation.Moreover,if we put $p = 0, k = 1$, we obtain the solution of non-linear heat equation.

## 3.1 Main Results

**Theorem 3.1.1.** Let $L$ be the operator defined by

$$L = \frac{\partial}{\partial t} - c^2(-\otimes)^k \tag{3.1.1}$$

where $\otimes^k$ is the operator iterated $k$-times defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k,$$

$p+q=n$ is the dimension of $\mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0,\infty)$, $k$ is a positive integer and $c$ is the positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega} \exp\left[c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k + i(\xi,x)\right] d\xi \tag{3.1.2}$$

as the elementary solution of (3.1.1) in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$, where $\sum_{j=p+1}^{p+q}\xi_j^2 > \sum_{i=1}^{p}\xi_i^2$.

**Proof.** Let $LE(x,t) = \delta(x,t)$ where $E(x,t)$ is the kernel or the elementary solution of the operator $L$ and $\delta$ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) - c^2(-\otimes)^k E(x,t) = \delta(x)\delta(t)$$

take the Fourier transform defined by (**??**) to both sides of the equation

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}}\exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right]$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right],$$

so we have

$$E(\xi,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}d\xi.$$

By (**??**),

$$E(\xi,t) = \frac{1}{(2\pi)^{n/2}}\int_{\Omega} e^{i(\xi,x)}\widehat{E(\xi,t)}d\xi$$

where $\Omega$ is the spectrum of $E(x,t)$. Thus

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k + i(\xi,x)\right]d\xi.$$

for $t > 0$. □

**Theorem 3.1.2. (The properties of $E(x,t)$)**
*The kernel $E(x,t)$ defined by (3.1.2) have the following properties*

*(1) $E(x,t) \in C^\infty$ - the space of continuous function for $x \in \mathbb{R}^n$, $t > 0$ with infinitely ditterentiable.*

*(2) $\left(\frac{\partial}{\partial t} - c^2(-\otimes)^k\right)E(x,t) = 0$ for $t > 0$.*

*(3) $|E(x,t)| \leq \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(p/2)\Gamma(q/2)}$ for $t > 0$ where $M(t)$ is a function of $t$ in the spectrum and $\Gamma$ denote the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t > 0$.*

*(4)* $\lim_{t\to 0} E(x,t) = \delta$.

**Proof.** (1) From (**??**)

$$\frac{\partial^n}{\partial x^n} E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \frac{\partial^n}{\partial x^n} \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right)^k + i(\xi, x)\right] d\xi.$$

Thus $E(x,t) \in C^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.
(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} - c^2(-\otimes)^k\right) E(x,t) = 0.$$

(3) We have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right)^k + i(\xi, x)\right] d\xi.$$

Thus

$$|E(x,t)| \le \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^3 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right)^k\right] d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = rw_1, \xi_2 = rw_2, \ldots, \xi_p = rw_p \text{ and } \xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \ldots, \xi_{p+q} = sw_{p+q}$$

where

$$\sum_{i=1}^{p} w_i^2 = 1 \quad \text{and} \quad \sum_{j=p+1}^{p+q} w_j^2 = 1$$

Thus

$$|E(x,t)| \le \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(r^6 - s^6\right)^k\right] r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q$$

where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and suppose $0 \le r \le R$ and $0 \le s \le L$ where $R$ and $L$ are constants. Thus we obtain

$$\begin{aligned} |E(x,t)| &\le \frac{\Omega_p \Omega_q}{(2\pi)^n} \int_0^R \int_0^L \exp\left[c^2 t \left(r^6 - s^6\right)^k\right] r^{p-1} s^{q-1} dr ds \\ &= \frac{\Omega_p \Omega_q}{(2\pi)^n} M(t) \qquad \text{for any fixed } t > 0 \text{ in the spectrum } \Omega \\ &= \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(p/2)\Gamma(q/2)} \qquad (3.1.3) \end{aligned}$$

where $M(t) = \int_0^R \int_0^L \exp\left[c^2 t \left(r^6 - s^6\right)^k\right] r^{p-1} s^{q-1} dr ds$ is a function for $t > 0$, $\Omega_p = \frac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \frac{2\pi^{q/2}}{\Gamma(q/2)}$. Thus for any fixed $t > 0$, $E(x,t)$ is bounded.

(4) From (2.5),

$$\lim_{t\to 0} E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega e^{i(\xi,x)} d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)} d\xi = \delta(x),$$

for $x \in \mathbb{R}^n$, [8, p. 396, Eq. (10.2.19b)]. □

**Theorem 3.1.3.** Given the nonlinear equation

$$\frac{\partial}{\partial t} u(x,t) - c^2(-\otimes)^k u(x,t) = f(x,t,u(x,t)) \tag{3.1.4}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$, $k$ is a positive number and with the following conditions on $u$ and $f$ as follows

(1) $u(x,t) \in C^{(6k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(6k)}(\mathbb{R}^n)$ is the space of continuous function with $6k$-derivative.

(2) $f$ satisfies the Lipchitz condition,

$$|f(x,t,u) - f(x,t,w)| \leq A|u - w|$$

where $A$ is constant with $0 < A < 1$.

(3) $\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\, dx dt < \infty$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $0 < t < \infty$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Then obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{3.1.5}$$

as a unique solution of (3.1.4) for $x \in \Omega$ where $\Omega$ is a compact subset of $\mathbb{R}^n$ and $0 \leq t \leq T$ with $T$ is a constant and $E(x,t)$ is an elementary solution defined by (2.3.8) and also $u(x,t)$ is bounded for any fixed $t > 0$. In particular, if we put $k = 1$ and $p = 0$ in (3.1.4), then (3.1.4) reduces to the nonlinear heat equation

$$\frac{\partial}{\partial t} u(x,t) + c^2 \triangle^3 u(x,t) = f(x,t,u(x,t))$$

which is related to the heat equation.

**Proof.** Convolving both sides of (3.1.4) with $E(x,t)$, that is

$$E(x,t) * \left[\frac{\partial}{\partial t} u(x,t) - c^2(-\otimes)^k u(x,t)\right] = E(x,t) * f(x,t,u(x,t))$$

or

$$\left[\frac{\partial}{\partial t} E(x,t) - c^2(-\otimes)^k E(x,t)\right] * u(x,t) = E(x,t) * f(x,t,u(x,t)),$$

so

$$\delta(x,t) * u(x,t) = E(x,t) * f(x,t,u(x,t)).$$

Thus

$$\begin{aligned} u(x,t) &= E(x,t) * f(x,t,u(x,t)) \\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} E(r,s) f(x-r,t-s,u(x-r,t-s))drds \end{aligned}$$

where $E(r,s)$ is given by definition (2.2.16). We next show that $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. We have

$$\begin{aligned} |u(x,t)| &\leq \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)||f(x-r,t-s,u(x-r,t-s))|drds \\ &\leq \frac{2^{2-n}NM(t)}{\pi^{n/2}\Gamma(p/2)\Gamma(q/2)} \qquad \text{by condition (3) and (\textbf{??})} \end{aligned}$$

where $N = \displaystyle\int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |f(x-r,t-s,u(x-r,t-s))|drds$. Thus $u(x,t)$ is bounded on $\mathbb{R}^n \times (0,\infty)$. To show that $u(x,t)$ is unique. Now, We next to show that $u(x,t)$ is unique. Let $w(x,t)$ be another solution of (3.1.4), then

$$w(x,t) = E(x,t) * f(x,t,w(x,t))$$

for $(x,t) \in \Omega_0 \times (0,T]$ and $E(x,t)$ is defined by (3.1.2).

Now, define $\|u(x,t)\| = \sup\limits_{\substack{x\in\Omega_0 \\ 0<t\leq T}} |u(x,t)|$.

Now,

$$\begin{aligned} |u(x,t)-w(x,t)| &= |E(x,t)*f(x,t,u(x,t)) - E(x,t)*f(x,t,w(x,t))| \\ &\leq \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |E(r,s)| \cdot |f(x-r,t-s,u(x-r,t-s)) \\ &\qquad\qquad - f(x-r,t-s,w(x-r,t-s))|drds \\ &\leq A|E(r,s)| \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} |u(x-r,t-s)-w(x-r,t-s)|drds \end{aligned}$$

by (2.3.8) and the condition (2) of the theorem. Now, for $(x,t) \in \Omega_0 \times (0,T]$ we have

$$\begin{aligned} |u-w| &\leq A|E(r,s)|\|u-w\| \int_0^T ds \int_{\Omega_0} dr \\ &= A|E(r,s)|TV(\Omega_0)\|u-w\| \end{aligned} \tag{3.1.6}$$

where $V(\Omega_0)$ is the volume of the surface on $\Omega_0$.

Choose $A|E(r,s)|TV(\Omega_0) \leq 1$ or $A \leq \dfrac{1}{|E(r,s)|TV(\Omega_0)}$.

Thus from (3.1.6),

$$\|u-w\| \leq \alpha\|u-w\| \text{ where } \alpha = A|E(r,s)|TV(\Omega_0) \leq 1.$$

It follows that $\|u-w\| = 0$, thus $u = w$.

That is the solution $u$ of (3.1.4) is unique.

In particular, if we put $k = 1$ and $p = 0$ in (3.1.4), then (3.1.4) reduces to the nonlinear heat equation

$$\frac{\partial}{\partial t}u(x,t) - c^2\triangle^3 u(x,t) = f(x,t,u(x,t))$$

which has solution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

where $E(x,t)$ is defined by (3.1.2) with $k = 1$ and $p = 0$. □