Chapter 4
The Generalized $\otimes^k$ operator related to Triharmonic Wave Equation

In this chapter, we study the generalized wave equation of the form

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(\otimes)^k u(x,t) = 0$$

with the initial conditions

$$u(x,0) = f(x), \quad \frac{\partial}{\partial t}u(x,0) = g(x)$$

where $u(x,t) \in \mathbb{R}^n \times (0,\infty)$, $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\otimes^k$ is the operator iterated $k-$times defined by

$$\otimes^k = \left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right]^k$$

c is a positive constant, k is a nonnegative integer, f and g are continuous and absolutely integrable functions. We obtain $u(x,t)$ as a solution for such equation. Moreover, by ϵ-approximation we also obtain the asymptotic solution $u(x,t) = O(\epsilon^{-n/3k})$. In particularly, if we put $k = 1$ and $p = 0$, the $u(x,t)$ reduces to the solution of the wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2(\triangle)^3 u(x,t) = 0.$$

which is related to the triharmonic wave equation.

4.1 Main Results

Theorem 4.1.1. Given the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(\otimes)^k u(x,t) = 0 \qquad (4.1.1)$$

with initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x) \qquad (4.1.2)$$

where $u(x,t) \in \mathbb{R}^n \times (0,\infty)$, $\otimes^k$ is the diamond operator iterated $k-$times, c is a positive constant, k is a nonnegative integer, f and g are continuous functions and absolutely integrable for $x \in \mathbb{R}^n$. Then (4.1.1) has a unique solution

$$u(x,t) = f(x) * \Psi_t(x) + g(x) * \Phi_t(x) \qquad (4.1.3)$$

and satisfy the condition (4.1.2) where Φ_t is an inverse Fourier transform of

$$\widehat{\Phi}_t(\xi) = \frac{\sin c\left(\sqrt{s^6 - r^6}\right)^k t}{c\left(\sqrt{s^6 - r^6}\right)^k}$$

and Ψ_t is an inverse Fourier transform of

$$\widehat{\Psi}_t(\xi) = \cos c\left(\sqrt{s^6 - r^6}\right)^k t = \frac{\partial}{\partial t}\widehat{\Phi}(\xi)$$

where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ and $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$.

Proof. By applying the Fourier transform defined by (2.2.16) to (4.1.1) and obtain

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi, t) + c^2\left(-\left(\sum_{i=1}^{p}\xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k \widehat{u}(\xi, t) = 0$$

and let $s > r$. Thus becomes

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi, t) + c^2\left(s^6 - r^6\right)^k \widehat{u}(\xi, t) = 0$$

$$\widehat{u}(\xi, t) = A(\xi)\cos c\left(\sqrt{s^6 - r^6}\right)^k t + B(\xi)\sin c\left(\sqrt{s^6 - r^6}\right)^k t.$$

By (4.1.2), $\widehat{u}(\xi, 0) = A(\xi) = \widehat{f}(\xi)$

$$\frac{\partial \widehat{u}(\xi, t)}{\partial t} = -c\left(\sqrt{s^6 - r^6}\right)^k A(\xi)\sin c\left(\sqrt{s^6 - r^6}\right)^k t + c\left(\sqrt{s^6 - r^6}\right)^k B(\xi)\cos c\left(\sqrt{s^6 - r^6}\right)^k t.$$

$$\frac{\partial \widehat{u}(\xi, 0)}{\partial t} = 0 + c\left(\sqrt{s^6 - r^6}\right)^k B(\xi) = \widehat{g}(\xi)$$

$$B(\xi) = \frac{\widehat{g}(\xi)}{c\left(\sqrt{s^6 - r^6}\right)^k}$$

$$\widehat{u}(\xi, t) = \widehat{f}(\xi)\cos c\left(\sqrt{s^6 - r^6}\right)^k t + \frac{\widehat{g}(\xi)}{c\left(\sqrt{s^6 - r^6}\right)^k}\sin c\left(\sqrt{s^6 - r^6}\right)^k t \qquad (4.1.4)$$

By applying the inverse Fourier transform (4.1.4), we obtain the solution $u(x, t)$ in the convolution form of (4.1.1). Now we need to show the existence of $\Phi_t(x)$ and $\Psi_t(x)$.

Let us consider the Fourier transform

$$\widehat{\Phi}_t(x) = \frac{\sin c\left(\sqrt{s^6 - r^6}\right)^k t}{c\left(\sqrt{s^6 - r^6}\right)^k} \quad \text{and} \quad \Psi_t(x) = \cos c\left(\sqrt{s^6 - r^6}\right)^k t.$$

They are all tempered distributions but they are not $L_1(\mathbb{R}^n)$ the space of integrable function. So we cannot compute the inverse Fourier transform $\Phi_t(x)$ and $\Psi_t(x)$ directly. Thus we compute the inverse $\Phi_t(x)$ and $\Psi_t(x)$ by using the method of $\epsilon-$approximation.

Let us defined

$$\widehat{\phi}_t^{\epsilon}(\xi) = e^{-\epsilon c\left(\sqrt{s^6 - r^6}\right)^k}\widehat{\phi}_t(\xi) = e^{-\epsilon c\left(\sqrt{s^6 - r^6}\right)^k}\frac{\sin c\left(\sqrt{s^6 - r^6}\right)^k t}{c\left(\sqrt{s^6 - r^6}\right)^k} \quad \text{for} \quad \epsilon > 0. \qquad (4.1.5)$$

We see that $\phi_t^\epsilon(x) \in L_1(\mathbb{R}^n)$ and $\widehat{\phi_t^\epsilon}(x) \to \widehat{\phi}_t(x)$ uniformly as $\epsilon \to 0$. So that $\phi_t(x)$ will be limit in the topology of tempered distribution of $\phi_t^\epsilon(x)$. Now

$$\begin{aligned}
\Phi_t^\epsilon(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{\Phi_t^\epsilon}(\xi) d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} e^{-\epsilon c\left(\sqrt{s^6-r^6}\right)^k} \frac{\sin c\left(\sqrt{s^6-r^6}\right)^k t}{c\left(\sqrt{s^6-r^6}\right)^k} d\xi \\
|\Phi_t^\epsilon(x)| &\leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{e^{-\epsilon c\left(\sqrt{s^6-r^6}\right)^k}}{c\left(\sqrt{s^6-r^6}\right)^k} d\xi \qquad (4.1.6)
\end{aligned}$$

By changing to bipolar coordinates. Now, put

$$\xi_1 = rw_1, \xi_2 = rw_2, \ldots, \xi_p = rw_p$$

and
$$\xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \ldots, \xi_p = sw_{p+q},\ p+q = n$$

where $w_1^2 + w_2^2 + \cdots + w_p^2 = 1$ and $w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1$,

$$|\Phi_t^\epsilon(x)| \leq \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \frac{e^{-\epsilon c\left(\sqrt{s^6-r^6}\right)^k}}{c\left(\sqrt{s^6-r^6}\right)^k} r^{p-1} s^{q-1} dr ds d\Omega_p d\Omega_q,$$

where $d\xi = r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q$, $d\Omega_p$ and $d\Omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively, where $\Omega_p = \frac{(2\pi)^{p/2}}{\Gamma(p/2)}$, $\Omega_q = \frac{(2\pi)^{q/2}}{\Gamma(q/2)}$,

$$|\Phi_t^\epsilon(x)| \leq \frac{\Omega_p \Omega_q}{(2\pi)^{n/2}} \int_0^\infty \int_0^s \frac{e^{-\epsilon c\left(\sqrt{s^6-r^6}\right)^k}}{c\left(\sqrt{s^6-r^6}\right)^k} r^{p-1} s^{q-1} dr ds,$$

put $r^3 = s^3 \sin\theta$, $3r^2 dr = s^3 \cos\theta d\theta$ and $0 \leq \theta \leq \frac{\pi}{2}$.

$$\begin{aligned}
|\Phi_t^\epsilon(x)| &\leq \frac{\Omega_p \Omega_q}{3(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} \frac{e^{-\epsilon c\left(\sqrt{s^6-s^6\sin^2\theta}\right)^k}}{c\left(\sqrt{s^6-s^6\sin^2\theta}\right)^k} (\sin\theta)^{\frac{p-3}{3}} s^{p+q-1} \cos\theta d\theta ds, \\
&= \frac{\Omega_p \Omega_q}{3c(2\pi)^{n/2}} \int_0^\infty \int_0^{\pi/2} \frac{e^{-\epsilon c\left(s^3\cos\theta\right)^k}}{\left(s^3\cos\theta\right)^k} s^{p+q-1} (\sin\theta)^{\frac{p-3}{3}} \cos\theta d\theta ds.
\end{aligned}$$

Put $y = \epsilon c\left(s^3\cos\theta\right)^k = \epsilon c s^{3k}\cos^k\theta$, $s^{3k} = \dfrac{y}{c\epsilon\cos^k\theta}$, $ds = \frac{sdy}{3ky}$, thus

$$
\begin{aligned}
|\Phi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{9c(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}s^{n-1}}{y/(\epsilon c)}(\sin\theta)^{\frac{p-3}{3}}\cos\theta\frac{s}{ky}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{9(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}\epsilon}{ky^2}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{n/3k}(\sin\theta)^{\frac{p-3}{3}}\cos\theta dyd\theta \\
&= \frac{\Omega_p\Omega_q}{9(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}y^{n/3k-2}}{c^{n/3k}k\epsilon^{n/3k-1}}(\sin\theta)^{\frac{p-3}{3}}(\cos\theta)^{\frac{3-n}{3}}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{9(2\pi)^{n/2}}\frac{\Gamma\left(\frac{n}{3k}-1\right)}{k\epsilon^{\frac{n}{3k}-1}c^{n/3k}}\int_0^{\pi/2}(\sin\theta)^{\frac{p-3}{3}}(\cos\theta)^{\frac{3-n}{3}}d\theta \\
&= \frac{\Omega_p\Omega_q}{18c^{n/3k}(2\pi)^{n/2}k\epsilon^{n/3k-1}}\Gamma\left(\frac{n}{3k}-1\right)\beta\left(\frac{p}{6},\frac{6-n}{6}\right) \\
|\Phi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{18c^{n/3k}(2\pi)^{n/2}k\epsilon^{n/3k-1}}\frac{\Gamma\left(\frac{n}{3k}-1\right)\Gamma\left(\frac{p}{6}\right)\Gamma\left(\frac{6-n}{6}\right)}{\Gamma\left(\frac{6-q}{6}\right)}.
\end{aligned}
$$

Similarly, we defined $\widehat{\Psi_t^\epsilon}(\xi) = e^{-\epsilon c\left(\sqrt{s^6-r^6}\right)^k}\cos c\left(\sqrt{s^6-r^6}\right)^k t$ and

$$
\begin{aligned}
\Psi_t^\epsilon(x) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{\Psi_t^\epsilon}(\xi)d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-\epsilon c\left(\sqrt{s^6-r^6}\right)^k}\cos c\left(\sqrt{s^6-r^6}\right)^k td\xi
\end{aligned}
$$

$$
\begin{aligned}
|\Psi_t^\epsilon(x)| &\leq \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-\epsilon c\left(\sqrt{s^6-r^6}\right)^k}d\xi \\
&= \frac{1}{(2\pi)^{n/2}}\int_0^\infty\int_0^s e^{-\epsilon c\left(\sqrt{s^6-r^6}\right)^k}r^{p-1}s^{q-1}drds,
\end{aligned}
$$

put $r^3 = s^3\sin\theta$, $3r^2dr = s^3\cos\theta d\theta$ and $0\leq\theta\leq\dfrac{\pi}{2}$

$$
\begin{aligned}
|\Psi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{3(2\pi)^{n/2}}\int_0^\infty\int_0^{\pi/2} e^{-\epsilon c\left(s^3\cos\theta\right)^k}(\sin\theta)^{\frac{p-3}{3}}s^{p+q-1}\cos\theta d\theta ds \\
&= \frac{\Omega_p\Omega_q}{3(2\pi)^{n/2}}\int_0^\infty\int_0^{\pi/2} e^{-\epsilon c\left(s^3\cos\theta\right)^k}s^{p+q-1}(\sin\theta)^{\frac{p-3}{3}}\cos\theta d\theta ds,
\end{aligned}
$$

put $y = \epsilon c(s^3\cos\theta)^k$, $ds = s\dfrac{dy}{3ky}$,

$$
\begin{aligned}
|\Psi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{9k(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}}{y}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{n/3k}(\sin\theta)^{\frac{p-3}{3}}\cos\theta dyd\theta \\
&= \frac{\Omega_p\Omega_q}{9k(2\pi)^{n/2}}\int_0^{\pi/2}\int_0^\infty \frac{e^{-y}y^{n/3k-1}}{c^{n/3k}\epsilon^{n/3k}}(\sin\theta)^{\frac{p-3}{3}}(\cos\theta)^{\frac{3-n}{3}}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{9(2\pi)^{n/2}kc^{n/3k}\epsilon^{n/3k}}\Gamma\left(\frac{n}{3k}\right)\int_0^{\pi/2}(\sin\theta)^{\frac{p-3}{3}}(\cos\theta)^{\frac{3-n}{3}}d\theta \\
|\Psi_t^\epsilon(x)| &\leq \frac{\Omega_p\Omega_q}{18(2\pi)^{n/2}kc^{n/3k}\epsilon^{n/3k}}\frac{\Gamma\left(\frac{n}{3k}\right)\Gamma\left(\frac{p}{6}\right)\Gamma\left(\frac{6-n}{6}\right)}{\Gamma\left(\frac{6-q}{6}\right)}.
\end{aligned}
$$

Set

$$u^{\epsilon}(x,t) = f(x) * \Psi_t^{\epsilon}(x) + g(x) * \Phi_t^{\epsilon}(x) \tag{4.1.7}$$

which $\epsilon-$approximation of $u(x,t)$ in (4.1.7) for $\epsilon \to 0$, $u^{\epsilon}(x,t) \to u(x,t)$ uniformly. Now

$$u^{\epsilon}(x,t) = \int_{\mathbb{R}^n} f(r)\Psi_t^{\epsilon}(x-r)dr + \int_{\mathbb{R}^n} g(r)\Phi_t^{\epsilon}(x-r)dr$$

Thus

$$\begin{aligned} |u^{\epsilon}(x,t)| &\leq |\Psi_t^{\epsilon}(x-r)| \int_{\mathbb{R}^n} |f(r)|dr + |\Phi_t^{\epsilon}(x-r)| \int_{\mathbb{R}^n} |g(r)|dr \\ &\leq \frac{\Omega_p\Omega_q}{18(2\pi)^{n/2}kc^{n/3k}\epsilon^{n/3k}} \frac{\Gamma\left(\frac{n}{3k}\right)\Gamma\left(\frac{p}{6}\right)\Gamma\left(\frac{6-n}{6}\right)}{\Gamma\left(\frac{6-q}{6}\right)} M + \\ &\quad \frac{\Omega_p\Omega_q}{18(2\pi)^{n/2}kc^{n/3k}\epsilon^{n/3k-1}} \frac{\Gamma\left(\frac{n}{3k}-1\right)\Gamma\left(\frac{p}{6}\right)\Gamma\left(\frac{6-n}{6}\right)}{\Gamma\left(\frac{6-q}{6}\right)} N \end{aligned}$$

$$\begin{aligned} \epsilon^{n/3k}|u^{\epsilon}(x,t)| &\leq \frac{\Omega_p\Omega_q}{18(2\pi)^{n/2}kc^{n/3k}} \frac{\Gamma\left(\frac{n}{3k}\right)\Gamma\left(\frac{p}{6}\right)\Gamma\left(\frac{6-n}{6}\right)}{\Gamma\left(\frac{6-q}{6}\right)} M + \\ &\quad \frac{\Omega_p\Omega_q\epsilon}{18(2\pi)^{n/2}kc^{n/3k}} \frac{\Gamma\left(\frac{n}{3k}-1\right)\Gamma\left(\frac{p}{6}\right)\Gamma\left(\frac{6-n}{6}\right)}{\Gamma\left(\frac{6-q}{6}\right)} N \end{aligned}$$

where $M = \int_{\mathbb{R}^n} |f(r)|dr$ and $N = \int_{\mathbb{R}^n} |g(r)|dr$, since f and g are absolutely integrable.

$$\lim_{\epsilon\to 0} \epsilon^{n/3k}|u^{\epsilon}(x,t)| \leq \frac{\Omega_p\Omega_q}{18(2\pi)^{n/2}kc^{n/3k}} \frac{\Gamma\left(\frac{n}{3k}\right)\Gamma\left(\frac{p}{6}\right)\Gamma\left(\frac{6-n}{6}\right)}{\Gamma\left(\frac{6-q}{6}\right)} = K.$$

It follows that $u(x,t) = O\left(\epsilon^{-n/3k}\right)$ for $n \neq k$ as $\epsilon \to 0$.

In particular, if we put $k = 1$ and $p = 0$ then (4.1.1) reduces to the solution of the equation,

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2\triangle^3 u(x,t) = 0.$$

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x)$$

where f and g are continuous and absolutely integrable for $x \in \mathbb{R}^n$.
Which is related Triharmonic wave equation. And, if we put $k = 1, n = 1$ and $p = 0$ then (4.1.1) reduces to the solution of the one dimensional equation,

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2\left(\frac{\partial^2}{\partial x^2}\right)^3 u(x,t) = 0.$$

Thus we obtain $u(x,t) = O(\epsilon^{-1/3})$ which is a solution of such one dimensional Triharmonic wave equation.