Chapter 5
On the Green Function of the Operator $\left(\otimes + m^6\right)^k$ Related to the Diamond Operator

In this chapter, we study the Green function of the operator $\left(\otimes + m^6\right)^k$ which is iterated $k-$times and m is positive real number. At first we find the Green function of the operator $\left(\otimes + m^6\right)^k$ and after that we apply such a Green function to find the solution of the equation $\left(\otimes + m^6\right)^k G(x) = f(x)$ where f is a generalized function and $G(x)$ is an unknown for $x \in \mathbb{R}^n$.

5.1 Main Results

Theorem 5.1.1. *Given the equation*

$$\left(\otimes + m^6\right)^k G(x) = \delta(x) \tag{5.1.1}$$

then

$$G(x) = \left(W^H_{6k}(u,m) * W^e_{2k}(v,m) * H^{*k}(x)\right) * \left(S^{*k}(x)\right)^{*-1} \tag{5.1.2}$$

is a Green function for the operator $\left(\otimes + m^6\right)^k$ iterated $k-$times where $\otimes$ is defined by (1.0.28), m is a nonnegative real number and

$$S(x) = \frac{3}{4}\delta - \frac{3}{4}m^2\left(W^H_4(u,m) * W^e_2(v,m) * \left(R^e_{-2}(v) + R^H_{-4}(u)\right)\right) * H(x) + \frac{1}{4}W^e_2(v,m) * H(x)$$

$S^{*k}(x)$ denotes the convolution of S itself $k-$times , $\left(S^{*k}(x)\right)^{*-1}$ denotes the inverse of $S^{*k}(x)$ in the convolution algebra. $H(x)$ is defined by (2.3.32) and for $k = 1$. Moreover $G(x)$ is a tempered distribution.

Proof. We have,

$$\begin{aligned}
\left(\otimes + m^6\right)^k &= \left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3 + m^6\right)^k \\
&= \left(\frac{3}{4}\left(\Diamond\triangle + m^6\right) + \frac{1}{4}\left(\Box^3 + m^6\right)\right)^k \\
&= \left(\frac{3}{4}\left(\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - m^4\triangle - m^2\Diamond\right) + \frac{1}{4}\left(\Box^3 + \left(m^2\right)^3\right)\right)^k \\
&= \left(\frac{3}{4}\left(\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - m^4\triangle - m^2\Diamond\right) + \frac{1}{4}\left(\Box + m^2\right)\left(\Box^2 - \Box m + m^2\right)\right)^k \\
&= \left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right) + \frac{1}{4}\left(\Box + m^2\right)^3\right)^k \qquad (5.1.3)
\end{aligned}$$

where $\Box+m^2$ is the ultra-hyperbolic Klein-Gordon operator and $\triangle+m^2$ is the Helmoltz operator and are defined by

$$\Box + m^2 = \sum_{i=1}^{p} \frac{\partial^2}{\partial {x_i}^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial {x_j}^2} + m^2 \qquad (5.1.4)$$

and

$$\triangle + m^2 = \sum_{i=1}^{n} \frac{\partial^2}{\partial {x_i}^2} + m^2 \qquad (5.1.5)$$

Thus

$$\begin{aligned}\left(\otimes + m^6\right)^k G(x) &= \left(\frac{3}{4}\Diamond\triangle + \frac{1}{4}\Box^3 + m^6\right)^k G(x) = \delta(x)\\ &= \left(\frac{3}{4}\Diamond\triangle + \frac{3}{4}m^6 + \frac{1}{4}\Box^3 + \frac{1}{4}m^6\right)^k G(x) = \delta(x)\end{aligned}$$

or we can write

$$\left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right) + \frac{1}{4}\left(\Box + m^2\right)^3\right)^k G(x) = \delta(x).$$

$$\left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right) + \frac{1}{4}\left(\Box + m^2\right)^3\right)\cdot$$
$$\left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right) + \frac{1}{4}\left(\Box + m^2\right)^3\right)^{k-1} G(x) = \delta(x)$$

by Lemma 2.2.9 with $k = 1$.

Convolving both sides of the above equation by $W_6^H(u,m) * W_2^e(v,m)$,

$$\begin{aligned}&\Bigg(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right)\left(W_6^H(u,m) * W_2^e(v,m)\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right)\cdot\\ &\qquad\left(W_6^H(u,m) * W_2^e(v,m)\right) + \frac{1}{4}\left(\Box + m^2\right)^3\left(W_6^H(u,m) * W_2^e(v,m)\right)\Bigg)\cdot\\ &\left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right) + \frac{1}{4}\left(\Box + m^2\right)^3\right)^{k-1} G(x)\\ &\qquad = \delta(x) * W_6^H(u,m) * W_2^e(v,m)\end{aligned}$$

$$\begin{aligned}&\frac{3}{4}\left(\Diamond + m^4\right)W_6^H(u,m) * \left(\triangle + m^2\right)W_2^e(v,m) - \frac{3}{4}m^2\Big(\triangle W_4^H(u,m) * W_2^e(v,m)\\ &\qquad +\Box W_4^H(u,m) * W_2^e(v,m)\Big) * \left(\Box + m^2\right)W_2^H(u,m)\\ &+\frac{1}{4}\Big(\left(\Box + m^2\right)W_2^H(u,m) * \left(\Box + m^2\right)W_2^H(u,m) * \left(\Box + m^2\right)W_2^H(u,m)\Big) * W_2^e(v,m)\\ &\left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right) + \frac{1}{4}\left(\Box + m^2\right)^3\right)^{k-1} G(x)\\ &\qquad = W_6^H(u,m) * W_2^e(v,m) \qquad (5.1.6)\end{aligned}$$

Since by (2.3.9) we have $(\Box + m^2)\, W_2^H(u,m) = \delta$ and by (2.3.10), $(\triangle + m^2)\, W_2^e(v,m) = \delta$ for $k = 1$.

Now, by (2.2.14)

$$W_4^H(u,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(1+r)}{r!\Gamma(1)} \left(m^2\right)^r R_{4+2r}^H(u).$$

Thus,

$$\Box W_4^H(u,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(1+r)}{r!\Gamma(1)} \left(m^2\right)^r \Box R_{4+2r}^H(u).$$

By Trione (1988) and Telles (1995) we have

$$\Box R_{4+2r}^H(u) = R_{4+2r-4}^H(u) = R_{4+2r}^H(u) * R_{-4}^H(u)$$

Similarly, by (2.3.11),

$$\triangle W_2^e(v,m) = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(1+r)}{r!\Gamma(1)} \left(m^2\right)^r 2(-1)^{k+r} \triangle R_{2+2r}^e(v)$$

we have

$$\triangle R_{2+2r}^e(v) = R_{2+2r-2}^e(v) = R_{2+2r}^e(v) * R_{-2}^e(v).$$

So equation (5.1.6) becomes

$$\left(\frac{3}{4}\left(\Diamond + m^4\right) * \delta(x) - \frac{3}{4}m^2\left(W_4^H(u,m) * W_2^e(v,m) * \left(R_{-2}^e(v) + R_{-4}^e(v)\right) * \delta(x)\right.\right.$$
$$\left.\left. + \frac{1}{4} * \delta(x) * \delta(x) * \delta(x) * W_2^e(v,m)\right)\left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right)\right.$$
$$\left. + \frac{1}{4}\left(\Box + m^2\right)^3\right)^{k-1} G(x) = W_6^H(u,m) * W_2^e(v,m)$$

$$\left(\frac{3}{4}\left(\Diamond + m^4\right) - \frac{3}{4}m^2\left(W_4^H(u,m) * W_2^e(v,m) * \left(R_{-2}^e(v) + R_{-4}^e(v)\right)\right.\right.$$
$$\left.\left. + \frac{1}{4}W_2^e(v,m)\right)\left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\left(\Box + m^2\right)\right.$$
$$\left. + \frac{1}{4}\left(\Box + m^2\right)^3\right)^{k-1} G(x) = W_6^H(u,m) * W_2^e(v,m) \qquad (5.1.7)$$

Convolving both sides of (5.1.7) by $H(x)$ for $k = 1$ where $H(x)$ is defined by (2.3.32)

$$\left(\frac{3}{4}\left(\Diamond + m^4\right)H(x) * -\frac{3}{4}m^2\left(W_4^H(u,m) * W_2^e(v,m) * \left(R_{-2}^e(v) + R_{-4}^e(v)\right) *\right.\right.$$
$$\left.\left. W_2^e(v,m) * H(x) + \frac{1}{4}W_2^e(v,m) * H(x)\right)\left(\frac{3}{4}\left(\Diamond + m^4\right)\left(\triangle + m^2\right) - \frac{3}{4}m^2\left(\triangle + \Box\right)\cdot\right.$$
$$\left.\left(\Box + m^2\right) + \frac{1}{4}\left(\Box + m^2\right)^3\right)^{k-1} G(x) = W_6^H(u,m) * W_2^e(v,m) * H(x). \qquad (5.1.8)$$

By Lemma 2.3.12 and by (5.1.8) we write,

$$S(x) * \left(\frac{3}{4} \left(\Diamond + m^4 \right) \left(\triangle + m^2 \right) - \frac{3}{4} m^2 \left(\triangle + \Box \right) \left(\Box + m^2 \right) + \frac{1}{4} \left(\Box + m^2 \right)^3 \right)^{k-1} G(x) = W_6^H(u,m) * W_2^e(v,m) * H(x).$$

Keeping on convolving both sides of the above equation by $W_6^H(u,m) * W_2^e(v,m) * H(x)$ up to $k-1$ times, we obtain

$$S^{*k}(x) * G(x) = \left(W_6^H(u,m) * W_2^e(v,m) * H(x) \right)^{*k}, \tag{5.1.9}$$

the symbol $*k$ denotes the convolution of itself $k-$times.

By Telles (1995-1996), we have

$$\left(W_6^H(u,m) * W_2^e(v,m) \right)^{*k} = W_{6k}^H(u,m) * W_{2k}^e(v,m).$$

Thus,

$$S^{*k}(x) * G(x) = W_{6k}^H(u,m) * W_{2k}^e(v,m) * H^{*k}(x).$$

Now, consider the function $S^{*k}(x)$, since $W_6^H(u,m) * W_2^e(v,m)$ by Lemma **??** and also $\delta(x)$ is a tempered distribution. By Kananthai (1997), Lemma 2.3.12 $R_{-2}^e(v) + R_{-4}^H(u)$ is a tempered distribution with compact support. Thus $S(x)$ defined by (5.1.2) is a tempered distribution and by Lemma **??** again, we obtain $S^{*k}(x)$ is a tempered distribution.

Now, $W_{6k}^H(u,m) * W_{2k}^e(v,m) \in \mathcal{S}'$, the space of tempered distribution. Choose $\mathcal{S}' \subset \mathcal{D}'_{\mathcal{R}}$ where $\mathcal{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathcal{D}'$ of distribution.

Thus $W_{6k}^H(u,m) * W_{2k}^e(v,m) \in \mathcal{D}'_{\mathcal{R}}$. It follow that $W_{6k}^H(u,m) * W_{2k}^e(v,m)$ is an element of convolution algebra, since $\mathcal{D}'_{\mathcal{R}}$ is a convolution algebra. Hence, by Zemanian (1965, pp.150-151), the equation (5.1.8) has a unique solution

$$G(x) = \left(W_{6k}^H(u,m) * W_{2k}^e(v,m) * H^{*k}(x) \right) * \left(S^{*k}(x) \right)^{*-1}$$

where $\left(S^{*k}(x) \right)^{*-1}$ is an inverse of S^{*k} in the convolution algebra, $G(x)$ is called the Green function of the operator $\left(\otimes + m^6 \right)^k$.

Since $W_{6k}^H(u,m) * W_{2k}^e(v,m) * H^{*k}(x)$ and $\left(S^{*k}(x) \right)^{*-1}$ are tempered distribution, then by Donoghue [see 5, p.152] $\left(W_{6k}^H(u,m) * W_{2k}^e(v,m) * H^{*k}(x) \right) * \left(S^{*k}(x) \right)^{*-1}$ is tempered distribution, It follows that $G(x)$ is a tempered distribution.

Theorem 5.1.2. *(An application of Green function)*
Given the equation

$$\left(\otimes + m^6 \right)^k K(x) = f(x) \tag{5.1.10}$$

where $f(x)$ is a generalized function, $K(x)$ is an unknown function and $x \in \mathbb{R}^n$. Then

$$K(x) = G(x) * f(x)$$

is a unique solution of the equation (5.1.10) where $G(x)$ is a Green function for $\left(\otimes + m^6 \right)^k$.

Proof. By convolving both sides of (5.1.10) by $G(x)$ where $G(x)$ is a Green function for $(\otimes + m^6)^k$ in theorem 3.1, we have

$$G(x) * \left(\otimes + m^6\right)^k K(x) = G(x) * f(x)$$

$$\left(\otimes + m^6\right)^k G(x) * K(x) = G(x) * f(x)$$

$$\delta(x) * K(x) = G(x) * f(x).$$

Thus,

$$K(x) = G(x) * f(x).$$

Sine $G(x)$ is unique, by theorem 3.1. It follows that $K(x) = G(x) * f(x)$ is unique.