Chapter 6
On the $\boxtimes^k$ Operator and Nonlinear $\boxtimes^k$ Operator Related to the Wave Equation

In this paper, we study the $\boxtimes^k$ operator iterated $k-$ times and is deined by

$$\boxtimes^k = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^6 - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^6\right)^k,$$

where $p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x)$ is an unknown function for $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$, $f(x)$ is the generalized function , k is a positive integer. Firstly, we study the solution of the equation $\boxtimes^k u(x)=f(x)$. It was found that the solution $u(x)$ depends on the condition of p and q and a solution is related to the solution of the Laplace equation and the wave equation. Finally, we study the solution of the nonlinear equation $\boxtimes^k u(x)=f(x,\square^{k-1}L^k\circledast^k u(x)$. It was found that the existence of the solution $u(x)$ of such an equation depends on the condition of f and $\square^{k-1}L^k\circledast^k u(x)$. Moreover a solution $u(x)$ related the inhomogeneous equation depends on the condition of p,q and k.

6.1 Main Results

Theorem 6.1.1. Given the equation

$$\boxtimes^k u(x)=0, \tag{6.1.1}$$

where $\boxtimes^k$ is the operator iterated $k-$times defined by (1.0.38), $u(x)$ is an unknown function. Then we obtain,

$$u(x)=\left((-1)^{5k}R^e_{10k}(x)*R^H_{8k}(x)\right)*(R^H_{2(k-1)}(V))^{(m)}*\left(C^{*k}(x)\right)^{*-1}\left(S^{*k}(x)\right)^{*-1} \tag{6.1.2}$$

is a solution of (6.1.1) where $S(x)$ and $C(x)$ defined by (2.3.24), (2.3.28) respectively, $R^e_{10k}(x)$ defined by (1.0.22) with $\alpha=10k$, $R^H_{8k}(x)$ defined by (1.0.19) with $\alpha=8k$. The function $\left(R^H_{2(k-1)}(V)\right)^{(m)}$ is defined by (1.0.19) with m derivative, $\alpha=2(k-1)$ and V is defined by (2.2.5).

Proof. Consider the homogeneous equation

$$\boxtimes^k u(x)=0.$$

The above equation can be written

$$\square^k L_1^k\circledast^k=0$$

where $\square^k$, L_1^k and $\circledast^k$ defined by (1.0.6), (2.3.26) and (1.0.29) respectively.
By Lemma 2.3.6, we obtain

$$L_1^k\circledast^k u(x)=\left(R^H_{2(k-1)}(V)\right)^{(m)}. \tag{6.1.3}$$

By Lemma 2.3.10 and Lemma 2.3.9, we have $K(x)$ and $H(x)$ are the elementary solution of the L_1^k operator and the $\circledast^k$ operator respectively. That is

$$L_1^k K(x) = \delta(x) \quad and \quad \circledast^k H(x) = \delta(x). \tag{6.1.4}$$

Convolving both sides of (6.1.3) by $K(x) * H(x)$, we obtain

$$K(x) * H(x) * \left(L_1^k \circledast^k u(x)\right) = K(x) * H(x) * \left(R^H_{2(k-1)}(V)\right)^{(m)}.$$

By properties of convolution

$$L_1^k K(x) * \circledast^k H(x) * u(x) = K(x) * H(x) * \left(R^H_{2(k-1)}(V)\right)^{(m)}.$$

By (6.1.4), we obtain

$$\delta(x) * \delta(x) * u(x) = K(x) * H(x) * \left(R^H_{2(k-1)}(V)\right)^{(m)}.$$

Thus

$$u(x) = K(x) * H(x) * \left(R^H_{2(k-1)}(V)\right)^{(m)} \tag{6.1.5}$$

Putting (2.3.27) and (2.3.23) in (6.1.5), we obtain

$$u(x) = \left(\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1}\right) * \left(\left(R^H_{4k}(x) * (-1)^{3k} R^e_{6k}(x)\right) * \left(S^{*k}(x)\right)^{*-1}\right) * \left(R^H_{2(k-1)}(V)\right)^{(m)}.$$

By Lemma 2.2.6, we obtain

$$u(x) = \left(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * \left(R^H_{2(k-1)}(V)\right)^{(m)} * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1}.$$

is a solution of (6.1.1). □

Theorem 6.1.2. Given the equation

$$\boxtimes^k u(x) = f(x), \tag{6.1.6}$$

where $\boxtimes^k$ is the operator iterated $k-$ times defined by (**??**), $f(x)$ is a generalized function, $u(x)$ is an unknown function and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and n is even, then we obtain

$$u(x) = \left(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}\right) * \left(R^H_{2(k-1)}(V)\right)^{(m)} * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1} + \left(R^H_{10k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1} * f(x). \tag{6.1.7}$$

or

$$u(x) = \left((R^H_{2(k-1)}(V))^{(m)} + R^H_{2k}(x) * f(x)\right) * \left(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1}. \tag{6.1.8}$$

is a solution of (6.1.6). Where $(R^H_{2(k-1)}(x))^{(m)}$ is a function with m-derivatives defined by (1.0.19) and $\alpha = 2(k-1)$. If we put $q = 0$, we obtain the solution of Laplacian equation and if we put $p = 1$ and $x_1 = t$ where t_1 is time then we obtain the solution of wave equation.

Proof. From(6.1.6), we have

$$\boxtimes^k u(x) = f(x).$$

The above equation can be written

$$\otimes^k \circledast^k u(x) = f(x).$$

By Lemma 2.3.8 and Lemma 2.3.22, we have $G(x)$ and $H(x)$ are the elementary solution of the $\otimes^k$ operator and the $\circledast^k$ operator respectively. That is

$$\otimes^k G(x) = \delta(x) \quad , \quad \circledast^k H(x) = \delta(x). \tag{6.1.9}$$

Convolving both sides of (6.1.6)the above equation by $G(x) * H(x)$, we obtain,

$$(G(x) * H(x)) * \otimes^k \circledast^k u(x) = G(x) * H(x) * f(x)$$

By properties of convolution, we obtain

$$\otimes^k G(x) * \circledast^k H(x) * u(x) = G(x) * H(x) * f(x).$$

By (6.1.9), we obtain

$$\delta(x) * \delta(x) * u(x) = G(x) * H(x) * f(x)$$

or

$$u(x) = G(x) * H(x) * f(x). \tag{6.1.10}$$

We put (**??**) and (**??**) in (6.1.10). Thus $u(x)$ becomes

$$u(x) = \left(\left(R^H_{6k}(x) * (-1)^{2k} R^e_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1}\right) \\ * \left(\left(R^H_{4k}(x) * (-1)^{3k} R^e_{6k}(x)\right) * \left(S^{*k}(x)\right)^{*-1}\right) * f(x).$$

By Lemma 2.2.6, we obtain

$$u(x) = \left(R^H_{10k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1} * f(x). \tag{6.1.11}$$

Next, consider homogeneous equation

$$\boxtimes^k u(x) = 0.$$

By Theorem 6.1.1, we have a solution of homogeneous equation

$$u(x) = \left(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * \left(R^H_{2(k-1)}(V)\right)^{(m)} * \left(C^{*k}(x)\right)^{*-1} \left(S^{*k}(x)\right)^{*-1}$$

Thus the general solution of (6.1.6) is

$$u(x) = \left(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * \left(R^H_{2(k-1)}(V)\right)^{(m)} * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1} \\ + \left(R^H_{10k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1} * f(x) \tag{6.1.12}$$

or

$$u(x) = \big((R^H_{2(k-1)}(V))^{(m)} + R^H_{2k}(x) * f(x)\big) * \big(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}(x)\big) * \big(C^{*k}(x)\big)^{*-1} * \big(S^{*k}(x)\big)^{*-1}. \quad (6.1.13)$$

In particular, if $q = 0$ the equation (6.1.6) becomes the Laplace equation

$$\triangle^{6k} u(x) = f(x) \quad (6.1.14)$$

where $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}_p$ and p is even. Now, from (6.1.1) for $q = 0$ we have

$$\triangle^{6k} u(x) = 0 \quad or \quad \triangle^k(\triangle^{5k} u(x)) = 0.$$

By Lemma 2.3.7, we obtain

$$\triangle^{5k} u(x) = (-1)^{k-1} (R^e_{2(k-1)}(x))^{(m)}. \quad (6.1.15)$$

Since $(-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\triangle^k$ that is

$$\triangle^k (-1)^k R^e_{2k}(x) = \delta(x).$$

Convolving both sides of (6.1.15) by $(-1)^{5k} R^e_{10k}(x)$, we obtain

$$\begin{aligned} u(x) &= (-1)^{5k} R^e_{10k}(x) * (-1)^{k-1} (R^e_{2(k-1)}(x))^{(m)} \\ &= (-1)^{6k-1} (R^e_{12k-2}(x))^{(m)} \quad for \quad x = (x_1, x_2, \dots, x_p) \in \mathbb{R}_p. \end{aligned} \quad (6.1.16)$$

is a solution homogeneous equation of (6.1.14). Next, we convolve both sides of (6.1.14) by $(-1)^{6k} R^e_{12k}(x)$, we obtain

$$(-1)^{6k} R^e_{12k}(x) * \triangle^{6k} u(x) = (-1)^{6k} R^e_{12k}(x) * f(x)$$

$$\triangle^{6k} (-1)^{6k} R^e_{12k}(x) * u(x) = (-1)^{6k} R^e_{12k}(x) * f(x).$$

By Lemma 2.3.3, we obtain

$$\delta(x) * u(x) = u(x) = (-1)^{6k} R^e_{12k}(x) * f(x). \quad (6.1.17)$$

By (6.1.16) and (6.1.17) we obtain the general solution of equation (6.1.14) is

$$u(x) = (-1)^{6k-1} (R^e_{12k-2}(x))^{(m)} + (-1)^{6k} R^e_{12k}(x) * f(x) \quad (6.1.18)$$

for $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}_p$ and p is even.

It follows that (6.1.18) is the general solution of the Laplace equation

$$\triangle^{6k} u(x) = f(x),$$

where $\triangle^{6k}$ is the Laplace operator iterated $6k$-times defined by (1.0.22) for $x = (x_1, x_2, \dots, x_p) \in \mathbb{R}_p$ and p is even and if we put $k = 1$, then the equation (6.1.18) becomes

$$u(x) = (-1)^5 (R^e_{10}(x))^{(m)} + (-1)^6 R^e_{12}(x) * f(x) \quad (6.1.19)$$

is the general solution of the Laplace equation $\triangle^6 u(x) = f(x)$.

Now, consider the case for the wave equation. Given the equation

$$\Box^k T(x) \;=\; f(x), \tag{6.1.20}$$

where $\Box^k$ is defined by (1.0.18). $T(x)$ is an unknown function and $f(x)$ is a generalized function. By definition 2.2.4 and (2.2.8), we obtain

$$T(x) \;=\; R^H_{2k}(x) * f(x) \tag{6.1.21}$$

is a solution of (6.1.20) where $R^H_{2k}(x)$ is defined by (**??**) with $\beta = 2k$.

Now, from (6.1.11) we have

$$u(x) = \big(R^H_{10k}(x) * (-1)^{5k} R^e_{10k}(x)\big) * \big(C^{*k}(x)\big)^{*-1} * \big(S^{*k}(x)\big)^{*-1} * f(x)$$

is a solution of $\boxtimes^k u(x) \;=\; f(x)$.

Convolving both sides of the above equation by $(-1)^k R^H_{-8k}(x) * R^e_{-10k}(x) * \big(C^{*k}(x)\big) * \big(S^{*k}(x)\big)$. We obtain

$$\begin{aligned}(-1)^k R^H_{-8k}(x) * R_{-10k}(x) * \big(C^{*k}(x)\big) * \big(S^{*k}(x)\big) * u(x)\\ = (-1)^{6k}(R^H_{-8k}(x) * R^H_{8k}(x)) * (R^e_{-10k}(x) * (R^e_{10k}(x)) * R^H_{2k}(x) * f(x).\end{aligned}$$

By Lemma 2.2.7,

$$\begin{aligned}(-1)^k R^H_{-8k}(x) * R^e_{-10k}(x) * \big(S^{*k}(x)\big) * \big(C^{*k}(x)\big) * u(x) &= R^H_0(x) * R^e_0(x) * R^H_{2k}(x) * f(x)\\ &= \delta * \delta * R^H_{2k}(x) * f(x)\\ &= R^H_{2k}(x) * f(x).\end{aligned}$$

Thus it follows that

$$T(x) = (-1)^k R^e_{-10k}(x) * R^H_{-8k}(x) * \big(C^{*k}(x)\big) * \big(S^{*k}(x)\big) * u(x). \tag{6.1.22}$$

In particular, put $k = 1$ in (6.1.21), we have $T(x) = R^H_2(x) * f(x)$ is a solution of the equation

$$\Box T(x) = f(x). \tag{6.1.23}$$

If we put $p = 1$ and $x_1 = t$(where t is time), then $\Box = B_t - \sum_{i=2}^n B_{x_i}$ is the wave operator. Thus (6.1.23) becomes wave equation

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2}\right) T(x) = f(x). \tag{6.1.24}$$

Thus $T(x) = M_2(x) * f(x)$ is a solution of (6.1.24) and the general solution of (6.1.24) is

$$T(x) = \delta^{(m)}(V) + M_2(x) * f(x)$$

where $\delta^{(m)}(V)$ is a solution for $f(x) = 0$ and $M_2(x)$ is defined by (1.0.21) with $\alpha = 2$.

Now, put $k = 1$ in (6.1.6) and (6.1.7), we obtain

$$\boxtimes u(x) = f(x)$$

and

$$u(x) = \left(R_8^H(x) * (-1)^5 R_{10}^e(x)\right) * (R_0^H(V))^{(m)} * \left(C^{*1}(x)\right)^{*-1} * \left(S^{*1}(x)\right)^{*-1} + \left(R_{10}^H(x) * (-1)^5 R_{10}^e(x)\right) * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1} * f(x)\big) \quad (6.1.25)$$

or

$$u(x) = \left(\delta^{(m)}(V) + R_2^H(x) * f(x)\right) * \left(R_8^H(x) * (-1)^5 R_{10}^e(x)\right) * \left(C^{*1}(x)\right)^{*-1} * \left(S^{*1}(x)\right)^{*-1} \quad (6.1.26)$$

is a solution of $\boxtimes u(x) = f(x)$ and by (6.1.22) with $k = 1$, we obtain

$$T(x) = (-1)R_{-10}^e(x) * R_{-8}(x) * \left(C^{*1}(x)\right) * \left(S^{*1}(x)\right) * u(x)$$

is a solution of (6.1.23) where $u(x)$ is defined by (6.1.26). We put $u(x)$ where defined by (6.1.26) in $T(x)$, we obtain

$$T(x) = R_{-8}^H(x) * (-1)R_{-10}^e(x) * \left(C^{*1}(x)\right) * \left(S^{*1}(x)\right) * \left(\delta^{(m)}(V) + R_2^H(x) * f(x)\right) * \left(R_8^H(x) * (-1)^5 R_{10}^e(x)\right) * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1}.$$

or

$$T(x) = (R_{-10}^e(x) * R_{10}^e(x)) * (R_{-8}^H(x) * R_8^H(x)) * \left(\delta^{(m)}(V) + R_2^H(x) * f(x)\right).$$

By Lemma 2.2.7

$$\begin{aligned} T(x) &= R_0^e(x) * R_0^H(x) * \left(\delta^{(m)}(V) + R_2^H(x) * f(x)\right) \\ &= \delta(x) * \delta(x) * \left(\delta^{(m)}(V) + *R_2^H(x) * f(x)\right) \\ &= \delta^{(m)}(V) + R_2^H(x) * f(x), \end{aligned} \quad (6.1.27)$$

where $V = x_1^2 + x_2^2 + ... + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - ... - x_{p+q}^2$, $p + q = n$. Now, if we put $p = 1$ and $x_1 = t$ then (6.1.27) becomes $T(x) = \delta^{(m)}(V) + M_2(x) * f(x)$ for $V = t^2 - x_2^2 - x_3^2 - .... - x_n^2$ since $R_2^H(x)$ becomes $M_2^H(V)$ where $M_2^H(x)$ is defined by (**??**) with $\alpha = 2$.

Thus $T(x) = \delta^{(m)}(V) + R_2(x) * f(x)$ is the general solution of the wave equation of (6.1.24) and $\delta^{(m)}(V)$ is a solution of

$$\left(\frac{\partial^2}{\partial t^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2}\right) T(x) = 0. \quad (6.1.28)$$

Now $V = t^2 - x_2^2 - x_3^2 - .... - x_n^2$. Let $r^2 = x_2^2 + x_3^2 + ... + x_n^2$. Thus by [See 4, pp. 234-236] obtain

$$T(x, t) = \delta^{(m)}(t^2 - r^2)$$

is the solution of (6.1.28) with the initial condition $T(x, 0) = 0$ and $\frac{\partial T(x,0)}{\partial t} = (-1)^m 2\pi^{m+1}\delta(x)$ at $t = 0$ and $x = (x_2, x_3, ..., x_n) \in R^{n-1}$. □

Theorem 6.1.3. Consider the nonlinear equation

$$\boxtimes^k u(x) = f(x, \Box^{k-1} L_1^k \circledast_B^k u(x)) \tag{6.1.29}$$

where $\Box^{k-1}$ is the Ultra-hyperbolic operator iterated $k-1$ times defined by (1.0.18), L_1^k is the operator iterated k times defined by (2.3.26) and $\circledast^k$ is the operator iterated k times defined by (1.0.29).

Let f be bounded function and have continuous first derivatives for all $x \in \Omega \cup \partial\Omega$ is an open subset of R^n and $\partial\Omega$ denotes the boundary of Ω and n is even. That is

$$|f(x, \Box^{k-1} L_1^k \circledast^k u(x)| \leq N, \qquad x \in \Omega \tag{6.1.30}$$

and the boundary condition

$$\Box^{k-1} L_1^k \circledast^k u(x) = 0 \quad , \quad x \in \partial\Omega \tag{6.1.31}$$

then, we obtain

$$u(x) = R^H_{2(k-1)}(x) * \left(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * (C^{*k}(x))^{*-1} * (S^{*k}(x))^{*-1} * W(x) \tag{6.1.32}$$

as a solution of (6.1.29) with the boundary condition

$$u(x) = \left(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * (C^{*k}(x))^{*-1} * (S^{*k}(x))^{*-1} * (R_{2(k-1)}(V))^{(m)} \tag{6.1.33}$$

for $k = 2, 3, 4, 5, ...$ and $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, $R^H_{8k}(x)$ is defined by (**??**) with $\beta = 8k$ and $R^e_{10k}(x)$ is defined by (**??**) with $\alpha = 10k$. The function $(R_{2(k-1)}(V))^{(m)}$ is defined by (**??**) with m derivatives and $\beta = 2(k-1)$. $C^{*k}(x)$ and $S^{*k}(x)$ denoted the convolution itself $k-$ times where $C(x)$ and $S(x)$ is defined by (**??**) and (**??**) respectively.

Moreover, for $k = 1$, we have

$$u(x) = \left(R^H_8(x) * (-1)^5 R^e_{10}(x)\right) * (C^{*1}(x))^{*-1} * (S^{*1}(x))^{*-1} * W(x)$$

as a solution of (6.1.29) with boundary condition

$$u(x) = \delta^{(m)}(x) * \left(R^H_8(x) * (-1)^5 R^e_8(x)\right) * (C^{*1}(x))^{*-1} * (S^{*1}(x))^{*-1}$$

for $x \in \partial\Omega$. Where $\delta^{(m)}(x)$ is the Dirac-delta distribution with m derivatives. Also, if we put $k = 1, p = 1$ and $q = n - 1$, we obtain

$$u(x) = \left(M_8(x) * (-1)^5 R^e_{10}(x)\right) * (C^{*1}_r(x))^{*-1} * (S^{*1}_r(x))^{*-1} * W(x)$$

as a solution of the inhomogeneous equation

$$L_1^* \circledast^* u(x) = W(x)$$

with the boundary condition

$$L_1^* \circledast^* u(x) = 0 \quad for \quad x \in \partial\Omega$$

or for $x \in \partial\Omega$,

$$u(x) = \delta^{(m)}(x) * \left(I^H_8(x) * (-1)^4 R^e_8(x)\right) * (C^{*1}_r(x))^{*-1} * (S^{*1}_r(x))^{*-1}.$$

Where

$$L_1^* = \frac{3}{4}\triangle + \frac{1}{4}\Box^*$$

and

$$\Box^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \quad \triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$$

and

$$\circledast^* = \left(\frac{\partial^2}{\partial x_1^2}\right)^3 - \left(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right)^3,$$

where $M_8^H(x)$ defined by (1.0.21) with $\beta = 8$, $C_r(x)$ reduces from $C(x)$ where is defined by (1.0.21), that is $C_r(x) = \frac{3}{4}M_4(x) + \frac{1}{4}(-1)^2 R_4^e(x)$. And $S_r(x)$ reduces from $S(x)$ where is defined by (**??**), that is $S_r(x) = \frac{3}{4}(-1)^2 R_4^e(x) + \frac{1}{4}M_4^H(x)$, where $M_4^H(x)$ defined by (**??**) with $\alpha = 4$.

Proof.

$$\begin{aligned} \Box^k u(x) &= \otimes^k \circledast^k u(x) \\ &= \Box\Box^{k-1} L_1^k \circledast^k u(x) \\ &= f(x, \Box^{k-1} L_1^k \circledast^k u(x)). \end{aligned}$$

Since $u(x)$ has continuous derivatives up to order $12k$ for $k = 1, 2, 3, \ldots$ and we can assume

$$\Box^{k-1} L_1^k \circledast^k u(x) = W(x) \quad , \quad \forall x \in \Omega \tag{6.1.34}$$

Thus, (6.1.29) can be written in the form

$$\boxtimes^k u(x) = \Box w(x) = f(x, w(x)) \tag{6.1.35}$$

by (6.1.30)

$$|f(x, W)| \leq N \quad , \quad \forall x \in \Omega \tag{6.1.36}$$

and by (6.1.34), $W(x) = 0$ or $\Box^{k-1} L_1^k \circledast^k u(x) = 0$ for $x \in \partial\Omega$. Thus by Lemma 2.12, there exists a unique solution $W(x)$ of (6.1.34) which satisfies (6.1.35).

Now consider the Eq.(3.34)). By Lemma 2.3.4, Lemma 2.3.9 and Lemma 2.3.10, we have $(-1)^{k-1} R_{2(k-1)}^H(x)$, $H(x)$ and $K(x)$ are the elementary solution of the operators $\Box^{k-1}$, $\circledast^k$ and L_1^k respectively. That is

$$\Box^{k-1} R_{2(k-1)}^H(x) = \delta(x) \quad , \quad \circledast^k H(x) = \delta(x) \tag{6.1.37}$$

and

$$L_1^k K(x) = \delta(x) \tag{6.1.38}$$

where δ is the Dirac-delta function .

Convolving both sides of (6.1.34) by $R_{2(k-1)}^H(x) * H(x) * K(x)$. We obtain

$$R_{2(k-1)}^H(x) * H(x) * K(x) * \Box^{k-1} \circledast^k L_1^k u(x) \quad = R_{2(k-1)}^H(x) * H(x) * K(x) * W(x)$$

By the properties of convolution, we obtain

$$\square^{k-1}R^H_{2(k-1)}(x) * \circledast^k H(x) * L_1^k K(x) * u(x) \quad = R^H_{2(k-1)}(x) * H(x) * K(x) * W(x)$$

By (6.1.37) and (6.1.38), we obtain

$$\delta * \delta * \delta * u(x) = u(x) = R^H_{2(k-1)}(x) * H(x) * K(x) * W(x). \tag{6.1.39}$$

Put (2.3.23) and (2.3.27) in (6.1.39), we obtain

$$u(x) = R^H_{2(k-1)}(x) * \left(\left(R^H_{4k}(x) * (-1)^{3k} R^e_{6k}\right) * \left(S^{*k}(x)\right)^{*-1}\right) * \\ \left(\left(R^H_{4k}(x) * (-1)^{2k} R^e_{4k}\right) * \left(C^{*k}(x)\right)^{*-1}\right) * W(x) \tag{6.1.40}$$

By Lemma 2.3, (6.1.40) becomes

$$u(x) = R^H_{2(k-1)}(x) * \left(R^H_{8k}(x) * (-1)^{5k} R^e_{10k}(x)\right) * \left(C^{*k}(x)\right)^{*-1} * \left(S^{*k}(x)\right)^{*-1} \tag{6.1.41}$$

as required.

Next, consider the boundary condition

$$\square^{k-1} L_1^k \circledast^k u(x) = 0 \quad , \quad x \in \partial\Omega. \tag{6.1.42}$$

By Lemma 2.3.6, we have

$$L_1^k \circledast^k u(x) = (R_{2(k-2)}(V))^{(m)}.$$

Convolving both sides of the above equation by $K(x) * H(x)$. We obtain

$$K(x) * H(x) * L_1^k \circledast^k u(x) = K(x) * H(x) * (R_{2(k-1)}(V))^{(m)}$$

By the properties of convolution, we obtain

$$L_1^k K(x) * \circledast^k H(x) * u(x) = K(x) * H(x) * (R_{2(k-1)}(V))^{(m)}$$

By (6.1.37) and (6.1.38), the above equation becomes

$$\delta * \delta * u(x) = u(x) = K(x) * H(x) * (R^H_{2(k-1)}(V))^{(m)}. \tag{6.1.43}$$

Put (2.3.23) and (2.3.27) in (6.1.43), we obtain

$$u(x) = \left(R^H_{4k} * (-1)^{2k} R^e_{4k}(x)\right) * \left(C^{*k}(x)\right)^{*-1}\right) * \\ \left(R^H_{4k}(x) * (-1)^{3k} R^e_{6k}(x)\right) \left(S^{*k}(x)\right)^{*-1}\right) * (R_{2(k-1)}(V))^{(m)} \tag{6.1.44}$$

By Lemma 2.2.6, (6.1.44) becomes,

$$u(x) = \left(R^H_{8k} * (-1)^{5k} R^e_{10k}(x)\right) * (C^{*k}(x))^{*-1} * (S^{*k}(x))^{*-1} * (R_{2(k-1)}(V))^{(m)} \tag{6.1.45}$$

as required for $x \in \partial\Omega$, and $k = 2, 3, 4, 5, \ldots \ldots$.

Now, for $k = 1$ in (6.1.41), we obtain

$$u(x) = \delta(x) * \left(R^H_8(x) * (-1)^5 R^e_{10}(x)\right) * (C^{*1}(x))^{*-1} * (S^{*1}(x))^{*-1} * W(x) \tag{6.1.46}$$

Since $R_0^H(x) = \delta(x)$. Now consider the boundary condition for $k = 1$ in (6.1.42), we obtain

$$L_1 \circledast u(x) = 0, \quad for \quad x \in \partial\Omega.$$

By (1.0.2) the above equation can be written as

$$L_1 L_2 \triangle u(x) = \triangle L_1 L_2 u(x) = 0, \quad for \quad x \in \partial\Omega.$$

Thus by Lemma 2.3.7, for $k = 1$, we have

$$L_1 L_2 u(x) = (R_0^e(x))^{(m)} = \delta^{(m)}(x).$$

By Lemma 2.3.10, Lemma 2.3.11, we obtain

$$u(x) = \delta^{(m)}(x) * \left(R_4^H(x) * (-1)^2 R_4^e(x)\right) * (C^{*1}(x))^{*-1} * \\ \left(R_4^H(x) * (-1)^2 R_4^e(x)\right) * (S^{*1}(x))^{*-1}$$

or by Lemma 2.2.6,

$$u(x) = \delta^{(m)}(x) * \left(R_8^H(x) * (-1)^4 R_8^e(x)\right) * (C^{*1}(x))^{*-1} * (S^{*1}(x))^{*-1}. \qquad (6.1.47)$$

Now consider the case $k = 1, p = 1$ and $q = n - 1$, thus from (6.1.41), $R_8^H(x)$ reduce to $M_8^H(x)$, where $M_8^H(x)$ is defined by (1.0.21) with $\beta = 8$ and the operator $\circledast$ defined by (1.0.29) reduces to the operator

$$\circledast^* = \left(\frac{\partial^2}{\partial x_1^2}\right)^3 - \left(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}\right)^3$$

and the L_1 operator defined by (2.3.26) reduced to the L_1^* operator and L_1^* defined by

$$L_1^* = \frac{3}{4}\triangle + \frac{1}{4}\Box^*,$$

where$\Box^*$ and $\triangle$ are defined by

$$\Box^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \quad \triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

Thus the solution of (6.1.41) reduces to

$$u(x) = \left(M_8^H(x) * (-1)^5 R_{10}^e(x)\right) * (C_r^{*1}(x))^{*-1} * (S_r^{*1}(x))^{*-1} * W(x).$$

Which is the solution of the inhomogeneous equation

$$L^* \circledast^* u(x) = W(x)$$

with the boundary condition for $x \in \partial\Omega$

$$L^* \circledast^* u(x) = 0$$

or for $x \in \partial\Omega$

$$u(x) = \delta^{(m)}(x) * \left(M_8(x) * (-1)^4 R_8^e(x)\right) * (C_r^{*1}(x))^{*-1} * (S_r^{*1}(x))^{*-1}.$$

as required. □