CHAPTER 1

INTRODUCTION

A partial order on a semigroup $(S, \cdot)$ is called *natural* if it is defined by means of the multiplication of S. Nambooripad [4], and Hartwig [1], independently discovered the natural partial order on a regular semigroup. There are many equivalent formulations (see [3] for detail), one of which is the following due to P. R. Jones. On any semigroup S define the relation ν by

$$a \, \nu \, b \text{ if and only if there are } e, f \in E(S^1) \text{ such that } a = eb = bf.$$

If S is a regular semigroup, then ν is a natural partial order on S.

In 1986, H. Mitsch [3] showed that for any semigroup S the relation

$$a \leq b \text{ iff } a = xb = by, xa = a \text{ for some } x, y \in S^1,$$

is a partial order which specializes to the natural partial order on regular semigroups.

Let X be a set, we denote the set of all mappings from X into X by $T(X)$ and it is a semigroup under composition of mappings: if $\alpha, \beta \in T(X)$, then $\alpha \circ \beta \in T(X)$ is defined by

$$x(\alpha \circ \beta) = (x\alpha)\beta, \; x \in X.$$

We call $T(X)$ the *full transformation semigroup on a set* X.

G. Kowol and H. Mitsch [3] studied properties of the so-called natural partial order $\leq$ on $T(X)$. They gave a characterization of this ordering in terms of images and kernels, the maximal and minimal elements were determined and the covering elements were described.

Let E be a nontrivial equivalence on X. Write

$$T_E(X) = \{\alpha \in T(X) : \forall (x, y) \in E, (x\alpha, y\alpha) \in E\},$$

then $T_E(X)$ is a subsemigroup of $T(X)$.

L. Sun, H. Pei and Z. Cheng [6] endowed $T_E(X)$ with the natural order and determined when two elements of $T_E(X)$ are related under this order, then found out elements of $T_E(X)$ which are compatible with $\leq$ on $T_E(X)$. Also, the maximal and minimal elements and covering elements were described.

In addition, we can define a transformation semigroup with restricted range as follow. Let X be any set and $\emptyset \neq Y \subseteq X$. Write

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\},$$

then $T(X,Y)$ is a subsemigroup of $T(X)$. We see that if $X = Y$, then $T(X,Y) = T(X)$. In 2008, J. Sanwong and W. Sommanee [5] studied regularity and Green's relation on $T(X,Y)$.

By the definition of $T(X,Y)$ which is a generalization of $T(X)$, we will study this semigroup with the natural order for expanding the results of G. Kowol and H. Mitsch.

The purpose of this thesis are:

(1) To characterize when two elements of $T(X,Y)$ are related under the natural order $\leq$.

(2) To find out elements of $T(X,Y)$ which are compatible with $\leq$ on $T(X,Y)$.

(3) To give a necessary and sufficient condition for elements in $T(X,Y)$ to be maximal or minimal.

(4) To decribe the covering elements.

The thesis comprises of four chapters. Chapter 1 is for the introduction. In Chapter 2, we list some well-known results, definitions and notations that will be used throughout this thesis. In Chapter 3, we present the characterization of the natural order $\leq$ on $T(X,Y)$, then we show some results about compatibility, maximal, minimal and covering elements. Finally, Chapter 4 is for the conclusion.