CHAPTER 2

PRELIMINARIES

2.1 Semigroups

A *semigroup* S is a nonempty set S together with a binary operation $\cdot : S \times S \to S$ which satisfies the associative property: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$.

Let S be a semigroup. An element e of S is called a *left identity* if $e \cdot a = a$ for all $a \in S$, and a *right identity* if $a \cdot e = a$ for all $a \in S$. If e is both a left identity and a right identity, then it is called a two-sided identity, or simply an identity.

Every semigroup has at most one identity element. A semigroup with identity is called a *monoid*. A semigroup without identity may be embedded into a monoid simply by adjoining an element $1 \notin S$ to S and defining $1 \cdot s = s \cdot 1 = s$ for all $s \in S \cup \{1\}$. The notation S^1 denotes a monoid obtained from S by adjoining an identity if necessary ($S^1 = S$ for a monoid).

2.2 Partially Orders

A partial order is a binary relation $\leq$ over a nonempty set P which is reflexive, antisymmetric, and transitive, i.e., for all a, b, c in P, we have that:

(1) $a \leq a$ (reflexivity);

(2) if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry);

(3) if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).

A set with a partial order is called a *partially ordered set*.

An element $m \in P$ is a *maximal element* of P if for all $s \in P$, $m \leq s$ implies $m = s$. An element $g \in P$ is the *greatest* or *maximum element* of P if $s \leq g$, for all elements $s \in P$. The definition for *minimal elements* and the *least* or *minimum elements* are defined dually.

Let $(P, \leq)$ be a partially ordered set. An element $r \in P$ is called an *upper cover* for $p \in P$ if $p < r$ and there exists no $q \in P$ such that $p < q < r$. *Lower cover* is defined dually.

We note that the notation $p < q$ means $p \leq q$ and $p \neq q$.

Let $\leq$ be a partial order on a semigroup S. An element $c \in S$ is said to be *left compatible* with $\leq$ if $ca \leq cb$ for all $a, b \in S$ such that $a \leq b$. *Right compatibility* with $\leq$ is defined dually.

2.3 Cardinality

The *cardinality* of a set is a measure of the number of elements of the set. For example, the set $A = \{2, 4, 6\}$ contains 3 elements, and therefore A has a cardinality of 3.

The cardinality of a set A is denoted $|A|$.

The formal definition of cardinality depends on the notion mappings between sets:

(1) Two sets A and B have the same cardinality if there exists a bijection, that is, an injective and surjective function, from A to B. Symbolically, we write $|A| = |B|$.

(2) A has cardinality less than or equal to the cardinality of B if there exists an injective function from A into B. Symbolically, we write $|A| \leq |B|$.

(3) A has cardinality strictly less than the cardinality of B if there is an injective function, but no bijective function, from A to B. Symbolically, we write $|A| < |B|$.

2.4 Semigroups of Transformations

Let X be a set, we denote the set of all mappings from X into X by $T(X)$ and it is a semigroup under composition of mappings: if $\alpha, \beta \in T(X)$, then $\alpha \circ \beta \in T(X)$ is defined by

$$x(\alpha \circ \beta) = (x\alpha)\beta, \ x \in X.$$

For abbreviation, we always write $\alpha\beta$ for $\alpha \circ \beta$.

It is well-known that $T(X)$ is a regular semigroup, that is for each $\alpha \in T(X)$ there exists $\beta \in T(X)$ such that $\alpha = \alpha\beta\alpha$.

Here, if Y is a nonempty subset of X, we define

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}.$$

Since $\alpha, \beta \in T(X,Y)$, we have $X\alpha \subseteq Y$ and $X\beta \subseteq Y$. Then $X\alpha\beta \subseteq X\beta \subseteq Y$, so $\alpha\beta \in T(X,Y)$. Therefore $T(X,Y)$ is a subsemigroup of $T(X)$.

We note that for any $\alpha \in T(X,Y)$, $\pi_\alpha = \{(a,b) \in X \times X : a\alpha = b\alpha\}$ is an equivalence on X. The relation π_α is usually called the *kernel* of α.

J. Sanwong and W. Sommanee gave the following result.

Lemma 2.4.1 [5] *Let $\alpha, \beta \in T(X,Y)$. Then $\pi_\beta \subseteq \pi_\alpha$ if and only if $\alpha = \beta\gamma$ for some $\gamma \in T(X,Y)$.*

In general, $T(X,Y)$ is not a regular semigroup, so they introduced a very useful set $F \subseteq T(X,Y)$ which is defined by

$$F = \{\alpha \in T(X,Y) : X\alpha \subseteq Y\alpha\}$$

It is easy to see that $F = \{\alpha \in T(X,Y) : (X \setminus Y)\alpha \subseteq Y\alpha\} = \{\alpha \in T(X,Y) : X\alpha = Y\alpha\}$. They also proved that F is a right ideal and the largest regular subsemigroup of $T(X,Y)$.