CHAPTER 3

MAIN RESULTS

In this chapter, we endow $T(X, Y)$ with the natural order $\leq$ and determine when two elements of $T(X, Y)$ are related under this order, then find out elements of $T(X, Y)$ which are compatible with $\leq$ on $T(X, Y)$. Also, the maximal and minimal elements and the covering elements are described.

Recall that for any semigroup S, we define $\leq$ on S as follow:

$$a \leq b \text{ if and only if } a = xb = by, xa = a \text{ for some } x, y \in S^1.$$

3.1 Characterization

In this section, we investigate the condition under which $\alpha \leq \beta$ for two elements $\alpha, \beta \in T(X, Y)$.

Lemma 3.1.1 *Let $\alpha, \beta \in T(X, Y)$, then the following statements are equivalent:*
(1) $X\alpha \subseteq Y\beta$;
(2) $(x\alpha)\beta^{-1} \cap Y \neq \emptyset$ for all $x \in X$;
(3) there exists $\gamma \in T(X, Y)$ such that $\alpha = \gamma\beta$.

Proof. (1) implies (2). Assume that $X\alpha \subseteq Y\beta$. Let $x \in X$. Since $x\alpha \in X\alpha \subseteq Y\beta$, we have $x\alpha = y\beta$ for some $y \in Y$. Therefore $y \in (x\alpha)\beta^{-1} \cap Y \neq \emptyset$.

(2) implies (3). Suppose that $(x\alpha)\beta^{-1} \cap Y \neq \emptyset$ for all $x \in X$. Then for each $x \in X$, there exists $d_x \in (x\alpha)\beta^{-1} \cap Y$. We define γ by $x\gamma = d_x$ for all $x \in X$. So $X\gamma \subseteq Y$ and $x\gamma\beta = d_x\beta = x\alpha$ for all $x \in X$. Therefore $\alpha = \gamma\beta$.

(3) implies (1). Assume that $\alpha = \gamma\beta$ for some $\gamma \in T(X, Y)$. Let $x\alpha \in X\alpha$. Since $x\alpha = x\gamma\beta \in Y\beta$, therefore $X\alpha \subseteq Y\beta$.

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To prove the next theorem, we need the following characterization of the natural partial order on any semigroups which is taken from H. Mitsch [3].

Lemma 3.1.2 *For any semigroups S and its natural partial order, the following conditions are equivalent:*

(1) $a \leq b$;

(2) $a = sb = bt$, $at = a$ for some $s, t \in S^1$;

(3) $a = ub = bv$, $ua = av = a$ for some $u, v \in S^1$.

Theorem 3.1.3 *Let $\alpha, \beta \in T(X,Y)$ such that $\alpha \neq \beta$. Then $\alpha \leq \beta$ if and only if the following statements hold:*

(1) $X\alpha \subseteq Y\beta$;

(2) $\pi_\beta \subseteq \pi_\alpha$;

(3) if $x\beta \in X\alpha$, then $x\alpha = x\beta$.

Proof. Suppose that $\alpha \leq \beta$, then there exist $\gamma, \mu \in T(X,Y)^1$ such that $\alpha = \gamma\beta = \beta\mu$ and $\alpha = \alpha\mu$ by Lemma 3.1.2. Since $\alpha \neq \beta$, we get $\gamma \neq 1 \neq \mu$, thus by Lemma 3.1.1 and Lemma 2.4.1, we have $X\alpha \subseteq Y\beta$ and $\pi_\beta \subseteq \pi_\alpha$. If $x\beta \in X\alpha$, then $x\beta = y\alpha$ for some $y \in X$ and therefore $x\alpha = x\beta\mu = y\alpha\mu = y\alpha = x\beta$.

Conversely, assume that the assumptions hold. Since $X\alpha \subseteq Y\beta$ and $\pi_\beta \subseteq \pi_\alpha$, by Lemma 3.1.1 and Lemma 2.4.1 there exist $\gamma, \mu \in T(X,Y)$ such that $\alpha = \gamma\beta = \beta\mu$. For each $x \in X$, $x\alpha = x\gamma\beta = y\beta$ for some $y \in Y$ since $\gamma \in T(X,Y)$, so $y\beta \in X\alpha$. By (3), we have $y\alpha = y\beta$ and thus $x\alpha = y\beta = y\alpha = y\beta\mu = x\alpha\mu$, hence $\alpha = \alpha\mu$. By Lemma 3.1.2, it is concluded that $\alpha \leq \beta$.

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Example 3.1.4 (1) Let $X = \{1,2,3,4,5,6\}, Y = \{2,4,6\}$. We define $\alpha, \beta \in T(X,Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 6 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 4 & 4 & 6 & 6 \end{pmatrix}.$$

Then there are $\gamma, \mu \in T(X,Y)$ such that

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 6 \end{pmatrix}, \mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 2 & 2 & 2 & 6 & 6 \end{pmatrix},$$

and $\alpha = \gamma\beta = \beta\mu$, $\alpha = \alpha\mu$ which follows that $\alpha \leq \beta$. In addition, we can check this by Theorem 3.1.3. Consider:

(i) $X\alpha = \{2, 6\} \subseteq \{2, 4, 6\} = Y\beta$;

(ii) $\pi_\beta = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (2,1), (3,4), (4,3), (5,6), (6,5)\}$, and

$\pi_\alpha = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,2), (2,1), (1,3), (3,1), (1,4), (4,1), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3), (5,6), (6,5)\}$, thus $\pi_\beta \subseteq \pi_\alpha$;

(iii) $1\beta, 2\beta, 5\beta, 6\beta \in X\alpha$, and $1\alpha = 1\beta, 2\alpha = 2\beta, 5\alpha = 5\beta, 6\alpha = 6\beta$. Thus $\alpha \leq \beta$.

(2) Let X be the set of all natural numbers, $\mathbb{N}$, and Y the set of all positive odd integers. Let $n \in \mathbb{N}$. Then by division algorithm there exist unique $q_n, r_n \in \mathbb{Z}$ such that

$$n = 4q_n + r_n; 0 \leq r_n < 4.$$

Thus we define α as follow:

$$n\alpha = \begin{cases} n - r_n + 1 & \text{if } r_n \neq 0 \\ n - 3 & \text{if } r_n = 0. \end{cases}$$

Similarly, for each $n \in \mathbb{N}$ there exist unique $t_n, s_n \in \mathbb{N}$ such that

$$n = 2t_n + s_n; 0 \leq s_n < 2.$$

Thus we define β as follow:

$$n\beta = \begin{cases} n & \text{if } s_n = 1 \\ n - 1 & \text{if } s_n = 0. \end{cases}$$

So $\alpha, \beta \in T(X, Y)$. We check that $\alpha \leq \beta$ by Theorem 3.1.3. By computation, we get

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \dots \\ 1 & 1 & 1 & 1 & 5 & 5 & 5 & 5 & 9 & 9 & 9 & 9 & \dots \end{pmatrix},$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \dots \\ 1 & 1 & 3 & 3 & 5 & 5 & 7 & 7 & 9 & 9 & 11 & 11 & \dots \end{pmatrix}.$$

We see that $X\alpha = \{1, 5, 9, 13, 17, ...\} \subseteq \{1, 3, 5, 7, 9, 11, ...\} = Y\beta$. Let $(m, n) \in \pi_\beta$. Then $m\beta = n\beta$. If $s_m = 1 = s_n$ or $s_m = 0 = s_n$, then $m = m\beta = n\beta = n$ or $m-1 = m\beta = n\beta = n-1$, respectively. Thus $m = n$ which follows that $m\alpha = n\alpha$. If $s_m = 1$ and $s_n = 0$, then $m = m\beta = n\beta = n-1$. Since $s_n = 0$, we have $r_n = 0$ or 2. If $r_n = 0$, then $r_m = 3$ since $m = n-1$. We get $m\alpha = m-r_m+1 = m-3+1 = m-2 = (n-1)-2 = n-3 = n\alpha$. If $r_n = 2$, then $r_m = 1$ since $m = n-1$. We have $m\alpha = m-r_m+1 = m-1+1 = m = n-1 = n-2+1 = n-r_n+1 = n\alpha$. Therefore $\pi_\beta \subseteq \pi_\alpha$. Let $n\beta \in X\alpha$. If $r_n = 0$, then $s_n = 0$ which implies that $n\beta = n-1$. Since $r_n = 0$ we get $n = 4k$ for some integer $k \geq 1$. Then $n\beta = n-1 = 4k-1$ which follows that $n\beta = 3, 7, 11, 15, \ldots \notin X\alpha$ which is a contradiction. If $r_n = 3$, then $n = 4k+3$ for some integer $k \geq 0$. We have $n = 2(2k+1)+1$ which implies that $s_n = 1$. Then $n\beta = n = 4k+3$. Thus $n\beta = 3, 7, 11, 15, \ldots \notin X\alpha$ which is also a contradiction. So, it is concluded that $r_n = 1$ or 2. If $r_n = 1$, then $n = 4k+1$ for some integer $k \geq 0$. We have $n = 2(2k)+1$ which implies that $s_n = 1$. Then $n\alpha = n-r_n+1 = n-1+1 = n = n\beta$. If $r_n = 2$, then $n = 4k+2$ for some integer $k \geq 0$. We have $n = 2(2k+1)$ which follows that $s_n = 0$. Then $n\alpha = n-r_n+1 = n-2+1 = n-1 = n\beta$ and hence $n\alpha = n\beta$. Therefore $\alpha \leq \beta$.

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We consider the case when $X = Y$ which implies that $T(X, Y) = T(X)$. By Theorem 3.1.3, we get the following result.

Corollary 3.1.5 *Let $\alpha, \beta \in T(X)$ such that $\alpha \neq \beta$. Then $\alpha \leq \beta$ if and only if the following statements hold:*

(1) $X\alpha \subseteq X\beta$;

(2) $\pi_\beta \subseteq \pi_\alpha$;

(3) if $x\beta \in X\alpha$, then $x\alpha = x\beta$.

To prove the following corollary, we recall that $F = \{\alpha \in T(X, Y) : X\alpha \subseteq Y\alpha\} = \{\alpha \in T(X, Y) : X\alpha = Y\alpha\}$.

Corollary 3.1.6 *Let $\alpha \in F, \beta \in T(X,Y)$ such that $\alpha \neq \beta$ and let $K(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\}$. If the following properties hold:*
(1) $x\alpha \in Y\beta$ and $x\beta \notin X\alpha$ for all $x \in K(\alpha,\beta)$;
(2) $x\beta \neq y\beta$ for all $x, y \in K(\alpha,\beta)$ with $x \neq y$,
then $\alpha \leq \beta$.

Proof. Since $\alpha \neq \beta$, we have $K(\alpha,\beta)$ is nonempty. For each $a\alpha \in X\alpha$, if $a \in K(\alpha,\beta)$ then $a\alpha \in Y\beta$ by (1). If $a \notin K(\alpha,\beta)$, then $a\alpha = a\beta$. Since $\alpha \in F$, we have $a\alpha \in X\alpha \subseteq Y\alpha$ which implies that $a\alpha = b\alpha$ for some $b \in Y$. If $b \in K(\alpha,\beta)$, then $a\alpha = b\alpha \in Y\beta$ by (1). If $b \notin K(\alpha,\beta)$, then $b\alpha = b\beta$ which implies that $a\alpha = b\alpha = b\beta \in Y\beta$. Therefore $X\alpha \subseteq Y\beta$. Now, let $(p,q) \in \pi_\beta$. So $p\beta = q\beta$. If $p \in K(\alpha,\beta)$ and $q \notin K(\alpha,\beta)$, then $p\beta = q\beta = q\alpha \in X\alpha$. By (1), we have $p\beta \notin X\alpha$ which is a contradiction. If $p, q \in K(\alpha,\beta)$ and $p \neq q$, then $p\beta \neq q\beta$ by (2) and this contradicts $p\beta = q\beta$. It is concluded that $p, q \notin K(\alpha,\beta)$. So $p\alpha = p\beta = q\beta = q\alpha$ which follows that $(p,q) \in \pi_\alpha$ and therefore $\pi_\beta \subseteq \pi_\alpha$. Finally, if $w\beta \in X\alpha$ then we have $w \notin K(\alpha,\beta)$ by (1) which implies that $w\alpha = w\beta$. Therefore $\alpha \leq \beta$ by Theorem 3.1.3.

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Lemma 3.1.7 *If $\alpha, \beta \in T(X,Y)$ such that $\alpha \leq \beta$ and $\alpha \neq \beta$, then $\alpha \in F$.*

Proof. Let $\alpha, \beta \in T(X,Y)$ with $\alpha \leq \beta$ and $\alpha \neq \beta$. Then by Theorem 3.1.3 we have $X\alpha \subseteq Y\beta$ and $x\beta \in X\alpha$ implies $x\alpha = x\beta$. Suppose that $\alpha \notin F$, then there is $x \in X \setminus Y$ such that $x\alpha \notin Y\alpha$. Since $X\alpha \subseteq Y\beta$, we have $x\alpha = y\beta$ for some $y \in Y$, thus $y\beta = x\alpha \in X\alpha$ implies $y\alpha = y\beta$. Hence $x\alpha = y\alpha \in Y\alpha$ which is a contradiction. Therefore, $\alpha \in F$.

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3.2 Compatibility

Recall that an element $\gamma \in T(X,Y)$ is said to be *left compatible* with $\leq$ if $\gamma\alpha \leq \gamma\beta$ for all $\alpha, \beta \in T(X,Y)$ such that $\alpha \leq \beta$. *Right compatibility* with $\leq$ is defined dually. In this section, we will find out elements of $T(X,Y)$ which are compatible with $\leq$ on $T(X,Y)$.

We note that if $|Y| = 1$, then $|T(X,Y)| = 1$ which implies that an element in $T(X,Y)$ is left and right compatible. So we assume that $|Y| > 1$.

Theorem 3.2.1 *Let $\gamma \in T(X,Y)$. Then γ is left compatible with $\leq$ on $T(X,Y)$ if and only if $Y = Y\gamma$.*

Proof. We prove the only if part of the theorem by contrapositive. Assume that $Y \neq Y\gamma$, then there exists $y \in Y \setminus Y\gamma$. Since $|Y| > 1$, there is $z \in Y$ such that $z \neq y$. We define $\alpha, \beta \in T(X,Y)$ by $x\alpha = y$ for all $x \in X$ and

$$x\beta = \begin{cases} y & \text{if } x = y \\ z & \text{if } x \neq y. \end{cases}$$

We have $X\alpha = \{y\} \subseteq \{y,z\} = Y\beta, \pi_\beta \subseteq X \times X = \pi_\alpha$ and if $x\beta \in X\alpha = \{y\}$, then $x\beta = y = x\alpha$. Therefore $\alpha \leq \beta$ by Theorem 3.1.3. Since $X\gamma\alpha = \{y\} \nsubseteq \{z\} = Y\gamma\beta$ which implies that $\gamma\alpha \nleq \gamma\beta$, we have γ is not left compatible with $\leq$ on $T(X,Y)$.

Conversely, assume that $Y = Y\gamma$. Let $\alpha, \beta \in T(X,Y)$ such that $\alpha \leq \beta$. We have $X\gamma\alpha \subseteq X\alpha \subseteq Y\beta = Y\gamma\beta$. Let $(x,y) \in \pi_{\gamma\beta}$, then $x\gamma\beta = y\gamma\beta$. So $(x\gamma, y\gamma) \in \pi_\beta \subseteq \pi_\alpha$, then $x\gamma\alpha = y\gamma\alpha$. Thus $(x,y) \in \pi_{\gamma\alpha}$, and that $\pi_{\gamma\beta} \subseteq \pi_{\gamma\alpha}$. Let $x\gamma\beta \in X\gamma\alpha$. Then $x\gamma\beta \in X\alpha$. So $x\gamma\alpha = x\gamma\beta$. Hence $\gamma\alpha \leq \gamma\beta$ by Theorem 3.1.3, therefore γ is left compatible with $\leq$ on $T(X,Y)$.

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Example 3.2.2 Let $X = \{1,2,3,4,5,6\}, Y = \{1,2,3,4\}$. We define $\alpha \in T(X,Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 1 & 3 & 2 & 4 \end{pmatrix}.$$

We see that $Y\alpha = \{1,2,3,4\} = Y$. Thus α is left compatible with $\leq$ on $T(X,Y)$ by Theorem 3.2.1.

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Lemma 3.2.3 *If $|Y| = 2$, then γ is right compatible with $\leq$ on $T(X,Y)$ for all $\gamma \in T(X,Y)$.*

Proof. Assume that $|Y| = 2$ and $\alpha, \beta \in T(X,Y)$ with $\alpha \leq \beta$. We first prove that $\alpha = \beta$ or $|X\alpha| = 1$. Suppose that $|X\alpha| \geq 2$. So $2 \leq |X\alpha| \leq |Y| = 2$, then $X\alpha = Y$. For each $x \in X$, $x\beta \in X\beta \subseteq Y = X\alpha$ and hence $x\alpha = x\beta$ by Theorem 3.1.3. Thus $\alpha = \beta$. Now, let γ be any element in $T(X,Y)$. If $\alpha = \beta$, then $\alpha\gamma = \beta\gamma$. If $|X\alpha| = 1$, then α is a constant map, this implies that $\alpha\gamma$ is also a constant map and that $\pi_{\alpha\gamma} = X \times X$, so $\pi_{\beta\gamma} \subseteq \pi_{\alpha\gamma}$. Since $\alpha \leq \beta$, we have $X\alpha \subseteq Y\beta$ and thus $X\alpha\gamma \subseteq Y\beta\gamma$. If $x\beta\gamma \in X\alpha\gamma$, then $x\beta\gamma = x\alpha\gamma$ since $\alpha\gamma$ is a constant map. Hence $\alpha\gamma \leq \beta\gamma$.

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In the proof of the following theorem, we shall use the notation

$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}$$

to mean that $\alpha \in T(X,Y)$ and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\} \subseteq Y$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

Theorem 3.2.4 *Let $|Y| > 2$ and $\gamma \in T(X,Y)$. γ is right compatible with $\leq$ on $T(X,Y)$ if and only if $|Y\gamma| = 1$ or $\gamma|_Y$ is injective.*

Proof. Let $\alpha, \beta \in T(X,Y)$ be such that $\alpha \leq \beta$. If $|Y\gamma| = 1$, then for each $x \in X$ we have $x\alpha\gamma = (x\alpha)\gamma \in Y\gamma$ and $x\beta\gamma = (x\beta)\gamma \in Y\gamma$ which implies that $x\alpha\gamma = x\beta\gamma$ since $|Y\gamma| = 1$ and that $\alpha\gamma = \beta\gamma$. Next, we prove that if $\gamma|_Y$ is injective, then $\alpha\gamma \leq \beta\gamma$. Since $\alpha \leq \beta$, we get $X\alpha \subseteq Y\beta$ which follows that $X\alpha\gamma \subseteq Y\beta\gamma$. Let $(x,y) \in \pi_{\beta\gamma}$. Then $x\beta\gamma = y\beta\gamma$. Since $\gamma|_Y$ is injective, we have $x\beta = y\beta$ which implies that $(x,y) \in \pi_\beta \subseteq \pi_\alpha$. So $x\alpha = y\alpha$, then $x\alpha\gamma = y\alpha\gamma$ which implies that $(x,y) \in \pi_{\alpha\gamma}$. Therefore $\pi_{\beta\gamma} \subseteq \pi_{\alpha\gamma}$. Let $x\beta\gamma \in X\alpha\gamma$. Then $x\beta\gamma = y\alpha\gamma$ for some $y \in X$. Since $\gamma|_Y$ is injective, we have $x\beta = y\alpha \in X\alpha$. By Theorem 3.1.3, $x\alpha = x\beta$. Thus $x\alpha\gamma = x\beta\gamma$. Therefore $\alpha\gamma \leq \beta\gamma$ by Theorem 3.1.3.

Conversely, we prove by contrapositive. Assume that $|Y\gamma| > 1$ and $\gamma|_Y$ is not injective. Since $\gamma|_Y$ is not injective, there exist $b, c \in Y$ such that $b \neq c$ and $b\gamma = c\gamma = y$ for some $y \in Y$. Since $|Y\gamma| > 1$ and $|Y| > 2$, so there exists $a \in Y, b \neq a \neq c$ such that $a\gamma = x$ for some $x \in Y$ and $x \neq y$. We write

$$\gamma = \begin{pmatrix} A_1 & A_2 & X_i \\ x & y & y_i \end{pmatrix},$$

where $a \in A_1$ and $b, c \in A_2$. We define $\alpha, \beta \in T(X, Y)$ by

$$\alpha = \begin{pmatrix} \{a,b\} & X \setminus \{a,b\} \\ a & c \end{pmatrix}, \beta = \begin{pmatrix} a & b & X \setminus \{a,b\} \\ a & b & c \end{pmatrix}.$$

Next, we show that $\alpha \leq \beta$. We see that $X\alpha = \{a, c\} = Y\alpha$ which follows that $\alpha \in F$. Then we can prove by Corollary 3.1.6. We see that $K(\alpha, \beta) = \{x \in X : x\alpha \neq x\beta\} = \{b\}$. Let $x \in K(\alpha, \beta) = \{b\}$. Then $x\beta = b\beta = b \notin X\alpha$ and $x\alpha = b\alpha = a = a\beta \in Y\beta$. Thus $\alpha \leq \beta$ and we can see that

$$\alpha\gamma = \begin{pmatrix} \{a,b\} & X \setminus \{a,b\} \\ x & y \end{pmatrix}, \beta\gamma = \begin{pmatrix} a & X \setminus \{a\} \\ x & y \end{pmatrix}.$$

We see that $b\beta\gamma = y \in X\alpha\gamma$ but $b\alpha\gamma = x \neq y = b\beta\gamma$. Therefore $\alpha\gamma \nleq \beta\gamma$ which implies that γ is not right compatible with $\leq$ on $T(X, Y)$.

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Example 3.2.5 Let X be the set of all intergers, Y a set of all nonnegative integers.

(1) We define $\alpha \in T(X, Y)$ by $n\alpha = |n|$. Then

$$\alpha = \begin{pmatrix} \ldots & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \ldots & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \ldots \end{pmatrix}.$$

We see that $\alpha|_Y$ is injective. Thus α is right compatible with $\leq$ on $T(X, Y)$ by Theorem 3.2.4.

(2) Let $\beta \in T(X,Y)$ which is defined by

$$n\beta = \begin{cases} 1 & \text{if } n \geq 0 \\ n & \text{otherwise.} \end{cases}$$

Then

$$\beta = \begin{pmatrix} \dots & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \dots & -6 & -5 & -4 & -3 & -2 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \dots \end{pmatrix}.$$

Then $Y\beta = \{1\}$ which follows that $|Y\beta| = 1$. Thus β is right compatible with $\leq$ on $T(X,Y)$ by Theorem 3.2.4.

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Now, let $X = Y$. We get the following corollaries which are from Lemma 3.2.3, Theorem 3.2.1 and Theorem 3.2.4. The second corollary below first appeared in [2].

Corollary 3.2.6 *If $|X| = 2$, then the following statements hold:*
(1) γ is left compatible with $\leq$ on $T(X)$ if and only if γ is surjcetive;
(2) γ is right compatible with $\leq$ on $T(X)$ for all $\gamma \in T(X)$.

Corollary 3.2.7 *Let $|X| \geq 3$ and $\gamma \in T(X)$. Then the following statements hold:*
(1) γ is left compatible with $\leq$ on $T(X)$ if and only if γ is surjcetive;
(2) γ is right compatible with $\leq$ on $T(X)$ if and only if γ is injective or constant.

3.3 Maximal and Minimal Elements

In this section, we will study the maximal and minimal elements of the semigroup $T(X,Y)$ with the natural order. We also prove that every element in $T(X,Y)$ must lie between maximal and minimal.

Lemma 3.3.1 *Let $\alpha \in T(X,Y)$. If $\alpha \notin F$ or α is surjective or α is injective, then α is a maximal element.*

Proof. Let $\beta \in T(X,Y)$ be such that $\alpha \leq \beta$. If $\alpha \notin F$, then $\alpha = \beta$ by Lemma 3.1.7. If α is surjective, then we have $x\beta \in Y = X\alpha$ for all $x \in X$, thus by Theorem 3.1.3 $x\alpha = x\beta$, hence $\alpha = \beta$. Now, consider the case α is injective. For each $x \in X$, we have $x\alpha \in X\alpha \subseteq Y\beta$ since $\alpha \leq \beta$. That is $x\alpha = y\beta$ for some $y \in Y$ which implies that $y\beta \in X\alpha$. By Theorem 3.1.3, we have $y\alpha = y\beta$. Thus $x\alpha = y\alpha$. Since α is injective, we get $x = y$. It follows that $x\alpha = x\beta$, and that $\alpha = \beta$. Therefore α is a maximal element.

■

Theorem 3.3.2 *Let $\alpha \in T(X,Y)$. Then α is maximal if and only if $\alpha \notin F$ or α is surjective or α is injective.*

Proof. Assume that $\alpha \in F, X\alpha \neq Y$ and α is not injective. Since $X\alpha \neq Y$, we have there exists $a \in Y$ such that $a \notin X\alpha$.

Case I: $\alpha|_Y : Y \to Y$ is injective. Then $X \neq Y$. We choose $x \in X \setminus Y$ and define $\beta \in T(X,Y)$ by

$$z\beta = \begin{cases} z\alpha & \text{if } z \neq x \\ a & \text{if } z = x. \end{cases}$$

Since $x\beta = a \notin X\alpha$, it follows that $\alpha \neq \beta$. We show that $\alpha < \beta$ by Theorem 3.1.3. Since $\alpha \in F$, $X\alpha = Y\alpha = Y\beta$. Let $(m,n) \in \pi_\beta$. Then $m\beta = n\beta$. If $m = x$ and $n \neq x$, we have $n\alpha = n\beta = m\beta = a \notin X\alpha$ which is a contradiction. It is concluded that $m = x = n$ or $m \neq x \neq n$, thus $m\alpha = n\alpha$. So $(m,n) \in \pi_\alpha$. Let $y\beta \in X\alpha$. If $y = x$, then $y\beta = a \notin X\alpha$ which is a contradiction. Thus $y \neq x$, then $y\beta = y\alpha$. Therefore $\alpha < \beta$.

Case II: $\alpha|_Y : Y \to Y$ is not injective. Then there exist $p, q \in Y$ such that $p\alpha = q\alpha$ and $p \neq q$. We define $\beta \in T(X,Y)$ by

$$z\beta = \begin{cases} z\alpha & \text{if } z \neq p \\ a & \text{if } z = p. \end{cases}$$

Since $a \in X\beta$ but $a \notin X\alpha$, we get $\alpha \neq \beta$. We show that $\alpha < \beta$ by Corollary 3.1.6. It is not hard to see that $K(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\} = \{p\}$. Let $x \in K(\alpha,\beta) = \{p\}$. We get $x\beta = p\beta = a \notin X\alpha$ and $x\alpha = p\alpha = q\alpha = q\beta \in Y\beta$. Therefore $\alpha < \beta$. It is concluded that α is not maximal.

The converse is Lemma 3.3.1.

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Example 3.3.3 Let X be the set of all integers and Y the set of all even numbers.

(1) We define $\alpha \in T(X,Y)$ by

$$n\alpha = \begin{cases} n-1 & \text{if } 2 \nmid n \\ n & \text{otherwise.} \end{cases}$$

We see that

$$\alpha = \begin{pmatrix} \dots & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \dots & -6 & -6 & -4 & -4 & -2 & -2 & 0 & 0 & 2 & 2 & 4 & 4 & 6 & \dots \end{pmatrix}.$$

Then α is surjective. By Theorem 3.3.2, α is maximal.

(2) Consider $\beta \in T(X,Y)$ which is defined by $n\beta = 4n$ for all integers n. Then β is injective. By Theorem 3.3.2, we have β is maximal.

(3) Let $\gamma \in T(X,Y)$ which is defined by

$$n\gamma = \begin{cases} 2 & \text{if } 2 \mid n \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\gamma = \begin{pmatrix} \dots & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \dots & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & \dots \end{pmatrix}.$$

We see that $Y\gamma = \{2\} \subsetneq \{0,2\} = X\gamma$. Then $\gamma \notin F$ which follows that γ is maximal by Theorem 3.3.2.

■

Theorem 3.3.4 *Let $\alpha \in T(X,Y)$. α is minimal if and only if $|X\alpha| = 1$.*

Proof. Suppose that $\alpha : X \to \{a\}$ for some $a \in Y$. Let $\beta \in T(X,Y)$ be such that $\beta \leq \alpha$. By Theorem 3.1.3, we have $X\beta \subseteq Y\alpha$. Let $x \in X$. Then $x\beta \in X\beta \subseteq Y\alpha = \{a\}$. Hence $x\beta = a = x\alpha$, then $\alpha = \beta$.

Conversely, we prove by contrapositive. Assume that $|X\alpha| > 1$. We choose $y \in Y\alpha$ and define $\beta \in T(X,Y)$ by $z\beta = y$ for all $z \in X$. Since $|X\beta| = |\{y\}| = 1 < |X\alpha|$, it follows that $\beta \neq \alpha$. We show that $\beta \leq \alpha$ by Theorem 3.1.3. We can see that $X\beta = \{y\} \subseteq Y\alpha$, $\pi_\alpha \subseteq X \times X = \pi_\beta$. Let $x\alpha \in X\beta = \{y\}$. Then $x\alpha = y = x\beta$. Thus $\beta \leq \alpha$ which follows that α is not minimal.

■

Example 3.3.5 Let X be the set of all real numbers, Y the set of all natural numbers. Let $n \in \mathbb{N}$. We define $\alpha_n \in T(X,Y)$ by $x\alpha_n = n$ for all $x \in X$. Thus $|X\alpha_n| = |\{n\}| = 1$ which follows that α_n is minimal by Theorem 3.3.4.

■

If $X = Y$, it follows that $T(X) = F$. Thus by Theorem 3.3.2 and Theorem 3.3.4 we have the following corollary which first appeared in [2].

Corollary 3.3.6 *An element $\alpha \in T(X)$ is maximal with $\leq$ on $T(X)$ if and only if α is surjective or injective; α is minimal if and only if α is a constant map.*

Lemma 3.3.7 *Let $|Y| \geq 2$ and $\alpha \in T(X,Y)$, then there exists $\beta \in T(X,Y)$ such that $\beta \nleq \alpha$.*

Proof. We consider α in two cases:

Case I: α is not injective. Then there exist $x, y \in X$ such that $x\alpha = y\alpha = a$ for some $a \in Y$ and $x \neq y$. Since $|Y| \geq 2$, there exists $b \in Y$ such that $b \neq a$. We define $\beta \in T(X, Y)$ by

$$z\beta = \begin{cases} z\alpha & \text{if } z \neq x \\ b & \text{if } z = x. \end{cases}$$

We see that $x\alpha = y\alpha = y\beta \in X\beta$ but $x\alpha = a \neq b = x\beta$. Therefore $\beta \nleq \alpha$.

Case II: α is injective. Since $|Y| \geq 2$, we choose $p, q \in Y$ such that $p \neq q$. Then $p\alpha \neq q\alpha$. We define $\beta \in T(X, Y)$ by

$$z\beta = \begin{cases} q\alpha & \text{if } z = p \\ p\alpha & \text{if } z \neq p. \end{cases}$$

We see that $p\alpha \in X\beta$ but $p\beta = q\alpha \neq p\alpha$, then $\beta \nleq \alpha$.

■

Remark 3.3.8 *If $|Y| \geq 2$, then $T(X, Y)$ has no maximum element.*

Proof. Suppose that $\alpha \in T(X, Y)$ is a maximum element, then $\beta \leq \alpha$ for all $\beta \in T(X, Y)$ which contradicts Lemma 3.3.7.

■

Lemma 3.3.9 *Let $|Y| \geq 2$ and $\alpha \in T(X, Y)$, then there exists $\beta \in T(X, Y)$ such that $\alpha \nleq \beta$.*

Proof. If $|X\alpha| \geq 2$, then there exist $x, y \in X$ and $x \neq y$ such that $x\alpha \neq y\alpha$. We define $\beta \in T(X, Y)$ by $z\beta = x\alpha$ for all $z \in X$, then $y\alpha \in X\alpha$ but $y\alpha \notin \{x\alpha\} = Y\beta$. Therefore $X\alpha \nsubseteq Y\beta$ which follows that $\alpha \nleq \beta$.

If $|X\alpha| = 1$, then $\alpha : X \to \{a\}$ for some $a \in Y$. Since $|Y| \geq 2$, there exists $b \in Y$ such that $b \neq a$. We define $\beta \in T(X, Y)$ by $z\beta = b$ for all $z \in X$. Then $X\alpha = \{a\} \nsubseteq \{b\} = Y\beta$ which implies that $\alpha \nleq \beta$.

■

Remark 3.3.10 *If $|Y| \geq 2$, then $T(X,Y)$ has no minimum element.*

Proof. Similarly to Remark 3.3.8.

$\blacksquare$

Theorem 3.3.11 *Let $\alpha \in T(X,Y)$. Then there exists a maximal element $\beta \in T(X,Y)$ such that $\alpha \leq \beta$.*

Proof. If α is a maximal element, then we let $\beta = \alpha$ and $\alpha \leq \beta$. Now, suppose that α is not maximal. We have $\alpha \in F$ and α is not surjective and injective by Theorem 3.3.2. Let $C(\alpha) = \{x\alpha^{-1} : x \in Y \text{ and } |x\alpha^{-1}| > 1\}$. Since α is not injective, we have $C(\alpha)$ is nonempty. Since $\alpha \in F$ and α is not surjective, we get $Y\alpha = X\alpha \subsetneq Y$, thus $Y \setminus X\alpha \neq \emptyset$. For each $C \in C(\alpha)$, choose $d_C \in C \cap Y$, then $C \setminus \{d_C\} \neq \emptyset$. We consider in two cases.

Case I: $|\bigcup_{C \in C(\alpha)}(C \setminus \{d_C\})| \geq |Y \setminus X\alpha|$. Then there exists an injection γ such that

$$\gamma : Y \setminus X\alpha \to \bigcup_{C \in C(\alpha)} (C \setminus \{d_C\}).$$

For each $z \in \text{im}\gamma$, $|z\gamma^{-1}| = 1$ since γ is injective, so let $z\gamma^{-1} = \{g_z\}$. We define $\beta \in T(X,Y)$ by

$$z\beta = \begin{cases} g_z & \text{if } z \in \text{im}\gamma \\ z\alpha & \text{otherwise.} \end{cases}$$

Then $K(\alpha,\beta) = \text{im}\gamma$. Let $x \in K(\alpha,\beta)$. We have $x\beta = g_x \in Y \setminus X\alpha$. Since $x \in K(\alpha,\beta)$, we get $x \in C \setminus \{d_C\}$ for some $C \in C(\alpha)$. Then $x\alpha = d_C\alpha = d_C\beta \in Y\beta$ since $d_C \notin \text{im}\gamma$. For each $p, q \in K(\alpha,\beta) = \text{im}\gamma$ with $p \neq q$, we have $p\beta = g_p \in Y \setminus X\alpha$, and $p\beta = g_p \neq g_q = q\beta$ since γ is a function. Therefore $\alpha \leq \beta$ by Corollary 3.1.6. Next, we show that β is surjective by letting $y \in Y$. If $y \in X\alpha$, then $y = x\alpha$ for some $x \in X$. For if $x \in \text{im}\gamma$, then $y = x\alpha = d_{C_0}\alpha = d_{C_0}\beta$ for some $C_0 \in C(\alpha)$, but if $x \notin \text{im}\gamma$, then $y = x\alpha = x\beta$. In the other hand, if $y \in Y \setminus X\alpha$, then $y\gamma \in \text{im}\gamma$ and that $(y\gamma)\beta = g_{y\gamma} \in (y\gamma)\gamma^{-1} = \{y\}$, thus $(y\gamma)\beta = y$. Therefore

β is surjective which implies that β is maximal by Theorem 3.3.2.

Case II: $|\bigcup_{C\in C(\alpha)}(C\setminus\{d_C\})| < |Y\setminus X\alpha|$. Then there exists an injection γ such that

$$\gamma : \bigcup_{C\in C(\alpha)} (C\setminus\{d_C\}) \to Y\setminus X\alpha.$$

We define $\beta \in T(X,Y)$ by

$$z\beta = \begin{cases} z\gamma & \text{if } z \in \mathrm{dom}\gamma \\ z\alpha & \text{otherwise.} \end{cases}$$

In this case we have $K(\alpha,\beta) = \mathrm{dom}\gamma$. Let $x \in K(\alpha,\beta)$. Then $x\beta = x\gamma \in Y\setminus X\alpha$ and $x\alpha = d_C\alpha = d_C\beta \in Y\beta$ for some $C \in C(\alpha)$. For each $p,q \in K(\alpha,\beta)$ with $p \neq q$ we have , $p\beta = p\gamma \neq q\gamma = q\beta$ since γ is injective. Therefore $\alpha \leq \beta$ by Corollary 3.1.6. Next, we show that β is injective. Let $x\beta = y\beta$. If $x \in \mathrm{dom}\gamma$, then $y\beta = x\beta = x\gamma \in Y\setminus X\alpha$, so $y \in \mathrm{dom}\gamma$ (if $y \notin \mathrm{dom}\gamma$, then $y\beta = y\alpha \in X\alpha$) and thus $x\gamma = x\beta = y\beta = y\gamma$, hence $x = y$ since γ is injective. If $x \notin \mathrm{dom}\gamma$, then $y\beta = x\beta = x\alpha \in X\alpha$, thus $y \notin \mathrm{dom}\gamma$ (if $\in \mathrm{dom}\gamma$, then $y\beta = y\gamma \notin X\alpha$) and hence $x\alpha = x\beta = y\beta = y\alpha$. From $x,y \notin \mathrm{dom}\gamma$, we get $x,y \in \{d_C : C \in C(\alpha)\} \cup \{x : |x\alpha^{-1}| = 1\}$, thus $x = y$. Therefore, β is injective and β is maximal by Theorem 3.3.2.

■

Example 3.3.12 Let $X = \{1,2,3,4,5,6\}$, $Y = \{1,3,5\}$. We define $\alpha \in T(X,Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 1 & 5 & 5 \end{pmatrix}.$$

We have $X\alpha = \{1,5\} = Y\alpha$ which follows that $\alpha \in F$. And we see that α is not surjective and injective. By Theorem 3.3.2, α is not maximal. Then there is $\beta \in T(X,Y)$ such that

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 3 & 3 & 5 & 5 \end{pmatrix}.$$

We see that $\alpha \leq \beta$ and $X\beta = \{1, 3, 5\} = Y$ which follow that β is surjective. By Theorem 3.3.2, β is maximal. Therefore α lies below some maximal elements.

Theorem 3.3.13 *Let $\alpha \in T(X, Y)$. Then there exists a minimal element $\beta \in T(X, Y)$ such that $\beta \leq \alpha$.*

Proof. Since $Y\alpha \neq \emptyset$, we choose $a \in Y\alpha$. Let β be the constant map with image $\{a\}$. Then β is a minimal element by Theorem 3.3.4. We see that $X\beta = \{a\} \subseteq Y\alpha$ and $\pi_\alpha \subseteq X \times X = \pi_\beta$. Let $x\alpha \in X\beta = \{a\}$. Then $x\alpha = a = x\beta$. Therefore $\beta \leq \alpha$ by Theorem 3.1.3.

■

Example 3.3.14 Let X be the set of all real numbers, Y the set of all nonnegative real numbers. We define $\alpha \in T(X, Y)$ by $x\alpha = |x|$ for all $x \in X$. We see that $|X\alpha| > 1$ which implies that α is not minimal by Theorem 3.3.4. Let c be a nonnegative real number. Consider $\beta_c \in T(X, Y)$ which is defined by $x\beta_c = c$ for all $x \in X$. We see that $\beta_c \leq \alpha$. Since $|X\beta_c| = |\{c\}| = 1$, we get β_c is minimal by Theorem 3.3.4. Therefore α lies above some minimal elements.

■

By Theorem 3.3.11 and Theorem 3.3.13, we have the following result immediately.

Corollary 3.3.15 *Every element in $T(X, Y)$ must lie below some maximal and lie above some minimal elements.*

3.4 Covering Elements

Recall that an element $\beta \in T(X, Y)$ is called an upper cover for $\alpha \in T(X, Y)$ if $\alpha < \beta$ and there exists no $\gamma \in T(X, Y)$ such that $\alpha < \gamma < \beta$. Lower cover is defined dually. In this section, we describe the covering elements in $T(X,Y)$ where $|Y|>1$. We first give the following remark.

Remark 3.4.1 *Let $\alpha,\beta\in T(X,Y)$. Then α is a lower cover for β if and only if β is an upper cover for α.*

Proof. Let α be a lower cover for β. Then $\alpha<\beta$ and there exists no $\gamma\in T(X,Y)$ such that $\alpha<\gamma<\beta$. Therefore β is an upper cover for α by the definition.

The converse is similar to the first part.

■

Lemma 3.4.2 *Let $\alpha,\beta\in T(X,Y)$ and $\alpha\leq\beta$. If $X\alpha=X\beta$, then $\alpha=\beta$.*

Proof. Let $x\in X$. Since $x\beta\in X\beta=X\alpha$, we have $x\alpha=x\beta$ by Theorem 3.1.3. Thus $\alpha=\beta$.

■

Lemma 3.4.3 *Let $\alpha,\beta\in T(X,Y)$. If β is an upper cover for α, then $|Y\beta\backslash X\alpha|=0$ or 1.*

Proof. Let β be an upper cover for α. It follows that $\alpha\leq\beta$ and that $X\alpha\subseteq Y\beta$. Suppose that $|Y\beta\setminus X\alpha|\geq 2$ which implies that there exist $a,b\in Y\beta\setminus X\alpha$ such that $a\neq b$. We define $\gamma\in T(X,Y)$ by

$$z\gamma=\begin{cases} z\alpha & \text{if } z\notin a\beta^{-1}\\ a & \text{if } z\in a\beta^{-1}.\end{cases}$$

Since $a\in X\gamma$ but $a\notin X\alpha$, we have $\alpha\neq\gamma$. Since $b\in Y\beta$, we have $b\in X\beta$. Since $X\gamma\subseteq X\alpha\cup\{a\}$, we get $b\notin X\gamma$, then $\gamma\neq\beta$. Therefore $\alpha\neq\gamma\neq\beta$. Next, we show that $\alpha\leq\gamma\leq\beta$.

Firstly, we prove that $\alpha\leq\gamma$ by Theorem 3.1.3. Let $x\alpha\in X\alpha$. Since $X\alpha\subseteq Y\beta$, we have $x\alpha=y\beta$ for some $y\in Y$ which follows that $y\beta\in X\alpha$. Thus $y\alpha=y\beta$ by Theorem 3.1.3. Since $a\notin X\alpha$, we have $y\beta=x\alpha\neq a$. Then $y\notin a\beta^{-1}$ which implies that $x\alpha=y\beta=y\alpha=y\gamma$. Therefore $x\alpha\in Y\gamma$ which implies that

$X\alpha \subseteq Y\gamma$. Let $(x, y) \in \pi_\gamma$, then $x\gamma = y\gamma$. If $x \notin a\beta^{-1}$ and $y \in a\beta^{-1}$, then $x\gamma = x\alpha \neq a = y\gamma$ which is a contradiction. That is $x, y \notin a\beta^{-1}$; or $x, y \in a\beta^{-1}$. If $x, y \notin a\beta^{-1}$, then $x\alpha = x\gamma = y\gamma = y\alpha$ and $(x, y) \in \pi_\alpha$. If $x, y \in a\beta^{-1}$, then $x\beta = a = y\beta$ and $(x, y) \in \pi_\beta \subseteq \pi_\alpha$. Thus $\pi_\alpha \subseteq \pi_\gamma$. Let $x\gamma \in X\alpha$. If $x \in a\beta^{-1}$, then $x\gamma = a \notin X\alpha$ which is a contradiction, so $x \notin a\beta^{-1}$ which implies that $x\gamma = x\alpha$. Therefore $\alpha \leq \gamma$.

Finally, we show that $\gamma \leq \beta$ by Theorem 3.1.3. Let $x\gamma \in X\gamma$. We have $X\gamma \subseteq X\alpha \cup \{a\} \subseteq Y\beta \cup \{a\} = Y\beta$ since $a \in Y\beta$. Thus $X\gamma \subseteq Y\beta$. Let $(x, y) \in \pi_\beta$, then $x\beta = y\beta$. Since $\pi_\beta \subseteq \pi_\alpha$, we have $x\alpha = y\alpha$. If $x \notin a\beta^{-1}$ and $y \in a\beta^{-1}$, then $y\beta = a \neq x\beta$ which is a contradiction. So $x, y \notin a\beta^{-1}$; or $x, y \in a\beta^{-1}$. If $x, y \notin a\beta^{-1}$, then $x\gamma = x\alpha = y\alpha = y\gamma$ and that $(x, y) \in \pi_\gamma$. If $x, y \in a\beta^{-1}$, then $x\gamma = a = y\gamma$ and that $(x, y) \in \pi_\gamma$. Therefore $\pi_\beta \subseteq \pi_\gamma$. Let $x\beta \in X\gamma$. Then $x\beta \in X\alpha \cup \{a\}$. If $x\beta = a$, then $x \in a\beta^{-1}$ which implies that $x\gamma = a = x\beta$. If $x\beta \in X\alpha$, then $x\beta = x\alpha$ (since $\alpha \leq \beta$). Thus $x\beta = x\alpha = x\gamma$ (since $x \notin a\beta^{-1}$). Therefore $\gamma \leq \beta$.

It is concluded that $\alpha < \gamma < \beta$. This contradicts the hypothesis that β is an upper cover for α. Therefore $|Y\beta \setminus X\alpha| = 0$ or 1.

■

Lemma 3.4.4 *Let β be an upper cover for α. If $|Y\beta \setminus X\alpha| = 1$, then $|X\beta \setminus X\alpha| = 1$.*

Proof. Suppose that $|Y\beta \setminus X\alpha| = 1$ and $|X\beta \setminus X\alpha| \neq 1$. If $|X\beta \setminus X\alpha| = 0$, then $X\beta \subseteq X\alpha$. Since $\alpha \leq \beta$, we have $X\alpha \subseteq Y\beta \subseteq X\beta$, thus $X\beta = X\alpha$. By Lemma 3.4.2, we have $\alpha = \beta$ which is a contradiction, then $|X\beta \setminus X\alpha| \geq 2$. Let $y\beta \in Y\beta \setminus X\alpha$ for some $y \in Y$. We define $\gamma \in T(X, Y)$ by

$$z\gamma = \begin{cases} y\beta & \text{if } z\beta = y\beta \\ z\alpha & \text{otherwise.} \end{cases}$$

Since $y\beta \notin X\alpha$ but $y\beta = y\gamma \in X\gamma$, we have $\alpha \neq \gamma$. Since $|X\beta \setminus X\alpha| \geq 2$ and $y\beta \in Y\beta \setminus X\alpha \subseteq X\beta \setminus X\alpha$, there exists $x\beta \in X\beta \setminus X\alpha$ such that $x\beta \neq y\beta$ and $x \notin Y$ (if $x \in Y$, then $x\beta \in Y\beta \setminus X\alpha$ and $x\beta = y\beta$ since $|Y\beta \setminus X\alpha| = 1$). We have

$x\gamma = x\alpha \in X\alpha$ since $x\beta \neq y\beta$ but $x\beta \notin X\alpha$, we have $x\gamma \neq x\beta$ which implies that $\gamma \neq \beta$. Next, we show that $\alpha \leq \gamma \leq \beta$ by Theorem 3.1.3.

Firstly, we prove that $\alpha \leq \gamma$. Let $p\alpha \in X\alpha$. Since $X\alpha \subseteq Y\beta$, we have $p\alpha = q\beta$ for some $q \in Y$. Since $q\beta = p\alpha \in X\alpha$ and $\alpha \leq \beta$, we have $q\beta = q\alpha$. If $q\beta = y\beta$, then $p\alpha = q\beta = q\gamma \in Y\gamma$. If $q\beta \neq y\beta$, then $p\alpha = q\beta = q\alpha = q\gamma \in Y\gamma$. Therefore $X\alpha \subseteq Y\gamma$. Let $(u, v) \in \pi_\gamma$. Then $u\gamma = v\gamma$. Suppose that $u\beta = y\beta$ and $v\beta \neq y\beta$, then $u\gamma = y\beta$ and $v\gamma = v\alpha$. Since $u\gamma = y\beta \notin X\alpha$ and $v\gamma = v\alpha \in X\alpha$, we have $u\gamma \neq v\gamma$ which is a contradiction. Thus this case is impossible. So $u\beta = y\beta = v\beta$ or $u\beta \neq y\beta \neq v\beta$. If $u\beta = y\beta = v\beta$, then $(u, v) \in \pi_\beta \subseteq \pi_\alpha$. If $u\beta \neq y\beta \neq v\beta$, then $u\alpha = u\gamma = v\gamma = v\alpha$ which implies that $(u, v) \in \pi_\alpha$. Therefore $\pi_\gamma \subseteq \pi_\alpha$. Let $w\gamma \in X\alpha$. We have $w\beta \neq y\beta$ (if $w\beta = y\beta$, then $y\beta = w\gamma \in X\alpha$), then $w\gamma = w\alpha$.

Finally, we show that $\gamma \leq \beta$. Let $p\gamma \in X\gamma$. If $p\beta = y\beta$, then $p\gamma = y\beta \in Y\beta$. If $p\beta \neq y\beta$, then $p\gamma = p\alpha \in X\alpha \subseteq Y\beta$. Therefore $X\gamma \subseteq Y\beta$. Let $(u, v) \in \pi_\beta$. Then $u\beta = v\beta$. If $u\beta = y\beta = v\beta$, then $u\gamma = u\beta = v\beta = v\gamma$ which follows that $(u, v) \in \pi_\gamma$. If $u\beta \neq y\beta \neq v\beta$, then $u\gamma = u\alpha$ and $v\gamma = v\alpha$. Since $(u, v) \in \pi_\beta \subseteq \pi_\alpha$, we have $u\gamma = u\alpha = v\alpha = v\gamma$ which implies that $(u, v) \in \pi_\gamma$. Therefore $\pi_\beta \subseteq \pi_\gamma$. Let $w\beta \in X\gamma$. Since $X\gamma \subseteq X\alpha \cup \{y\beta\}$, we have $w\beta \in X\alpha$ or $w\beta = y\beta$. If $w\beta \in X\alpha$, then $w\beta \neq y\beta$ since $y\beta \notin X\alpha$. Thus $w\gamma = w\alpha$. Since $w\beta \in X\alpha$ and $\alpha \leq \beta$, we have $w\alpha = w\beta$. Then $w\gamma = w\beta$. If $w\beta = y\beta$, then $w\gamma = y\beta = w\beta$.

It is concluded that $\alpha < \gamma < \beta$ which contradicts the hypothesis that β is an upper cover for α. Therefore $|X\beta \setminus X\alpha| = 1$.

■

Theorem 3.4.5 *Let $\alpha, \beta \in T(X, Y)$. Then β is an upper cover for α if and only if the following statements hold:*

(1) $\alpha < \beta$;

(2) $|Y\beta \setminus X\alpha| = 0$ or $|X\beta \setminus X\alpha| = 1$.

Proof. Firstly, assume that (1) and (2) hold. Let $\gamma \in T(X, Y)$ be such that $\alpha \leq \gamma \leq \beta$. If $|Y\beta \setminus X\alpha| = 0$, then $Y\beta \subseteq X\alpha$. Since $\alpha < \beta$, we have $X\alpha \subseteq Y\beta$, thus $Y\beta = X\alpha$. We have $X\alpha \subseteq Y\gamma \subseteq X\gamma \subseteq Y\beta = X\alpha$, thus $X\alpha = X\gamma$ and that

$\alpha = \gamma$. Now, we consider the case $|X\beta \setminus X\alpha| = 1$. Since $\alpha \leq \gamma \leq \beta$, we have $X\alpha \subseteq Y\gamma \subseteq X\gamma \subseteq Y\beta \subseteq X\beta$. Since $|X\beta \setminus X\alpha| = 1$, it follows that $X\alpha = X\gamma$ or $X\gamma = X\beta$. Thus $\alpha = \gamma$ or $\gamma = \beta$ by Remark 3.4.2.

To prove the converse, assume that β is an upper cover for α. Then $\alpha < \beta$ and $|Y\beta \setminus X\alpha| = 0$ or 1 by Lemma 3.4.3. Suppose that $|Y\beta \setminus X\alpha| \neq 0$, then $|Y\beta \setminus X\alpha| = 1$. By Lemma 3.4.4, we have $|X\beta \setminus X\alpha| = 1$.

■

Theorem 3.4.6 *Let $\alpha, \beta \in T(X,Y)$. Then α is a lower cover for β if and only if the following statements hold:*
(1) $\alpha < \beta$;
(2) $|Y\beta \setminus X\alpha| = 0$ or $|X\beta \setminus X\alpha| = 1$.

Proof. If α is a lower cover for β, then $\alpha < \beta$. By Remark 3.4.1, β is an upper cover for α, and by Theorem 3.4.5 we get $|Y\beta \setminus X\alpha| = 0$ or $|X\beta \setminus X\alpha| = 1$.

If (1) and (2) hold, then by Theorem 3.4.5 we have β is an upper cover for α and Remark 3.4.1 gives α is a lower cover for β.

■

Example 3.4.7 Let $X = \{1,2,3,4\}$, $Y = \{1,2,3\}$. We define $\alpha \in T(X,Y)$ by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 1 \end{pmatrix}.$$

Consider $\beta, \gamma \in T(X,Y)$ which are defined by

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{pmatrix}.$$

We can see that $\alpha < \beta$ and $\alpha < \gamma$. Since $Y\beta = \{1,2\} = X\alpha$, we get $|Y\beta \setminus X\alpha| = 0$ which follows that β is an upper cover for α by Theorem 3.4.5. Since $X\gamma = \{1,2,3\}$ and $X\alpha = \{1,2\}$, then $|X\gamma \setminus X\alpha| = |\{3\}| = 1$ which follows that γ is also an upper cover for α by Theorem 3.4.5.

Conversely, we can see that α is a lower cover for β and γ by Theorem 3.4.6.

■

Theorem 3.4.8 *Every nonmaximal element in $T(X,Y)$ has an upper cover.*

Proof. Let α be a nonmaximal element in $T(X,Y)$. By Theorem 3.3.2, $\alpha \in F$ is not injective and surjective. Then there exist $u, v \in X$ such that $u \neq v$ and $u\alpha = v\alpha$. Since $X\alpha \neq Y$, there exists $w \in Y \setminus X\alpha$. We define $\beta \in T(X,Y)$ by

$$z\beta = \begin{cases} z\alpha & \text{if } z \neq x \\ w & \text{if } z = x. \end{cases}$$

Then $w \in X\beta$ but $w \notin X\alpha$, thus $\alpha \neq \beta$. We prove that $\alpha \leq \beta$ by Corollary 3.1.6. We see that $K(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\} = \{u\}$. Let $a \in K(\alpha,\beta) = \{u\}$. We have $a\beta = u\beta = w \notin X\alpha$ and $a\alpha = u\alpha = v\alpha = v\beta \in Y\beta$. Therefore $\alpha < \beta$.

Next, we show that $|Y\beta \setminus X\alpha| = 0$ or $|X\beta \setminus X\alpha| = 1$ by considering $u \in Y$ or $u \notin Y$.

If $u \in Y$, we show that $|X\beta \setminus X\alpha| = 1$. By the definition of β, we get $w \in X\beta \setminus X\alpha$. To prove the uniqueness, assume that there is $b \in X\beta \setminus X\alpha$, then $b = c\beta$ for some $c \in X$. If $c \neq u$, then $c\beta = c\alpha$. Thus $b = c\alpha \in X\alpha$ which is a contradiction. Hence $c = u$, then $b = c\beta = u\beta = w$. Therefore $|X\beta \setminus X\alpha| = 1$.

If $u \notin Y$. We prove that $|Y\beta \setminus X\alpha| = 0$. Let $y\beta \in Y\beta$ for some $y \in Y$. Since $u \notin Y$, we have $y \neq u$ which follows that $y\beta = y\alpha \in X\alpha$. Then $Y\beta \subseteq X\alpha$. Since $\alpha \leq \beta$, we have $X\alpha \subseteq Y\beta$ by Theorem 3.1.3. Therefore $Y\beta = X\alpha$ which implies that $|Y\beta \setminus X\alpha| = 0$.

By Theorem 3.4.5, β is an upper cover for α.

■

Theorem 3.4.9 *Every nonminimal element in $T(X,Y)$ has a lower cover.*

Proof. Let α be any nonminimal element in $T(X,Y)$. By Theorem 3.3.4, $|X\alpha| > 1$.

Case I: $\alpha \in F$. Since $|X\alpha| > 1$, there exist $x, y \in X\alpha$ such that $x \neq y$. We define $\beta \in T(X,Y)$ by

$$z\beta = \begin{cases} z\alpha & \text{if } z \notin x\alpha^{-1} \\ y & \text{if } z \in x\alpha^{-1}. \end{cases}$$

We show that $\beta \leq \alpha$ by Theorem 3.1.3. It is obvious that $X\beta \subseteq X\alpha = Y\alpha$ (since $\alpha \in F$). Let $(a,b) \in \pi_\alpha$. Then $a\alpha = b\alpha$. If $a, b \in x\alpha^{-1}$, then $a\beta = y = b\beta$. If $a, b \notin x\alpha^{-1}$, then $a\beta = a\alpha = b\alpha = b\beta$. Therefore $\pi_\alpha \subseteq \pi_\beta$. Let $a\alpha \in X\beta$. If $a \in x\alpha^{-1}$, then $a\alpha = x \notin X\beta$ which is a contradiction. Thus $a \notin x\alpha^{-1}$ which implies that $a\alpha = a\beta$. Therefore $\beta \leq \alpha$. Next, we show that $|X\alpha \setminus X\beta| = 1$. We know that $x \in X\alpha \setminus X\beta$. Assume that there is $u \in X\alpha \setminus X\beta$, then $u = v\alpha$ for some $v \in X$. If $v \notin x\alpha^{-1}$, then $v\beta = v\alpha$. Thus $u = v\beta \in X\beta$ which is a contradiction. Hence $v \in x\alpha^{-1}$ which follows that $u = v\alpha = x$. Therefore $|X\alpha \setminus X\beta| = 1$. Since $x \in X\alpha \setminus X\beta$, we have $\alpha \neq \beta$.

By Theorem 3.4.6, β is a lower cover for α.

Case II: $\alpha \notin F$. Then $Y\alpha \subsetneq X\alpha$. We choose $y \in Y\alpha$ and define $\beta \in T(X,Y)$ by

$$z\beta = \begin{cases} z\alpha & \text{if } z\alpha \in Y\alpha \\ y & \text{if } z\alpha \notin Y\alpha. \end{cases}$$

We show that $\beta \leq \alpha$ by Theorem 3.1.3. By the definition of β, we have $X\beta \subseteq Y\alpha$. Let $(a,b) \in \pi_\alpha$. Then $a\alpha = b\alpha$. If $a\alpha = b\alpha \in Y\alpha$, then $a\beta = a\alpha = b\alpha = b\beta$ which implies that $(a,b) \in \pi_\beta$. If $a\alpha = b\alpha \notin Y\alpha$, then $a\beta = y = b\beta$ which follows that $(a,b) \in \pi_\beta$. Hence $\pi_\alpha \subseteq \pi_\beta$. Let $a\alpha \in X\beta$. Since $X\beta \subseteq Y\alpha$, we have $a\alpha \in Y\alpha$ which implies that $a\alpha = a\beta$. Therefore $\beta \leq \alpha$. Next, we show that $|Y\alpha \setminus X\beta| = 0$. Let $a\alpha \in Y\alpha$. By the definition of β, we get $a\alpha = a\beta \in X\beta$. Hence $Y\alpha \subseteq X\beta$, and that $|Y\alpha \setminus X\beta| = 0$. Since $\alpha \notin F$, we have $Y\alpha \neq X\alpha$. Then there is $p \in X\alpha \setminus Y\alpha$. Since $\beta \leq \alpha$, we get $X\beta \subseteq Y\alpha$ and $p \notin Y\alpha$ implies $p \notin X\beta$. Thus $\alpha \neq \beta$.

Therefore β is a lower cover for α by Theorem 3.4.6.

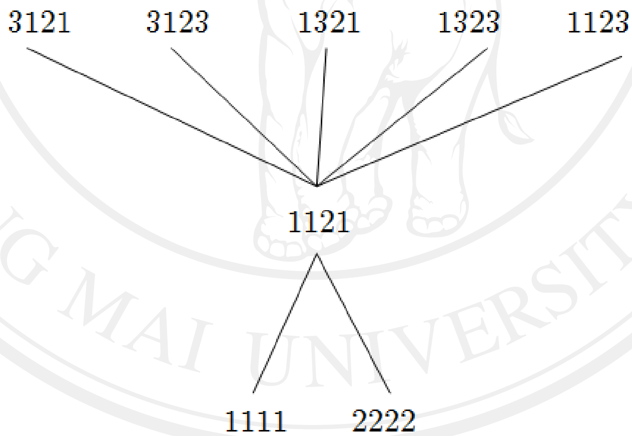
■

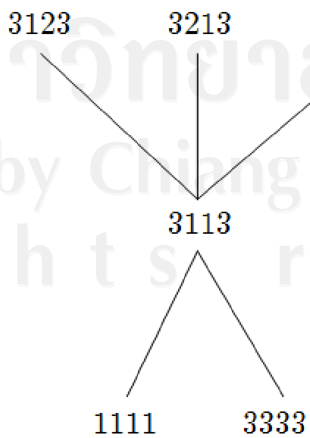
Example 3.4.10 Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2, 3\}$. The notation *abcd* for a map $\alpha \in T(X, Y)$ means that $1\alpha = a, 2\alpha = b, 3\alpha = c, 4\alpha = d$. We have:

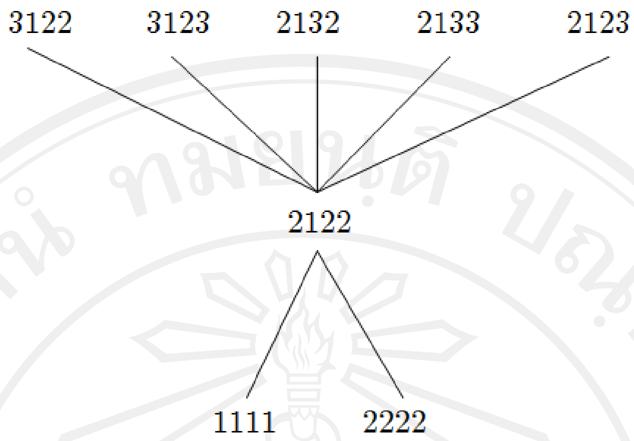
(1) the set of all maximal elements in $T(X, Y)$ is $\{1231, 1232, 1233, 1321, 1322, 1323, 2131, 2132, 2133, 2311, 2312, 2313, 3121, 3122, 3123, 3211, 3212, 3213, 1213, 1223, 1312, 1332, 2113, 2123, 2321, 2331, 3112, 3132, 3221, 3231, 1123, 1132, 2213, 2231, 3312, 3321, 2221, 3331, 1112, 3332, 1113, 2223\}$;

(2) the set of all minimal elements in $T(X, Y)$ is $\{1111, 2222, 3333\}$;

(3) the following diagrams show upper and lower covers for some nonmaximal and nonminimal elements:1121, 3113 and 2122.






3122
3123
2132
2133
2123
2122
1111
2222

■