Chapter 1

Introduction

In Universal Algebra we use identities to classify algebras into collections called varieties. We can also use hyperidentities to classify varieties into collections called hypervarieties. Hyperidentities and hypervarieties were introduced by W. Taylor in 1981 [33]. Hyperidentities in a variety V are identities which have the property that after substituting the operation symbols which occur in these identities by any terms of the same arity, it is still satisfied in the variety. The main tool to study hyperidentities is the concept of a hypersubstitution which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [11]. A hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which maps n_i-ary operation symbols to n_i-ary terms. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ. For all $\sigma \in Hyp(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$. Indeed, $(Hyp(\tau); \circ_h, \sigma_{id})$ is a monoid where $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and σ_{id} is the identity element. In 1998, K. Denecke and S.L. Wismath [15] characterized idempotent elements of the monoid of all hypersubstitutions of type $\tau = (2)$. They found the order of each element and studied Green's relations of this monoid. In 2004, Th. Changphas [4] characterized regular elements of the monoid of all hypersubstitutions of type $\tau = (n)$. Idempotent elements and the Green's relations, have been studied for this monoid by S.L. Wismath [34].

In 2000, S. Leeratanavalee and K. Denecke [21] generalized the concept of a hypersubstitution to a generalized hypersubstitution. We used it as a tool to study strong hyperidentities and used strong hyperidentities to classify varieties into collections called strong hypervarieties. Varieties which are closed under arbitrary application of generalized hypersubstitutions are called strongly solid. A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which maps each n_i-ary operation symbol of type τ to a term of this type which does not necessarily preserve the arity. We denoted the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. Any generalized hypersubstitutions σ of type τ uniquely determines a mapping $\hat{\sigma}$ on the set of all terms of this type. Indeed, $(Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid where $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and σ_{id} is the identity element and $(Hyp(\tau); \circ_h, \sigma_{id})$ is a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

In this work, we are interested in the complexity of terms, generalized superpositions and generalized hypersubstitutions. Then we are interested in semigroup properties of monoids of generalized hypersubstitutions of a given type. These included idempotent elements, regular elements, natural partial ordering on the set of all idempotent elements, order of generalized hypersubstitutions, and Green's relations.

This thesis is divided into seven chapters. Chapter 1 is an introduction to the research problems. Chapter 2 deals with some preliminaries and some useful results those will be used in later chapters. Chapter 3 to Chapter 6 are the main results of this research work and the conclusion is in Chapter 7. Chapter 3 presents the four useful measurements of the complexity of a term, called the maximum depth, the minimum depth, the variable count, and the operation count. We generalize the concept of complexity of compositions and hypersubstitutions which were studied by K. Denecke and S. L. Wismath [13] to complexity of generalized superpositions and generalized hypersubstitutions. We also obtain formulas for the complexity of $\hat{\sigma}[t]$ in terms of the complexity of t where t is a compound term and σ is a regular generalized hypersubstitution. We apply these formulas to the theory of M-strongly solid varieties, examining the k-normalization chains of a variety with respect to these complexity measurements. In Chapter 4, we characterize all idempotent , all primitive idempotent and all regular elements of the monoid of all generalized hypersustitutions of type $\tau = (2)$. We determine the order of generalized hypersustitutions of this type. We give the complete characterization of the natural partial ordering on the set of all idempotent elements of this monoid. We describe some classes in this monoid under Green's relations. In Chapter 5, we give the complete characterization of idempotent pre-generalized hypersubstitutions of type $\tau = (2, 2)$. In Chapter 6, we characterize idempotent and regular elements of the monoid of generalized hypersubstitutions of type $\tau = (n)$. We describe some classes in this monoid under Green's relations.