Chapter 2

Preliminaries

In this chapter, we will briefly review some concepts and some results of Semigroup Theory and Universal Algebra, without any proofs and which are basis for this thesis. For more details see [1], [2], [5], [6], [7], [14], [16], [17], [18], [20], and [26].

2.1 Semigroups

2.1.1 Elementary Concepts of Semigroups

An element e in a semigroup S is called *idempotent* if $e^2 = e$, and we set $E(S)$ to be the set of all idempotent elements in S. An *idempotent semigroup* or *band* is a semigroup in which every element is idempotent. A *semilattice* is a commutative band.

An element a in a semigroup S is called *regular* if $a = aba$ for some b in S. A semigroup S is *regular* if every element in S is regular.

An element a in a semigroup S is called *left zero* (*right zero*) if $ab = a$ ($ba = a$) for every b in S. An element a in S is called *zero* if it is both left zero and right zero. A *left zero* (*right zero*) *semigroup* is a semigroup in which every element is left zero (right zero).

Let S be a semigroup and $\emptyset \neq A \subseteq S$. A is called a *left ideal* (*right ideal*) of S if $SA \subseteq A$ ($AS \subseteq A$). A is an (*two-sided*) *ideal* of S if it is both a left and a right ideal of S.

A semigroup S is called a *left simple semigroup* (*right simple semigroup, simple semigroup*) if S is the only left ideal (right ideal, ideal) of S.

From the definition, we see that every left simple semigroup and every right simple semigroup are simple semigroups.

A monoid is a semigroup with an identity 1.

The subsemigroup $< a >$ of a semigroup S generated by a consists of all positive integral powers of a: $< a > := \{a, a^2, \dots\}$. If $S =< a >$ for some $a \in S$, then S is called a *cyclic semigroup*. In general case, we call $< a >$ the *cyclic subsemigroup* of S *generated* by a. The *order* of a is defined to be the order of $< a >$.

2.1.2 Green's Relations

Let S be a semigroup and $1 \notin S$. We extend the binary operation from S to $S \cup \{1\}$ by define $x1 = 1x = x$ for all $x \in S \cup \{1\}$. Then $S \cup \{1\}$ is a semigroup with an identity 1.

Let S be a semigroup. Then we define,

$$S^1 = \begin{cases} S & \text{if } S \text{ has an identity,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Let S be a semigroup and $\emptyset \neq A \subseteq S$. We now set

$$\begin{aligned} (A)_l &= \cap\{L | L \text{ is a left ideal of } S \text{ containing } A\}, \\ (A)_r &= \cap\{R | R \text{ is a right ideal of } S \text{ containing } A\}, \\ (A)_i &= \cap\{I | I \text{ is an ideal of } S \text{ containing } A\}. \end{aligned}$$

Then $(A)_l$,$(A)_r$ and $(A)_i$ are left ideal, right ideal and ideal of S, respectively. We call $(A)_l$ ($(A)_r, (A)_i$) the *left ideal* (*right ideal, ideal*) *of* S *generated by* A.

It is easy to see that

$$\begin{aligned} (A)_l &= S^1A = SA \cup A, \\ (A)_r &= AS^1 = A \cup SA, \\ (A)_i &= S^1AS^1 = SAS \cup SA \cup AS \cup A. \end{aligned}$$

For $a_1, a_2, \ldots, a_n \in S$, we write $(a_1, a_2, \ldots, a_n)_l$ instead of $(\{a_1, a_2, \ldots, a_n\})_l$ and call it the *left ideal of* S *generated by* $a_1, a_2, \ldots, a_n$. Similarly, we can define $(a_1, a_2, \ldots, a_n)_r$ and $(a_1, a_2, \ldots, a_n)_i$. If A is a left ideal of S and $A = (a)_l$ for some $a \in S$, we then call A the *principal left ideal generated by* a. We can define principal right ideal and principal ideal in the same manner.

Let S be a semigroup. We define the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ on S as follows:

$$\begin{aligned}
a\mathcal{L}b &\Leftrightarrow (a)_l = (b)_l,\\
a\mathcal{R}b &\Leftrightarrow (a)_r = (b)_r,\\
\mathcal{H} &= \mathcal{L} \cap \mathcal{R},\\
\mathcal{D} &= \mathcal{L} \circ \mathcal{R},\\
a\mathcal{J}b &\Leftrightarrow (a)_i = (b)_i.
\end{aligned}$$

Then we have, for all $a, b \in S$

$$\begin{aligned}
a\mathcal{L}b &\Leftrightarrow Sa \cup \{a\} = Sb \cup \{b\}\\
&\Leftrightarrow S^1a = S^1b\\
&\Leftrightarrow a = xb \text{ and } b = ya \text{ for some } x, y \in S^1,\\
a\mathcal{R}b &\Leftrightarrow aS \cup \{a\} = bS \cup \{b\}\\
&\Leftrightarrow aS^1 = bS^1\\
&\Leftrightarrow a = bx \text{ and } b = ay \text{ for some } x, y \in S^1,\\
a\mathcal{H}b &\Leftrightarrow a\mathcal{L}b \text{ and } a\mathcal{R}b,\\
a\mathcal{D}b &\Leftrightarrow (a, c) \in \mathcal{L} \text{ and } (c, b) \in \mathcal{R} \text{ for some } c \in S,\\
a\mathcal{J}b &\Leftrightarrow SaS \cup Sa \cup aS \cup \{a\} = SbS \cup Sb \cup bS \cup \{b\}\\
&\Leftrightarrow S^1aS^1 = S^1bS^1\\
&\Leftrightarrow a = xby \text{ and } b = zau \text{ for some } x, y, z, u \in S^1.
\end{aligned}$$

Remark 2.1.1. Let S be a semigroup. Then the following statements hold.

(i) $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ are equivalence relations.

(ii) $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. ■

We call the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ the *Green's relations on S*. For each $a \in S$, we denote $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class containing a by L_a, R_a, H_a, D_a and J_a, respectively.

2.2 Universal Algebra

2.2.1 Algebras

Let A be a non-empty set and $n \in \mathbb{N}$, where $\mathbb{N} = \{1, 2, \ldots, \}$. We define $A^n = \{(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in A\}$. A function $f^A : A^n \to A$ is called an *n-ary operation* defined on A, and is said to have *arity n.*

Let $(f_i)_{i \in I}$ be a sequence of operation symbols, where I is an indexed set. To each f_i we assign an integer $n_i \geq 1$ as its arity. A *type* τ is the sequence of arity of f_i for all $i \in I$. We always write $\tau := (n_i)_{i \in I}$.

Let $\tau := (n_i)_{i \in I}$ be a type with the sequence of operation symbols $(f_i)_{i \in I}$. An *algebra of type* τ is an ordered pair $\mathcal{A} := (A; (f_i^A)_{i \in I})$, where A is a non-empty set and $(f_i^A)_{i \in I}$ is a sequence of operations on A indexed by a non-empty indexed set I such that to each n_i-ary operation symbol f_i there is a corresponding n_i-ary operation f_i^A on A.

The set A is called the *universe* of $\mathcal{A}$ and the sequence $(f_i^A)_{i \in I}$ is called the *sequence of fundamental operations* of $\mathcal{A}$. An algebra is called a *trivial algebra* if its universe is a singleton set. We denoted by $Alg(\tau)$ the class of all algebras of type τ.

Let $\mathcal{A} := (A; (f_i^A)_{i \in I})$ and $\mathcal{B} := (B; (f_i^B)_{i \in I})$ be algebras of type τ. Then an algebra $\mathcal{B}$ is called a *subalgebra* of $\mathcal{A}$, written as $\mathcal{B} \subseteq \mathcal{A}$, if the following conditions are satisfied:

(i) $B \subseteq A$,

(ii) for every $i \in I$, $f_i^A \mid_B = f_i^B$.

A relation $\leq$ on a non-empty set P is called a *partial ordering* on P if (1) $a \leq a$, (2) $a \leq b$ and $b \leq a$ imply $a = b$, and (3) $a \leq b$ and $b \leq c$ imply $a \leq c$. If $\leq$ is a partial ordering on P, the ordered pair $(P, \leq)$ is called a *partially ordered set.* A partially ordered set $(P, \leq)$ is called a *lattice* if for every $x, y \in P$ both $sup\{x, y\}$ (*supremum* of x and y) and $inf\{x, y\}$ (*infimum* of x and y) exist in P. Let L be a non-empty subset of P. Then $L := (L, \leq)$ is called a *sublattice* of $P := (P, \leq)$ if $x, y \in L$ implies $sup\{x, y\} \in L$ and $inf\{x, y\} \in L$. A partially ordered set $(P, \leq)$ is called a *complete lattice* if for every non-empty subset L of P both $supL$ and $infL$ exist in L.

Let S be a semigroup and $E(S) \neq \emptyset$. Define $e \leq f$ (e, f in $E(S)$) iff $ef = fe = e$. Then $\leq$ is a partial ordering on $E(S)$. If $e \leq f$, we say that e is under f and that f

is over e. Then $\leq$ is a partial ordering on $E(S)$. We shall call $\leq$ the *natural partial ordering* on $E(S)$. An idempotent element $e \neq 0$ in a semigroup S is called *primitive* if $f \leq e$ implies $f = e$ for all f in $E(S)$.

Note that the lattice $(P, \leq)$ can be considered as an algebra of type $\tau = (2,2)$. Indeed, we define two binary operations, denoted by $\vee$ and $\wedge$, then so-called *join* and *meet*, respectively, by $a \vee b := sup\{a,b\}$ and $a \wedge b := inf\{a,b\}$ for all $a, b \in P$. This algebra satisfies a list of axioms containing the associative laws, the commutative laws, the idempotency laws for both operations and the absorption laws, i.e. $\forall a, b \in P, a \vee (a \wedge b) = a = a \wedge (a \vee b)$.

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ and $\mathcal{B} = (B; (f_i^B)_{i\in I})$ be algebras of the same type $\tau = (n_i)_{i\in I}$. A function $h : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism* from A to B if for all $i \in I$,

$$h(f_i^A(a_1, \ldots, a_{n_i})) = f_i^B(h(a_1), \ldots, h(a_{n_i})),$$

for all $a_1, \ldots, a_{n_i} \in A$. If the function h is onto, then the homomorphism $h : \mathcal{A} \to \mathcal{B}$ is called an *epimorphism*. If the function h is one-to-one, then the homomorphism $h : \mathcal{A} \to \mathcal{B}$ is called an *embedding*, and is called an *isomorphism* if h is bijective. We call $\mathcal{A}$ *isomorphic* to $\mathcal{B}$ denoted by $\mathcal{A} \cong \mathcal{B}$ if there is an isomorphism h from $\mathcal{A}$ to $\mathcal{B}$. A homomorphism $h : \mathcal{A} \to \mathcal{A}$ is called an *endomorphism* of $\mathcal{A}$, and is called an *automorphism* if h is an isomorphism from $\mathcal{A}$ onto $\mathcal{A}$.

Let A be a non-empty set, $\theta \subseteq A \times A$ an equivalence relation on A and f an n-ary operation on A. Then f is said to be *compatible* with θ, if for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$,

$$(a_1, b_1) \in \theta, \ldots, (a_n, b_n) \in \theta \Rightarrow (f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \in \theta.$$

Let $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be an algebra of type $\tau = (n_i)_{i\in I}$. An equivalence relation θ on A is called a *congruence relation* on $\mathcal{A}$ if all fundamental operations f_i^A are compatible with θ. We denote by $Con(\mathcal{A})$ the set of all congruence relations on the algebra $\mathcal{A}$.

Let θ be a congruence relation on an algebra $\mathcal{A}$. For each $i \in I$, we define an n_i-ary operation $f_i^{A/\theta}$ on the quotient set A/θ, the set of all equivalence classes on A with respect to θ,

$$f_i^{A/\theta} : (A/\theta)^{n_i} \longrightarrow A/\theta$$

defined by

$$([a_1]_\theta, \ldots, [a_{n_i}]_\theta) \mapsto f_i^{A/\theta}(([a_1]_\theta, \ldots, [a_{n_i}]_\theta)) := [f_i^A(a_1, \ldots, a_{n_i})]_\theta.$$

The algebra $\mathcal{A}/\theta = (A/\theta; (f_i^{A/\theta})_{i\in I})$ is called the *quotient algebra* or *factor algebra* of $\mathcal{A}$ by θ.

Let θ be a congruence relation on an algebra $\mathcal{A}$ and $\mathcal{A}/\theta$ be the quotient algebra of $\mathcal{A}$ by θ. Then the mapping $h : \mathcal{A} \to \mathcal{A}/\theta$ defined by $a \mapsto [a]_\theta$ is a homomorphism from the algebra $\mathcal{A}$ onto the quotient algebra $\mathcal{A}/\theta$. We call this homomorphism the *natural homomorphism* induced by θ on $\mathcal{A}$, and it is usually denoted by $nat(\theta)$. The *kernel* of a homomorphism $h : \mathcal{A} \to \mathcal{B}$, denoted by $ker(h)$, is a binary relation which is defined by:

$$ker(h) := \{(a, b) \in A^2 \mid h(a) = h(b)\}.$$

Theorem 2.2.1. (*General Homomorphism Theorem*) *Let $h : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{A} \to \mathcal{C}$ be homomorphisms and let g be surjective. Then there exists a homomorphism $f : \mathcal{C} \to \mathcal{B}$ which satisfies $f \circ g = h$ if and only if $ker(g) \subseteq ker(h)$. If this f exists, it has the following properties:*

(i) *The homomorphism f is unique.*

(ii) *f is injective if and only if $ker(g) = ker(h)$.*

(iii) *f is surjective if and only if h is surjective.* ■

Theorem 2.2.2. (*Homomorphic Image Theorem*) *Let $h : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism. Then there exists a unique homomorphism f from $\mathcal{A}/ker(h)$ onto $\mathcal{B}$ with $f \circ nat(ker(h)) = h$.* ■

A congruence relation θ on an algebra $\mathcal{A}$ of type τ is called *fully invariant* if whenever $(x, y) \in \theta$ we also have $(\varphi(x), \varphi(y)) \in \theta$, for every endomorphism φ of $\mathcal{A}$.

Let $\{\mathcal{A}_j \mid j \in J\}$ be a class of algebras of type τ. The *direct product* $\prod\limits_{j\in j} \mathcal{A}_j$ of the $\mathcal{A}_j$ is defined as an algebra

$$\mathcal{P} := (P; (f_i^P)_{i\in I}),$$

where $P := \prod\limits_{j\in j} A_j$ is the cartesian product of $A_j, j \in J$ and for each $i \in I$,

$$f_i^P((a_{1j})_{j\in J}, \ldots, (a_{n_ij})_{j\in J}) := (f_i^{A_j}(a_{1j}, \ldots, a_{n_ij}))_{j\in J}.$$

2.2.2 Closure Operators and Galois Connections

Let A be a non-empty set and $\mathcal{P}(A)$ be the power set of A. A mapping $\gamma : \mathcal{P}(A) \to \mathcal{P}(A)$ is called a *closure operator* on A if for any $X, Y \in \mathcal{P}(A)$, the following conditions hold:

(i) $X \subseteq \gamma(X)$ (extensively),

(ii) $X \subseteq Y \Rightarrow \gamma(X) \subseteq \gamma(Y)$ (monotonicity),

(iii) $\gamma(\gamma(X)) = \gamma(X)$ (idempotency).

A subset X of A is called a *closed set* with respect to the closure operator γ if $\gamma(X) = X$. Let $\mathcal{H}_\gamma$ denote the set of all closed sets with respect to the closure operator γ. In fact, $\mathcal{H}_\gamma$ forms a complete lattice.

Proposition 2.2.3. *Let $\gamma : \mathcal{P}(A) \to \mathcal{P}(A)$ be a closure operator on A. Then $\mathcal{H}_\gamma$ is a complete lattice with respect to the set inclusion. For any set $\{H_i \in \mathcal{H}_\gamma \mid i \in I\}$, the meet and the join operators are defined by*

$$\wedge\{H_i \in \mathcal{H}_\gamma \mid i \in I\} := \bigcap_{i\in I} H_i,$$
$$\vee\{H_i \in \mathcal{H}_\gamma \mid i \in I\} := \cap\{H \in \mathcal{H}_\gamma \mid H \supseteq \bigcup_{i\in I} H_i\} = \gamma(\bigcup_{i\in I} H_i).$$ ■

The concepts of a closure operator is closely connected to the next concept of a Galois connection.

A *Galois connection* between sets A and B is a pair (μ, ι) of mappings $\mu : \mathcal{P}(A) \to \mathcal{P}(B)$ and $\iota : \mathcal{P}(B) \to \mathcal{P}(A)$ such that for any $X, X' \in \mathcal{P}(A)$ and $Y, Y' \in \mathcal{P}(B)$ the following conditions hold:

(i) $X \subseteq X' \Rightarrow \mu(X) \supseteq \mu(X')$ and $Y \subseteq Y' \Rightarrow \iota(Y) \supseteq \iota(Y')$,

(ii) $X \subseteq \iota\mu(X)$ and $Y \subseteq \mu\iota(Y)$.

Proposition 2.2.4. *Let (μ, ι) with $\mu : \mathcal{P}(A) \to \mathcal{P}(B)$ and $\iota : \mathcal{P}(B) \to \mathcal{P}(A)$ be a Galois connection between sets A and B. Then*

(i) *$\mu\iota\mu = \mu$ and $\iota\mu\iota = \iota$.*

(ii) *$\iota\mu$ and $\mu\iota$ are closure operators on A and B, respectively.*

(iii) *The closed sets under* $\iota\mu$ *are exactly the sets of the form* $\iota(Y)$ *for some* $Y \subseteq B$ *and the closed sets under* $\mu\iota$ *are exactly the sets of the form* $\mu(X)$ *for some* $X \subseteq A$.

(iv) $\mu(\bigcup\limits_{i\in I} X_i) = \bigcap\limits_{i\in I} \mu(X_i)$, *where* $X_i \subseteq A$ *for all* $i \in I$.

(v) $\iota(\bigcup\limits_{i\in I} Y_i) = \bigcap\limits_{i\in I} \iota(Y_i)$, *where* $Y_i \subseteq B$ *for all* $i \in I$. ■

Note that any relation $R \subseteq A \times B$ between sets A and B induces a Galois connection (μ_R, ι_R) between A and B as follows:

We can define the mappings $\mu_R : \mathcal{P}(A) \to \mathcal{P}(B)$ and $\iota_R : \mathcal{P}(B) \to \mathcal{P}(A)$ by

$$\mu_R(X) := \{y \in B \mid \forall x \in X((x, y) \in R)\},$$

$$\iota_R(Y) := \{x \in A \mid \forall y \in Y((x, y) \in R)\}.$$

Conversely, for any Galois connection (μ, ι) between sets A and B, we define a relation $R_{\mu,\iota}$ by

$$R_{\mu,\iota} := \cup\{X \times \mu(X) \mid X \subseteq A\}.$$

In fact, there is a one-to-one correspondence between Galois connections and relations between sets A and B.

2.2.3 Terms and Term Operations

Let $n \in \mathbb{N}$ and $X_n := \{x_1, \ldots, x_n\}$ be an n-elements set. The set X_n is called an *alphabet* and its elements are called *variables*. Let $\tau = (n_i)_{i\in I}$ be a type such that the set of operation symbols $\{f_i \mid i \in I\}$ is disjoint with X_n. An *n-ary term* of type τ is defined inductively as follows:

(i) Every variable $x_i \in X_n$ is an *n-ary term* of type τ.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an *n-ary term* of type τ.

The set $W_\tau(X_n)$ of all n-ary terms of type τ is the smallest set containing $x_1, \ldots, x_n$ that is closed under finite application of (ii). The set of all terms of type τ over the alphabet $X := \{x_1, x_2, \ldots\}$ is defined as $W_\tau(X) := \bigcup\limits_{n=1}^{\infty} W_\tau(X_n)$. For any $t \in W_\tau(X)$, the set of all variables occurring in the term t is denoted by $var(t)$.

By using step (ii) in the definition of terms of type τ, the term algebra

$$\mathcal{F}_\tau(X) := (W_\tau(X), (\bar{f}_i)_{i\in I}).$$

of type τ, the so-called *absolutely free algebra*, can be defined by

$$\bar{f}_i(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i})$$

for each operation symbol f_i and $t_1, \ldots, t_{n_i} \in W_\tau(X)$.

Note that for $\mathcal{A} \in Alg(\tau)$, any mapping $\varphi : X \to A$ can be uniquely extend a homomorphism $\hat{\varphi} : \mathcal{F}_\tau(X) \to \mathcal{A}$. Up to isomorphism, the algebra $\mathcal{F}_\tau(X)$ is uniquely determined by the alphabet X.

The following concept will be used to define identities. Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$. Let $t \in W_\tau(X_n)$ for $n \in \mathbb{N}$ and $\mathcal{A} = (A; (f_i^A)_{i\in I})$ be an algebra of type τ. The *n-ary term operation* $t^{\mathcal{A}} : A^n \to A$ of type τ is inductively defined by

(i) $t^{\mathcal{A}}(a_1, \ldots, a_n) := a_i$ if $t = x_i \in X_n$.

(ii) $t^{\mathcal{A}}(a_1, \ldots, a_n) := f_i^A(t_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_{n_i}^{\mathcal{A}}(a_1, \ldots, a_n))$ if t is a compound term $f_i(t_1, \ldots, t_{n_i})$.

We will denote by $W_\tau(X_n)^{\mathcal{A}}$ the set of all n-ary term operations of the algebra $\mathcal{A}$, and by $W_\tau(X)^{\mathcal{A}}$ the set of all (finitary) term operations on $\mathcal{A}$. Make a remark that the elements of $W_\tau(X_n)^{\mathcal{A}}$ are also called n-ary term operations induced by terms from $W_\tau(X_n)$.

2.2.4 Varieties and Identities

Let $\tau = (n_i)_{i\in I}$ be a type. Let s, t be n-ary terms of type τ and $\mathcal{A}$ be an algebra of type τ. An *equation* of type τ is a pair (s, t); such pair are commonly written as $s \approx t$. The set of all equations of type τ is denoted by $E_\tau(X)$.

An equation $s \approx t$ is an *identity* of $\mathcal{A}$, denoted by $\mathcal{A} \models s \approx t$ if $s^{\mathcal{A}} = t^{\mathcal{A}}$.

Let K be a class of algebras of type τ. The class K satisfies an equation $s \approx t$, denoted by $K \models s \approx t$, if for every $\mathcal{A} \in K$, $\mathcal{A} \models s \approx t$.

Let Σ be a set of equations of type τ. The class K is said to satisfy Σ, denoted by $K \models \Sigma$, if $K \models s \approx t$, for every $s \approx t \in \Sigma$. Let

$$IdK := \{s \approx t \in E_\tau(X) \mid K \models s \approx t\},$$

$$Mod\Sigma := \{\mathcal{A} \in Alg(\tau) \mid \mathcal{A} \models \Sigma\}.$$

Then Id is a function from the power set of $Alg(\tau)$ to the power set of $E_\tau(X)$ and Mod is a function from the power set of $E_\tau(X)$ to the power set of $Alg(\tau)$. We obtain

Theorem 2.2.5. *Let $K, K_1, K_2 \subseteq Alg(\tau)$ and $\Sigma, \Sigma_1, \Sigma_2 \subseteq E_\tau(X)$. Then*

(i) *$K \subseteq ModIdK$ and $\Sigma \subseteq IdMod\Sigma$.*

(ii) *$K_1 \subseteq K_2 \Rightarrow IdK_2 \subseteq IdK_1$ and $\Sigma_1 \subseteq \Sigma_2 \Rightarrow Mod\Sigma_2 \subseteq Mod\Sigma_1$.* ■

Then we have the ordered pair (Id, Mod) is a Galois connection between $Alg(\tau)$ and $E_\tau(X)$.

By the properties of Galois connection, we obtain

Theorem 2.2.6. *The following statements hold:*

(i) *$ModId$ and $IdMod$ are closure operators on $Alg(\tau)$ and $E_\tau(X)$, respectively.*

(ii) *The closed sets under $IdMod$ are exactly the sets of the form $IdK, K \subseteq Alg(\tau)$, and the closed sets under $ModId$ are exactly the sets of the form $Mod\Sigma, \Sigma \subseteq E_\tau(X)$.* ■

Let V be a non-empty subset of $Alg(\tau)$. V is called a *variety* if $V = ModIdV$. Let Σ be a non-empty subset of $E_\tau(X)$. Σ is called an *equational theory* if $\Sigma = IdMod\Sigma$. We obtain

Theorem 2.2.7. *A non-empty subset V of $Alg(\tau)$ is a variety if and only if $V = Mod\Sigma$ for some $\Sigma \subseteq E_\tau(X)$.* ■

2.2.5 Hypersubstitutions

The concept of a hypersubstitution was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [11]. They used it as the tool to study hyperidentities and solid varieties. Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$.

A *hypersubstitution* of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ where $\sigma(f_i) \in W_\tau(X_{n_i})$. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ.

For all $\sigma \in Hyp(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ as follows, for any $t \in W_\tau(X)$, $\hat{\sigma}[t]$ is inductively defined by

(i) $\hat{\sigma}[t] := t$ if $t \in X$.

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, if t is a compound term $f_i(t_1, \ldots, t_{n_i})$.

Using the induced maps $\hat{\sigma}$, a binary operation $\circ_h$ can be defined on the set $Hyp(\tau)$. For any hypersubstitutions $\sigma_1, \sigma_2 \in Hyp(\tau)$, $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ i.e.

$$\forall i \in I, (\sigma_1 \circ_h \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)].$$

Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $\underline{Hyp(\tau)} = (Hyp(\tau); \circ_h, \sigma_{id})$ is a monoid where σ_{id} is the identity element.

Let $\underline{M}$ be a submonoid of $\underline{Hyp(\tau)} = (Hyp(\tau); \circ_h, \sigma_{id})$ and V be a variety of type τ. The variety V is called *M-solid variety* if

$$\forall s \approx t \in IdV, \forall \sigma \in M(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV).$$

An identity $s \approx t \in IdV$ is called *M-hyperidentity* if

$$\forall \sigma \in M(V \models \hat{\sigma}[s] \approx \hat{\sigma}[t]).$$

If $\underline{M} = \underline{Hyp(\tau)}$, then we speak of *solid variety* and *hyperidentity*, respectively.

2.2.6 Generalized Hypersubstitutions

In 2000, S. Leeratanavalee and K. Denecke [21] generalized the concept of a hypersubstitution to a generalized hypersubstitution. We used it as a tool to study strong hyperidentities and used strong hyperidentities to classify varieties into collections called strong hypervarieties. Varieties which are closed under arbitrary application of generalized hypersubstitutions are called strongly solid.

Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$. A generalized hypersubstitution of type τ, for short, a generalized hypersubstitution is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which maps each n_i-ary operation symbol of type τ to a term of this type which does not necessarily preserve the arity. We denoted the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. Firstly, we define inductively the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ by the following steps:

(i) If $t = x_j, 1 \le j \le m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Example 2.2.8. Let $\tau = (2, 3)$ be a type, i.e. we have one binary operation symbol and one ternary operation symbol, say f and g, respectively. Then we have

$S^3(x_1, x_2, f(x_3, x_5), x_3) = x_2$,
$S^3(x_2, x_2, f(x_3, x_5), x_3) = f(x_3, x_5)$,
$S^3(x_3, x_2, f(x_3, x_5), x_3) = x_3$,
$S^3(x_7, x_2, f(x_3, x_5), x_3) = x_7$,
$S^3(g(x_1, f(x_2, x_7), x_3), x_2, f(x_3, x_5), x_3)$
$=g(S^3(x_1, x_2, f(x_3, x_5), x_3), S^3(f(x_2, x_7), x_2, f(x_3, x_5), x_3), S^3(x_3, x_2, f(x_3, x_5), x_3))$
$=g(x_2, f(S^3(x_2, x_2, f(x_3, x_5), x_3), S^3(x_7, x_2, f(x_3, x_5), x_3)), x_3)$
$=g(x_2, f(f(x_3, x_5), x_7), x_3)$. ■

To define a binary operation on $Hyp_G(\tau)$, we extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$.

(ii) $\hat{\sigma}[t] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if t is a compound term, $f_i(t_1, \ldots, t_{n_i})$.

Example 2.2.9. Let $\tau = (2, 3)$ be a type, i.e. we have one binary operation symbol and one ternary operation symbol, say f and g, respectively. Let $\sigma : \{f, g\} \to W_{(2,3)}(X)$

where $\sigma(f) = f(g(x_1, x_2, x_1), x_1)$ and $\sigma(g) = f(x_3, x_5)$. Then σ is a generalized hypersubstitution of type $\tau = (2, 3)$ which is not a hypersubstitution of type $\tau = (2, 3)$ since $\sigma(g) \notin W_{(2,3)}(X_3)$. Then we have

$$\begin{aligned}
\hat{\sigma}[f(x_1, g(x_2, x_3, x_7))] &= S^2(\sigma(f), \hat{\sigma}[x_1], \hat{\sigma}[g(x_2, x_3, x_7)]) \\
&= S^2(f(g(x_1, x_2, x_1), x_1), x_1, S^3(\sigma(g), \hat{\sigma}[x_2], \hat{\sigma}[x_3], \hat{\sigma}[x_7])) \\
&= S^2(f(g(x_1, x_2, x_1), x_1), x_1, S^3(f(x_3, x_5), x_2, x_3, x_7)) \\
&= S^2(f(g(x_1, x_2, x_1), x_1), x_1, f(x_7, x_5)) \\
&= f(g(x_1, f(x_7, x_5), x_1), x_1).
\end{aligned}$$

■

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$.

We proved the following propositions.

Proposition 2.2.10. ([21]) *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$.

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$. ■

Proposition 2.2.11. ([21]) $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid where σ_{id} is the identity element and the set of all hypersubstitutions of type τ forms a submonoid of $\underline{Hyp_G(\tau)}$. ■

Let $\underline{M}$ be a submonoid of $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ and V be a variety of type τ. The variety V is called *M-strongly solid variety* if

$$\forall s \approx t \in IdV, \forall \sigma \in M(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV).$$

An identity $s \approx t \in IdV$ is called *M-strong hyperidentity* if

$$\forall \sigma \in M(V \models \hat{\sigma}[s] \approx \hat{\sigma}[t]).$$

If $\underline{M} = \underline{Hyp(\tau)}$, then we speak of *strongly solid variety* and *strong hyperidentity*, respectively.