Chapter 3
Complexity of Terms, Generalized Superpositions and Generalized Hypersubstitutions

In this chapter, we consider the four useful measurements of the complexity of a term, called the maximum depth, the minimum depth, the variable count, and the operation count. We construct a formula for the complexity of the generalized superposition $S^m(s, t_1, \ldots, t_m)$ in terms of complexity of the inputs $s, t_1, \ldots, t_m$ for each of these measurements. We also obtain formulas for the complexity of $\hat{\sigma}[t]$ in terms of the complexity of t where t is a compound term and σ is a generalized hypersubstitution. We apply these formulas to the theory of M-strongly solid varieties, examining the k-normalization chains of a variety with respect to these complexity measurements.

In Section 3.1, we recall the definition of the measurements of the complexity of a term which was defined by K. Denecke and S. L. Wismath [13]. We then consider the complexity of generalized superpositions and generalized hypersubstitutions and construct a formula for the complexity of the generalized superposition $S^m(s, t_1, \ldots, t_m)$ in terms of the complexity of the inputs $s, t_1, \ldots, t_m$ for each of these measurements. We also obtain formulas for the complexity of $\hat{\sigma}[t]$ in terms of the complexity of t where t is a compound term and σ is a generalized hypersubstitution. In Section 3.3, we apply these formulas to the theory of M-strongly solid varieties. We examine the chains obtained by taking the k-normalizations of a given variety V, as defined in [12], and show that under suitable choices of a monoid N, each variety of this chain is $M \cap N$-strongly solid when the variety V is M-strongly solid. This can be used to construct an infinite chain of $M \cap N$-strongly solid varieties of any type.

3.1 Complexity of Terms

In this section, we recall the definition of measurements of the complexity of terms which was defined by K. Denecke and S. L. Wismath [13]. At first, we consider the following example.

Example 3.1.1. Let $\tau = (2, 3)$ be a type, i.e. we have one binary operation symbol and one ternary operation symbol, say f and g, respectively. Consider the term $t = g(f(x_1, x_5), f(x_5, g(f(x_6, x_6), x_9, x_1)), f(x_1, x_1))$ which can be represented by a tree as in Figure 1 below.

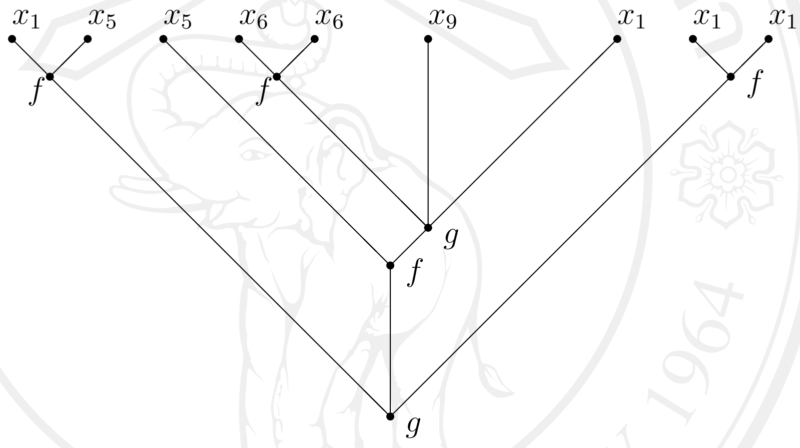

Figure 1.

There are several numbers we can associate with the term t, each measuring a different aspect of how complex this term is as follows:

(i) the length of the longest path (from root to vertex) in t is 4,

(ii) the length of the shortest path (from root to vertex) in t is 2,

(iii) the total number of occurrences of variable symbols in t is 9,

(iv) the number of distinct variables occurring in t is 4,

(v) the total number of occurrences of operation symbols in t is 6. ■

Definition 3.1.2. ([13]) Let $\tau = (n_i)_{i \in I}$ be a type and $t \in W_\tau(X)$.

(a) The maximum depth of a term t, which is denoted by $maxdepth(t)$, is the length of the longest path from the root to a vertex in the tree. It is defined inductively by

(i) $maxdepth(t) = 0$ if t is a variable.

(ii) $maxdepth(t) = 1 + max\{maxdepth(t_j) \mid 1 \leq j \leq n_i\}$ if t is a compound term, $t = f_i(t_1, \ldots, t_{n_i})$.

(b) The minimum depth of a term t, which is denoted by $mindepth(t)$, is the length of the shortest path from the root to a vertex in the tree and is defined inductively by

(i) $mindepth(t) = 0$ if t is a variable.

(ii) $mindepth(t) = 1 + min\{mindepth(t_j) \mid 1 \leq j \leq n_i\}$ if t is a compound term, $t = f_i(t_1, \ldots, t_{n_i})$.

(c) The variable count or the length of a term t, denoted by $vb(t)$, is the total number of occurrences of variables in t (including multiplicities). This can be defined inductively by

(i) $vb(t) = 1$ if t is a variable.

(ii) $vb(t) = \sum_{j=1}^{n_i} vb(t_j)$ if t is a compound term, $t = f_i(t_1, \ldots, t_{n_i})$.

(d) The operation symbol count of a term t, denoted by $op(t)$, is the total number of occurrences of operation symbols in t and is defined inductively by

(i) $op(t) = 0$ if t is a variable.

(ii) $op(t) = 1 + \sum_{j=1}^{n_i} op(t_j)$ if t is a compound term, $t = f_i(t_1, \ldots, t_{n_i})$.

Let $c : W_\tau(X) \to \mathbb{N} \cup \{0\}$ be a mapping from the set of all terms of type τ to the set of all non-negative natural numbers, which assigns to each term t a complexity number $c(t)$. They refer to such a function as a complexity mapping or a cost function.

They also need to measure, for each variable $x_i \in X$, both how many times it occurs in t and the maximum depth and the minimum depth at which it occurs.

Definition 3.1.3. ([13]) Let $t \in W_\tau(X_n)$ be an n-ary term. For each variable x_k, the maximum depth with respect to k of the term t denoted by $maxdepth_k(t)$ is defined inductively as follows:

(i) If t is a variable from X_n, then $maxdepth_k(t) = 0$.

(ii) If $x_k \notin var(t)$, then $maxdepth_k(t) = 0$.

(iii) If $t = f_i(t_1, \ldots, t_{n_i})$ and $x_k \in var(t)$, then
$maxdepth_k(t) = 1 + max\{maxdepth_k(t_j) | 1 \leq j \leq n_i, x_k \in var(t_j)\}$.

Similarly, they define the minimum depth with respect to k for any term t and any variable x_k.

Definition 3.1.4. ([13]) Let $t \in W_\tau(X_n)$ be an n-ary term. For each variable x_k, the minimum depth with respect to k of the term t denoted by $mindepth_k(t)$ is defined inductively as follows:

(i) If t is a variable from X_n, then $mindepth_k(t) = 0$.

(ii) If $x_k \notin var(t)$, then $mindepth_k(t) = 0$.

(iii) If $t = f_i(t_1, \ldots, t_{n_i})$ and $x_k \in var(t)$, then
$mindepth_k(t) = 1 + min\{mindepth_k(t_j) | 1 \leq j \leq n_i, x_k \in var(t_j)\}$.

They also need a function that counts the number of occurrences of a specific variable x_k in a term t.

Definition 3.1.5. ([13]) Let $t \in W_\tau(X_n)$ be an n-ary term. For each variable x_k, the x_k-variable count $vb_k(t)$ of t is defined inductively as follows:

(i) $vb_k(x_k) = 1$.

(ii) If $x_k \notin var(t)$, then $vb_k(t) = 0$.

(iii) If $t = f_i(t_1, \ldots, t_{n_i})$ and $x_k \in var(t)$, then $vb_k(t) = \sum_{j=1}^{n_i} vb_k(t_j)$.

3.2 Complexity of Generalized Superpositions and Generalized Hypersubstitutions

In this section, we generalize the concept of complexity of compositions and hypersubstitutions which were studied by K. Denecke and S. L. Wismath [13] to complexity of generalized superpositions and generalized hypersubstitutions. We have the following proposition.

Proposition 3.2.1. *Let $s, t_1, \ldots, t_m \in W_\tau(X)$. Then,*

(i) $mindepth(S^m(s, t_1, \ldots, t_m)) = min\{mindepth_j(s) + mindepth(t_j), mindepth_k(s) \mid 1 \leq j \leq m, k > m, x_j, x_k \in var(s)\}.$

(ii) $maxdepth(S^m(s, t_1, \ldots, t_m)) = max\{maxdepth_j(s) + maxdepth(t_j), maxdepth_k(s) \mid 1 \leq j \leq m, k > m, x_j, x_k \in var(s)\}.$

(iii) $vb(S^m(s, t_1, \ldots, t_m)) = \sum_{j=1}^{m} vb_j(s) vb(t_j) + \sum_{j>m} vb_j(s).$

(iv) $op(S^m(s, t_1, \ldots, t_m)) = \sum_{j=1}^{m} vb_j(s) op(t_j) + op(s).$

Proof. We will prove all of (i)-(iv) by induction on the complexity of the term s.
(i) If $s = x_l \in X$ for some $1 \leq l \leq m$, then

$$\begin{aligned} mindepth(S^m(s, t_1, \ldots, t_m)) &= mindepth(t_l) \\ &= min\{mindepth_j(s) + mindepth(t_j), mindepth_k(s) \mid \\ &\quad 1 \leq j \leq m, k > m, x_j, x_k \in var(s)\}. \end{aligned}$$

If $s = x_l \in X$ for some $l > m$, then

$$\begin{aligned} mindepth(S^m(s, t_1, \ldots, t_m)) &= 0 \\ &= min\{mindepth_j(s) + mindepth(t_j), mindepth_k(s) \mid \\ &\quad 1 \leq j \leq m, k > m, x_j, x_k \in var(s)\}. \end{aligned}$$

Let $s = f_i(s_1, \ldots, s_{n_i})$ and the formula is satisfied for $s_1, \ldots, s_{n_i}$. Then

$mindepth(S^m(s, t_1, \ldots, t_m))$

$$\begin{aligned}
&= mindepth(S^m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m)) \\
&= mindepth(f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))) \\
&= min\{mindepth(S^m(s_1, t_1, \ldots, t_m)), \ldots, mindepth(S^m(s_{n_i}, t_1, \ldots, t_m))\} + 1 \\
&= min\{min\{mindepth_j(s_1) + mindepth(t_j), mindepth_k(s_1) \mid 1 \le j \le m, k > m, \\
&\quad x_j, x_k \in var(s_1)\}, \ldots, min\{mindepth_j(s_{n_i}) + mindepth(t_j), mindepth_k(s_{n_i}) \mid \\
&\quad 1 \le j \le m, k > m, x_j, x_k \in var(s_{n_i})\}\} + 1 \\
&= min\{min\{mindepth_j(s_1) + 1 + mindepth(t_j), mindepth_k(s_1) + 1 \mid 1 \le j \le m, \\
&\quad k > m, x_j, x_k \in var(s_1)\}, \ldots, min\{mindepth_j(s_{n_i}) + 1 + mindepth(t_j), \\
&\quad mindepth_k(s_{n_i}) + 1 \mid 1 \le j \le m, k > m, x_j, x_k \in var(s_{n_i})\}\} \\
&= min\{min\{mindepth_j(s_t) \mid 1 \le t \le n_i, x_j \in var(s_t)\} + 1 + mindepth(t_j), \\
&\quad min\{mindepth_k(s_t) \mid 1 \le t \le n_i, x_k \in var(s_t)\} + 1 \mid 1 \le j \le m, k > m, x_j, x_k \\
&\quad \in \cup\{var(s_r) \mid 1 \le r \le n_i\}\} \\
&= min\{mindepth_j(s) + mindepth(t_j), mindepth_k(s) \mid 1 \le j \le m, k > m, x_j, x_k \\
&\quad \in var(s)\}.
\end{aligned}$$

(ii) The proof is similar to the proof of (i).

(iii) If $s = x_l \in X$ for some $1 \le l \le m$, then

$$\begin{aligned}
vb(S^m(s, t_1, \ldots, t_m)) &= vb(t_l) \\
&= \sum_{j=1}^{m} vb_j(s) vb(t_j) + \sum_{j>m} vb_j(s).
\end{aligned}$$

If $s = x_l \in X$ for some $l > m$, then

$$\begin{aligned}
vb(S^m(s, t_1, \ldots, t_m)) &= 1 \\
&= \sum_{j=1}^{m} vb_j(s) vb(t_j) + \sum_{j>m} vb_j(s).
\end{aligned}$$

Let $s = f_i(s_1, \ldots, s_{n_i})$ and the formula is satisfied for $s_1, \ldots, s_{n_i}$. Then

$$\begin{aligned}
vb(S^m(s, t_1, \ldots, t_m)) &= vb(S^m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m)) \\
&= vb(f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))) \\
&= \sum_{k=1}^{n_i} vb(S^m(s_k, t_1, \ldots, t_m)) \\
&= \sum_{k=1}^{n_i} (\sum_{j=1}^{m} vb_j(s_k) vb(t_j) + \sum_{j>m} vb_j(s_k)) \\
&= \sum_{k=1}^{n_i} (\sum_{j=1}^{m} vb_j(s_k) vb(t_j)) + \sum_{k=1}^{n_i} (\sum_{j>m} vb_j(s_k)) \\
&= \sum_{j=1}^{m} (\sum_{k=1}^{n_i} vb_j(s_k) vb(t_j)) + \sum_{j>m} (\sum_{k=1}^{n_i} vb_j(s_k)) \\
&= \sum_{j=1}^{m} ((\sum_{k=1}^{n_i} vb_j(s_k)) vb(t_j)) + \sum_{j>m} vb_j(s) \\
&= \sum_{j=1}^{m} vb_j(s) vb(t_j) + \sum_{j>m} vb_j(s).
\end{aligned}$$

(iv) If $s = x_l \in X$ for some $1 \leq l \leq m$, then

$$\begin{aligned}
op(S^m(s, t_1, \ldots, t_m)) &= op(t_l) \\
&= \sum_{j=1}^{m} vb_j(s) op(t_j) + op(s).
\end{aligned}$$

If $s = x_l \in X$ for some $l > m$, then

$$\begin{aligned}
op(S^m(s, t_1, \ldots, t_m)) &= 0 \\
&= \sum_{j=1}^{m} vb_j(s) op(t_j) + op(s).
\end{aligned}$$

Let $s = f_i(s_1, \ldots, s_{n_i})$ and the formula is satisfied for $s_1, \ldots, s_{n_i}$. Then

$$\begin{aligned}
op(S^m(s, t_1, \ldots, t_m)) &= op(S^m(f_i(s_1, \ldots, s_{n_i}), t_1, \ldots, t_m)) \\
&= op(f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))) \\
&= \sum_{k=1}^{n_i} op(S^m(s_k, t_1, \ldots, t_m)) + 1 \\
&= \sum_{k=1}^{n_i}(\sum_{j=1}^{m} vb_j(s_k)op(t_j) + op(s_k)) + 1 \\
&= \sum_{k=1}^{n_i}(\sum_{j=1}^{m} vb_j(s_k)op(t_j)) + \sum_{k=1}^{n_i} op(s_k) + 1 \\
&= \sum_{j=1}^{m}(\sum_{k=1}^{n_i} vb_j(s_k)op(t_j)) + op(s) \\
&= \sum_{j=1}^{m}((\sum_{k=1}^{n_i} vb_j(s_k))op(t_j)) + op(s) \\
&= \sum_{j=1}^{m} vb_j(s)op(t_j) + op(s).
\end{aligned}$$

■

Using the fact that $\hat{\sigma}[t]$ is defined by using generalized superposition, we have the following corollary.

Corollary 3.2.2. *Let $\tau = (n_i)_{i \in I}$ be a type and let t be a compound term of the form $t = f_i(t_1, \ldots, t_{n_i})$ where f_i is an n_i-ary operation symbol. Let σ be a generalized hypersubstitution of type τ. Then,*

(i) $mindepth(\hat{\sigma}[t]) = min\{mindepth_j(\sigma(f_i)) + mindepth(\hat{\sigma}[t_j]), mindepth_k(\sigma(f_i)) \mid 1 \leq j \leq n_i, k > n_i, x_j, x_k \in var(\sigma(f_i))\}$.

(ii) $maxdepth(\hat{\sigma}[t]) = max\{maxdepth_j(\sigma(f_i)) + maxdepth(\hat{\sigma}[t_j]), maxdepth_k(\sigma(f_i)) \mid 1 \leq j \leq n_i, k > n_i, x_j, x_k \in var(\sigma(f_i))\}$.

(iii) $vb(\hat{\sigma}[t]) = \sum_{j=1}^{n_i} vb_j(\sigma(f_i))vb(\hat{\sigma}[t_j]) + \sum_{j > n_i} vb_j(\sigma(f_i))$.

(iv) $op(\hat{\sigma}[t]) = \sum_{j=1}^{n_i} vb_j(\sigma(f_i))op(\hat{\sigma}[t_j]) + op(\sigma(f_i))$. ■

For the case of arity preserving hypersubstitutions is contained in this result.

3.3 M-Strongly Solid Varieties

Firstly, we give some notations which are used to discuss the k-normalization of a variety. Let V be a variety of type τ and let k be a non-negative natural number. Let c be one of the four complexity functions defined in Section 3.1. We define the k-normalization of V, with respect to the complexity function c, to be the variety $N_k^c(V) = Mod\{u \approx v \in IdV | c(u), c(v) \geq k\}$.

It is clear that $N_0^c(V) = V$ and that the k-normalization of V forms a chain

$$V = N_0^c(V) \leq N_1^c(V) \leq N_2^c(V) \leq \cdots .$$

The properties of these varieties, and of the operator N_k^c for $k \geq 0$, have been studied for $c = \textit{mindepth}$ in [10] and $c = \textit{maxdepth}$ in [12].

Next, we will consider the M-strongly solidity properties of the varieties $N_k^c(V)$. Suppose that we start with an M-strongly solid variety V of type τ for some monoid $\underline{M}$ of generalized hypersubstitutions of type τ. To show that $N_k^c(V)$ is also M-strongly solid where $k \geq 1$, we have to show that for any identity $u \approx v$ of $N_k^c(V)$ and any $\sigma \in M$, we have $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ also in $IdN_k^c(V)$. It suffices to consider an identity $u \approx v$ from the defining basis for $N_k^c(V)$, that is we may assume that $u \approx v$ is an identity of V with the property that both $c(u)$ and $c(v)$ are greater than or equal to k. Since V itself is M-strongly solid, we know that $\hat{\sigma}[u] \approx \hat{\sigma}[v]$ is in IdV. Thus it suffices to show that $c(\hat{\sigma}[u]) \geq k$ and $c(\hat{\sigma}[v]) \geq k$. In general, then, we need to compare the complexity of $\hat{\sigma}[t]$ and would like to be able to show that $c(\hat{\sigma}[t]) > c(t)$. However, this is not always the case as in the following example.

Example 3.3.1. (i) Let $\tau = (2)$ be a type, i.e. we have only one binary operation symbol, say f. Let t be the term $f(x_1, f(x_2, x_3))$ so that $\textit{maxdepth}(t) = 2$, $\textit{mindepth}(t) = 1$, $vb(t) = 3$ and $op(t) = 2$. Let σ be the generalized hypersubstitution mapping f to the term $f(x_1, x_1)$. Then, we have $\hat{\sigma}[t] = f(x_1, x_1)$, and this term has $\textit{maxdepth}(\hat{\sigma}[t]) = \textit{mindepth}(\hat{\sigma}[t]) = op(\hat{\sigma}[t]) = 1$ and $vb(\hat{\sigma}[t]) = 2$. Thus all but *mindepth* result in lower complexity for $\hat{\sigma}[t]$ than for t.

(ii) Let $\tau = (2, 2)$ be a type, i.e. we have two binary operation symbols, say f and g. Let t be the term $f(f(x_1, x_2), g(x_1, x_2))$. Let σ be the generalized hypersubstitution mapping f to the term $f(x_2, x_2)$ and g to the variable x_1. Then, although t has $\textit{mindepth}(t) = 2$, the term $\hat{\sigma}[t] = f(x_1, x_1)$ has $\textit{mindepth}(\hat{\sigma}[t])$ equal to 1. ■

Although not all generalized hypersubstitutions σ have the property that $\hat{\sigma}[t]$ has a complexity greater than or equal to the complexity of t, there are conditions we can put on σ to ensure this property. Next, we will consider a kind of generalized hypersubstitutions, i.e. regular generalized hypersubstitutions which was introduced by S. Leeratanavalee in [24]. A generalized hypersubstitution $\sigma \in Hyp_G(\tau)$ is called *regular* if for every $i \in I$, all the variables $x_1, \ldots, x_{n_i}$ occur in the term $\sigma(f_i)$. The set of all regular generalized hypersubstitutions of type τ is denoted by $\underline{Reg_G(\tau)}$. In [24] S. Leeratanavalee proved that $\underline{Reg_G(\tau)}$ forms a submonoid of $\underline{Hyp_G(\tau)}$, and a variety which is M-strongly solid for this submonoid $\underline{M}$ is called *regular-strongly solid*.

Note that the concept of regularity in this section is different from the concept of regularity that was defined in Chapter 2, Section 2.1.

Theorem 3.3.2. *Let $\tau = (n_i)_{i \in I}$ be a type, $t \in W_\tau(X)$ be a term, and $\sigma \in Hyp_G(\tau)$ be a generalized hypersubstitution of type τ. Then the following statements hold:*

(i) *If σ is a regular generalized hypersubstitution and $n_i > 1$ for all $i \in I$, then $maxdepth(\hat{\sigma}[t]) \geq maxdepth(t)$.*

(ii) *If σ is a regular generalized hypersubstitution, then $vb(\hat{\sigma}[t]) \geq vb(t)$.*

(iii) *If σ is a regular generalized hypersubstitution and $n_i > 1$ for all $i \in I$, then $op(\hat{\sigma}[t]) \geq op(t)$.*

Proof. We prove all of the three claims by induction on the complexity of the term t. In all cases, when t is a variable $x \in X$, we have $\hat{\sigma}[t] = x = t$, and both $\hat{\sigma}[t]$ and t have the same complexity.

(i) Let $t = f_i(t_1, \ldots, t_{n_i})$. Since σ is a regular generalized hypersubstitution and $n_i > 1$ for all $i \in I$, thus $\sigma(f_i) \notin X$ and $x_j \in var(\sigma(f_i))$ for all $1 \leq j \leq n_i$. So

$maxdepth_j(\sigma(f_i)) \geq 1$ for all $1 \leq j \leq n_i$. We have

$$\begin{aligned}
maxdepth(\hat{\sigma}[t]) &= max\{maxdepth_j(\sigma(f_i)) + maxdepth(\hat{\sigma}[t_j]), maxdepth_k(\sigma(f_i)) \mid \\
&\quad 1 \leq j \leq n_i, k > n_i, x_j, x_k \in var(\sigma(f_i))\} \\
&= max\{maxdepth_j(\sigma(f_i)) + maxdepth(\hat{\sigma}[t_j]), maxdepth_k(\sigma(f_i)) \mid \\
&\quad 1 \leq j \leq n_i, k > n_i, x_k \in var(\sigma(f_i))\} \quad \text{(since } \sigma \text{ is regular)} \\
&\geq max\{maxdepth_j(\sigma(f_i)) + maxdepth(\hat{\sigma}[t_j]) \mid 1 \leq j \leq n_i\} \\
&\geq max\{1 + maxdepth(\hat{\sigma}[t_j]) \mid 1 \leq j \leq n_i\} \\
&= 1 + max\{maxdepth(\hat{\sigma}[t_j]) \mid 1 \leq j \leq n_i\} \\
&\geq 1 + max\{maxdepth(t_j) \mid 1 \leq j \leq n_i\} \quad \text{(by induction)} \\
&= maxdepth(t).
\end{aligned}$$

(ii) Let $t = f_i(t_1, \ldots, t_{n_i})$. Since σ is a regular generalized hypersubstitution, thus $vb_j(\sigma(f_i)) \geq 1$ for all $1 \leq j \leq n_i$. We have

$$\begin{aligned}
vb(\hat{\sigma}[t]) &= \sum_{j=1}^{n_i} vb_j(\sigma(f_i)) vb(\hat{\sigma}[t_j]) + \sum_{j > n_i} vb_j(\sigma(f_i)) \\
&\geq \sum_{j=1}^{n_i} vb_j(\sigma(f_i)) vb(\hat{\sigma}[t_j]) \\
&\geq \sum_{j=1}^{n_i} 1 vb(\hat{\sigma}[t_j]) \\
&= \sum_{j=1}^{n_i} vb(\hat{\sigma}[t_j]) \\
&\geq \sum_{j=1}^{n_i} vb(t_j) \\
&= vb(t).
\end{aligned}$$

(iii) Let $t = f_i(t_1, \ldots, t_{n_i})$. Since σ is a regular generalized hypersubstitution and $n_i > 1$ for all $i \in I$, thus $\sigma(f_i) \notin X$ and $x_j \in var(\sigma(f_i))$ for all $1 \leq j \leq n_i$. So

$vb_j(\sigma(f_i)) \geq 1$ for all $1 \leq j \leq n_i$. Since $\sigma(f_i) \notin X$, thus $op(\sigma(f_i)) \geq 1$. Then

$$\begin{aligned} op(\hat{\sigma}[t]) &= \sum_{j=1}^{n_i} vb_j(\sigma(f_i)) op(\hat{\sigma}[t_j]) + op(\sigma(f_i)) \\ &\geq \sum_{j=1}^{n_i} 1 op(\hat{\sigma}[t_j]) + 1 \\ &= \sum_{j=1}^{n_i} op(\hat{\sigma}[t_j]) + 1 \\ &\geq \sum_{j=1}^{n_i} op(t_j) + 1 \\ &= op(t). \end{aligned}$$

■

The next example shows that if σ is a regular generalized hypersubstitution and t is a term, then $maxdepth(\hat{\sigma}[t])$ and $op(\hat{\sigma}[t])$ need not be greater than or equal to $maxdepth(t)$ and $op(t)$, respectively. Moreover, the example shows that if σ is a regular generalized hypersubstitution, τ is a type which does not contain a unary operation symbol and t is a term, then $mindepth(\hat{\sigma}[t])$ need not be greater than or equal to $mindepth(t)$.

Example 3.3.3. (i) Let $\tau = (1)$ be a type with one unary operation symbol f. Let t be the term $f(f(x_5))$. So that $maxdepth(t) = op(t) = 2$. Let σ be the generalized hypersubstitution mapping f to the term x_1. Then, we have σ is a regular generalized hypersubstitution and $\hat{\sigma}[t] = x_5$, and this term has $maxdepth(\hat{\sigma}[t]) = op(\hat{\sigma}[t]) = 0$. Hence $maxdepth(\hat{\sigma}[t]) < maxdepth(t)$ and $op(\hat{\sigma}[t]) < op(t)$.

(ii) Let $\tau = (2)$ be a type with one binary operation symbol f. Let t be the term $f(f(x_1, x_2), f(x_1, x_2))$. So that $mindepth(t) = 2$. Let σ be the generalized hypersubstitution mapping f to the term $f(f(x_1, x_2), x_3)$. Then, we have σ is a regular generalized hypersubstitution and $\hat{\sigma}[t] = f(f(f(f(x_1, x_2), x_3), f(f(x_1, x_2), x_3)), x_3)$ and this term has $mindepth(\hat{\sigma}[t]) = 1$. Hence $mindepth(\hat{\sigma}[t]) < mindepth(t)$. ■

Combining Theorem 3.3.2 with the discussion preceding Theorem 3.3.2 gives the following result.

Corollary 3.3.4. *Let $\tau = (n_i)_{i \in I}$ be a type and V be a non-trivial M-strongly solid variety of type τ. Let $k \geq 1$. Then the following statements hold:*

(i) *For the maximum depth c, if $n_i > 1$ for all $i \in I$, then each $N_k^c(V)$ is $(M \cap Reg)$-strongly solid.*

(ii) *For the variable count c, each* $N_k^c(V)$ *is* $(M \cap Reg)$*-strongly solid.*

(iii) *For the operation count c, if* $n_i > 1$ *for all* $i \in I$*, then each* $N_k^c(V)$ *is* $(M \cap Reg)$*-strongly solid.* ■