Chapter 4

Monoids of Generalized Hypersubstitutions of Type $\tau = (2)$

The order of hypersubstitutions and all idempotent elements of the monoid of all hypersubstitutions of type $\tau = (2)$ were studied by K. Denecke and S.L. Wismath [15]. All idempotent elements of the monoid of all hypersubstitutions of type $\tau = (2, 2)$ were studied by Th. Changphas and K. Denecke [3]. Green's relations on the monoid of all hypersubstitutions of type $\tau = (2)$ were studied by K. Denecke and S.L. Wismath [15]. We want to study similar problems for the monoid of all generalized hypersubstitutions of type $\tau = (2)$. In this chapter, we characterize all idempotent and all regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ and determine the order of generalized hypersubstitutions of this monoid. Then we study Green's relations, characterize all primitive idempotent elements of this monoid and characterize the natural partial ordering on the set of all idempotent elements of this monoid.

We assume that from now the type $\tau = (2)$, i.e. we have only one binary operation symbol, say f. By σ_t we denote the generalized hypersubstitution which maps f to the term t in $W_{(2)}(X)$. Firstly, we introduce some notations. For $s, f(c, d) \in W_{(2)}(X), x_i, x_j \in X, i, j \in \mathbb{N}$ and $S \subseteq W_{(2)}(X) \setminus X$ we denote :

$leftmost(s) :=$ the first variable (from the left) that occurs in s,

$rightmost(s) :=$ the last variable that occurs in s,

$W^G_{(2)}(\{x_1\}) := \{s \in W_{(2)}(X) | x_1 \in var(s), x_2 \notin var(s)\}$,

$W^G_{(2)}(\{x_2\}) := \{s \in W_{(2)}(X) | x_2 \in var(s), x_1 \notin var(s)\}$,

$W(\{x_1\}) := W^G_{(2)}(\{x_1\}) \setminus \{x_1\}$,

$W(\{x_2\}) := W^G_{(2)}(\{x_2\}) \setminus \{x_2\}$,

$W^G_{(2)}(\{x_1, x_2\}) := \{t \in W_{(2)}(X) | x_1, x_2 \in var(t)\}$,

$W^G := \{t \in W_{(2)}(X) | t \notin X, x_1, x_2 \notin var(t)\}$,

$P_G(2) := \{\sigma_{x_i} \in Hyp_G(2) | i \in \mathbb{N}, x_i \in X\}$,

$E^G_{x_1} := \{\sigma_{f(x_1,s)} \in Hyp_G(2) | s \in W_{(2)}(X), x_2 \notin var(s)\}$,

$E^G_{x_2} := \{\sigma_{f(s,x_2)} \in Hyp_G(2) | s \in W_{(2)}(X), x_1 \notin var(s)\}$,

$E^G(\{x_1\}) := \{\sigma_t \in Hyp_G(2) | t \in W(\{x_1\})\}$,

$E^G(\{x_2\}) := \{\sigma_t \in Hyp_G(2) | t \in W(\{x_2\})\}$,

$E^G(\{x_1, x_2\}) := \{\sigma_t \in Hyp_G(2) | t \in W^G_{(2)}(\{x_1, x_2\})\}$,

$G := \{\sigma_s \in Hyp_G(2) | s \in W_{(2)}(X) \setminus X, x_1, x_2 \notin var(s)\}$,

$\overline{f(c,d)}$:= the term obtained from $f(c,d)$ by interchanging all occurrences of the letters x_1 and x_2, i.e. $\overline{f(c,d)} = S^2(f(c,d), x_2, x_1)$ and $f(c,d) = S^2(\overline{f(c,d)}, x_2, x_1)$,

$f(c,d)'$:= the term defined inductively by $x_i' = x_i$ and $f(c,d)' = f(d', c')$,

${}_{x_i}C[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_1 by x_i i.e. ${}_{x_i}C[f(c,d)] = S^2(f(c,d), x_i, x_2)$,

$C_{x_i}[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_2 by x_i i.e. $C_{x_i}[f(c,d)] = S^2(f(c,d), x_1, x_i)$,

${}_{x_i}C_{x_j}[f(c,d)]$:= the term obtained from $f(c,d)$ by replacing each of the occurrences of the letter x_1 by x_i and the letter x_2 by x_j i.e. ${}_{x_i}C_{x_j}[f(c,d)] = S^2(f(c,d), x_i, x_j)$.

$\overline{S} := \{\overline{s} | s \in S\}$,

$S' := \{s' | s \in S\}$,

$\overline{H} := \{\sigma_{\overline{t}} | \sigma_t \in H\}$ where $H \subseteq Hyp_G(2) \setminus P_G(2)$,

$H' := \{\sigma_{t'} | \sigma_t \in H\}$ where $H \subseteq Hyp_G(2) \setminus P_G(2)$.

Then we have for any $t \in W_{(2)}(X) \setminus X$, $(t')' = t, \overline{\overline{t}} = t, \overline{t}' = \overline{t'}, \overline{t'}' = \overline{t}, \overline{\overline{t}'} = t'$, $\overline{f(c,d)} = f(\overline{c}, \overline{d})$, $\overline{\overline{S}} = S$, $(S')' = S$, $\overline{\overline{H}} = H$, $(H')' = H$, $\sigma_{f(x_2,x_1)} \circ_G \sigma_t = \sigma_{t'}$, $\sigma_t \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\overline{t}}$, $(E^G_{x_1})' = \overline{E^G_{x_2}}$ and $(E^G_{x_2})' = \overline{E^G_{x_1}}$.

4.1 Idempotent Elements in $Hyp_G(2)$

Now, we characterize all idempotent elements of $Hyp_G(2)$.

Proposition 4.1.1. *Let σ_t be a generalized hypersubstitution of type $\tau = (2)$. Then σ_t is idempotent if and only if $\hat{\sigma}_t[t] = t$.*

Proof. Assume that σ_t is idempotent, i.e. $\sigma_t^2 = \sigma_t$. Then $\hat{\sigma}_t[t] = \hat{\sigma}_t[\sigma_t(f)] = (\hat{\sigma}_t \circ \sigma_t)(f) = (\sigma_t \circ_G \sigma_t)(f) = \sigma_t^2(f) = \sigma_t(f) = t$. Conversely, let $\hat{\sigma}_t[t] = t$. We have $(\sigma_t \circ_G \sigma_t)(f) = (\hat{\sigma}_t \circ \sigma_t)(f) = \hat{\sigma}_t[\sigma_t(f)] = \hat{\sigma}_t[t] = t = \sigma_t(f)$. Thus $\sigma_t^2 = \sigma_t$. ∎

Proposition 4.1.2. *For every $x_i \in X$, σ_{x_i} and σ_{id} are idempotent.*

Proof. Since for every $i \in \mathbb{N}$ and $x_i \in X$, $\hat{\sigma}_{x_i}[x_i] = x_i$. By Proposition 4.1.1, we have σ_{x_i} is idempotent. Since σ_{id} is the identity element, thus σ_{id} is idempotent. ■

Proposition 4.1.3. *Let $t \in W_{(2)}(X)$. Then the following statements hold:*

(i) *If $x_2 \notin var(t)$, then $\sigma_{f(x_1,t)}$ is idempotent.*

(ii) *If $x_1 \notin var(t)$, then $\sigma_{f(t,x_2)}$ is idempotent.*

Proof. (i) Let $x_2 \notin var(t)$. Then $\hat{\sigma}_{f(x_1,t)}[f(x_1,t)] = S^2(\sigma_{f(x_1,t)}(f), x_1, \hat{\sigma}_{f(x_1,t)}[t]) = S^2(f(x_1,t), x_1, \hat{\sigma}_{f(x_1,t)}[t]) = f(x_1,t)$ since $x_2 \notin var(t)$.

(ii) Let $x_1 \notin var(t)$. Then $\hat{\sigma}_{f(t,x_2)}[f(t,x_2)] = S^2(\sigma_{f(t,x_2)}(f), \hat{\sigma}_{f(t,x_2)}[t], x_2) = S^2(f(t,x_2), \hat{\sigma}_{f(t,x_2)}[t], x_2) = f(t,x_2)$ since $x_1 \notin var(t)$. ■

Lemma 4.1.4. *Let $f(c,d) \in W_{(2)}(X) \setminus X$, $\sigma_{x_i} \in P_G(2)$, $\sigma_s \in Hyp_G(2)$ and $\sigma_t \in G$. Then the following statements hold:*

(i) $\sigma_s \circ_G \sigma_{x_i} = \sigma_{x_i}$.

(ii) $\sigma_{x_i} \circ_G \sigma_s \in P_G(2)$ ($\hat{\sigma}_{x_i}[s] \in X$).

(iii) $\sigma_t \circ_G \sigma_{f(c,d)} = \sigma_t$ (*G itself is a left zero band*).

Proof. (i) Consider $(\sigma_s \circ_G \sigma_{x_i})(f) = (\hat{\sigma}_s \circ \sigma_{x_i})(f) = \hat{\sigma}_s[\sigma_{x_i}(f)] = \hat{\sigma}_s[x_i] = x_i = \sigma_{x_i}(f)$. So $\sigma_s \circ_G \sigma_{x_i} = \sigma_{x_i}$.

(ii) We will prove by induction on the complexity of the term s. If $s \in X$, then by (i) we get $\sigma_{x_i} \circ_G \sigma_s = \sigma_s \in P_G(2)$. Assume that $s = f(u,v)$ and $\sigma_{x_i} \circ_G \sigma_u, \sigma_{x_i} \circ_G \sigma_v \in P_G(2)$. Thus $\hat{\sigma}_{x_i}[u], \hat{\sigma}_{x_i}[v] \in X$. Consider $(\sigma_{x_i} \circ_G \sigma_s)(f) = (\sigma_{x_i} \circ_G \sigma_{f(u,v)})(f) = S^2(x_i, \hat{\sigma}_{x_i}[u], \hat{\sigma}_{x_i}[v])$. If $x_i = x_1$, then $(\sigma_{x_i} \circ_G \sigma_s)(f) = \hat{\sigma}_{x_i}[u] \in X$. If $x_i = x_2$, then $(\sigma_{x_i} \circ_G \sigma_s)(f) = \hat{\sigma}_{x_i}[v] \in X$. If $i > 2$, then $(\sigma_{x_i} \circ_G \sigma_s)(f) = x_i \in X$. So $\sigma_{x_i} \circ_G \sigma_s \in P_G(2)$.

(iii) Since $x_1, x_2 \notin var(t)$, thus $(\sigma_t \circ_G \sigma_{f(c,d)})(f) = S^2(t, \hat{\sigma}_t[c], \hat{\sigma}_t[d]) = t$ (since there has nothing to substitute in the term t). So $\sigma_t \circ_G \sigma_{f(c,d)} = \sigma_t$. ■

Proposition 4.1.5. *Every $\sigma_t \in G$ is idempotent.*

Proof. By Lemma 4.1.4 (iii). ■

Proposition 4.1.6. *Let $t \in W_{(2)}(X)$. Then the following statements hold:*

(i) *If $x_2 \in var(t)$ and $t \neq x_2$, then $\sigma_{f(x_1,t)}$ is not idempotent.*

(ii) *If $x_1 \in var(t)$ and $t \neq x_1$, then $\sigma_{f(t,x_2)}$ is not idempotent.*

(iii) *If $t \neq x_1$, then $\sigma_{f(t,x_1)}$ is not idempotent.*

(iv) *If $t \neq x_2$, then $\sigma_{f(x_2,t)}$ is not idempotent.*

(v) *If $x_1 \in var(t)$ or $x_2 \in var(t)$, then $\sigma_{f(x_i,t)}$ and $\sigma_{f(t,x_i)}$ are not idempotent where $i \in \mathbb{N}$ with $i > 2$.*

Proof. (i) Let $x_2 \in var(t)$ and $t \neq x_2$. Then we have $\hat{\sigma}_{f(x_1,t)}[f(x_1,t)] = S^2(f(x_1,t), x_1, \hat{\sigma}_{f(x_1,t)}[t])$. Since $x_2 \in var(t)$, then we have to substitute x_2 in the term t by $\hat{\sigma}_{f(x_1,t)}[t]$. Thus $S^2(f(x_1,t), x_1, \hat{\sigma}_{f(x_1,t)}[t]) \neq f(x_1,t)$.

The proof of (ii), (iii), (iv) and (v) are similar to (i). ■

Proposition 4.1.7. *Let $t_1, t_2 \in W_{(2)}(X) \setminus X$. If $x_1 \in var(t_1) \cup var(t_2)$ or $x_2 \in var(t_1) \cup var(t_2)$, then $\sigma_{f(t_1,t_2)}$ is not idempotent.*

Proof. The proof is similar to the proof of Proposition 4.1.6. ■

Then we have the main result:

Theorem 4.1.8. *$P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\}$ is the set of all idempotent elements in $Hyp_G(2)$.*

Proof. By Proposition 4.1.2, Proposition 4.1.3 and Proposition 4.1.5, we get every element in $P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup G \cup \{\sigma_{id}\}$ is idempotent. Let $\sigma_t \in Hyp_G(2)$ be idempotent. If $t \in X$, then $\sigma_t \in P_G(2)$. Let $t = f(t_1, t_2)$. We consider into two cases:

Case 1: $x_1 \in var(t)$. Suppose that $t_1 = x_1$. If $t_2 \in X$, then $\sigma_t \in E^G_{x_1} \cup \{\sigma_{id}\}$. If $t_2 \notin X$, then by Proposition 4.1.6 (i) we get $x_2 \notin var(t_2)$. So $\sigma_t \in E^G_{x_1}$. Suppose that $t_1 = x_2$. By Proposition 4.1.6 (iv), we get $t_2 = x_2$, which contradicts to $x_1 \in var(t)$. Suppose that $t_1 = x_i$ where $i > 2$. Then $x_1 \in var(t_2)$. By Proposition 4.1.6 (v), we get σ_t is not idempotent. Suppose that $t_1 \notin X$. If $x_1 \in var(t_1)$, then by Proposition 4.1.6 (ii), (iii), (v) and Proposition 4.1.7, we get σ_t is not idempotent. If $x_1 \notin var(t_1)$, then $x_1 \in var(t_2)$. By Proposition 4.1.6 (iii) and Proposition 4.1.7, we get σ_t is not idempotent.

Case 2: $x_1 \notin var(t)$. The proof of this case is similar to the proof of Case 1. ■

4.2 The Order of Generalized Hypersubstitutions of Type $\tau = (2)$

In this section, we determine the order of generalized hypersubstitutions of type $\tau = (2)$.

Lemma 4.2.1. *Let $f(c,d), f(u,v) \in W_{(2)}(X)$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$. Then $vb(w) > vb(f(c,d))$ unless $f(c,d)$ and $f(u,v)$ match one of the following 16 possibilities:*

E(1) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(c,d)}$ *where* $\sigma_{f(c,d)} \in G$.

E(2) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{C_{x_1}[f(c,d)]}$.

E(3) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_2)} = \sigma_{{}_{x_2}C[f(c,d)]}$.

E(4) $\sigma_{f(c,d)} \circ_G \sigma_{id} = \sigma_{f(c,d)}$.

E(5) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_i)} = \sigma_{C_{x_i}[f(c,d)]}$ *where* $x_i \in X,\ i > 2$.

E(6) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\overline{f(c,d)}}$.

E(7) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_i)} = \sigma_{{}_{x_2}C_{x_i}[f(c,d)]}$ *where* $x_i \in X,\ i > 2$.

E(8) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_1)} = \sigma_{{}_{x_i}C_{x_1}[f(c,d)]}$ *where* $x_i \in X,\ i > 2$.

E(9) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_2)} = \sigma_{{}_{x_i}C[f(c,d)]}$ *where* $x_i \in X,\ i > 2$.

E(10) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_j)} = \sigma_{{}_{x_i}C_{x_j}[f(c,d)]}$ *where* $x_i, x_j \in X,\ i,j > 2$.

E(11) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,v)} = \sigma_{f(c,d)}$ *where* $v \notin X,\ f(c,d) \in W(\{x_1\})$.

E(12) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,v)} = \sigma_{\overline{f(c,d)}}$ *where* $v \notin X,\ f(c,d) \in W(\{x_1\})$.

E(13) $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,v)} = \sigma_{{}_{x_i}C[f(c,d)]}$ *where* $x_i \in X, i > 2,\ v \notin X,\ f(c,d) \in W(\{x_1\})$.

E(14) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_1)} = \sigma_{\overline{f(c,d)}}$ *where* $u \notin X,\ f(c,d) \in W(\{x_2\})$.

E(15) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_2)} = \sigma_{f(c,d)}$ *where* $u \notin X,\ f(c,d) \in W(\{x_2\})$.

E(16) $\sigma_{f(c,d)} \circ_G \sigma_{f(u,x_i)} = \sigma_{C_{x_i}[f(c,d)]}$ *where* $x_i \in X, i > 2,\ u \notin X,\ f(c,d) \in W(\{x_2\})$.

Proof. Assume that $f(c,d), f(u,v) \in W_{(2)}(X)$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$. We want to compare $vb(w)$ with $vb(f(c,d))$. From $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_w$, thus $w = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v])$. If $\sigma_{f(c,d)} \in G$, then by Lemma 4.1.4 (iii) we get $w = f(c,d)$ and we have E(1). Assume that $\sigma_{f(c,d)} \notin G$. Then $x_1 \in var(f(c,d))$ or $x_2 \in var(f(c,d))$. We will consider the following cases.

Case 1: $u, v \in X$. We have $\hat{\sigma}_{f(c,d)}[u] = u$ and $\hat{\sigma}_{f(c,d)}[v] = v$. This gives 9 possible subcases:

(1) $u = v = x_1$. We have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{C_{x_1}[f(c,d)]}$, which is E(2).

(2) $u = v = x_2$. We have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_2)} = \sigma_{{}_{x_2}C[f(c,d)]}$, which is E(3).

(3) $u = x_1, v = x_2$. We have $\sigma_{f(c,d)} \circ_G \sigma_{id} = \sigma_{f(c,d)}$, which is E(4).

(4) $u = x_1, v = x_i, i > 2$. We have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_1,x_i)} = \sigma_{C_{x_i}[f(c,d)]}$, which is E(5).

(5) $u = x_2, v = x_1$. We have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\overline{f(c,d)}}$, which is E(6).

(6) $u = x_2, v = x_i, i > 2$. We have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_2,x_i)} = \sigma_{{}_{x_2}C_{x_i}[f(c,d)]}$, which is E(7).

(7) $u = x_i, v = x_1, i > 2$. We have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_1)} = \sigma_{{}_{x_i}C_{x_1}[f(c,d)]}$, which is E(8).

(8) $u = x_i, v = x_2, i > 2$. We have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_2)} = \sigma_{{}_{x_i}C[f(c,d)]}$, which is E(9).

(9) $u = x_i, v = x_j, i, j > 2$. We have $\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_j)} = \sigma_{{}_{x_i}C_{x_j}[f(c,d)]}$, which is E(10).

Case 2: $u = x_1$ and $v \notin X$. We have $w = S^2(f(c,d), x_1, \hat{\sigma}_{f(c,d)}[v])$. If $f(c,d) \in W(\{x_1\})$, then $w = f(c,d)$, as in E(11). Assume that $x_2 \in var(f(c,d))$. Since $vb(\hat{\sigma}_{f(c,d)}[v]) > 1$ and we have to substitute x_2 in $f(c,d)$ by $\hat{\sigma}_{f(c,d)}[v]$ thus $vb(w) > vb(f(c,d))$.

Case 3: $u = x_2$ and $v \notin X$. In this case we get E(12) or $vb(w) > vb(f(c,d))$.

Case 4: $u = x_i, i > 2$ and $v \notin X$. In this case we get E(13) or $vb(w) > vb(f(c,d))$.

Case 5: $u \notin X$ and $v = x_1$. In this case we get E(14) or $vb(w) > vb(f(c,d))$.

Case 6: $u \notin X$ and $v = x_2$. In this case we get E(15) or $vb(w) > vb(f(c,d))$.

Case 7: $u \notin X$ and $v = x_i, i > 2$. In this case we get E(16) or $vb(w) > vb(f(c,d))$.

Case 8: $u, v \notin X$. We have $vb(\hat{\sigma}_{f(c,d)}[u]) > 1$ and $vb(\hat{\sigma}_{f(c,d)}[v]) > 1$. Since $vb(\hat{\sigma}_{f(c,d)}[u]) > 1$ and $vb(\hat{\sigma}_{f(c,d)}[v]) > 1$ and we have to substitute x_1 in $f(c,d)$ by $\hat{\sigma}_{f(c,d)}[u]$ or x_2 in $f(c,d)$ by $\hat{\sigma}_{f(c,d)}[v]$, thus $vb(w) > vb(f(c,d))$. ■

Lemma 4.2.2. *Let $s \in W_{(2)}(X) \backslash X, x_1, x_2 \in var(s), t \in W_{(2)}(X)$ and $x_i \in X$. If $x_i \in var(t)$, then $x_i \in var(\hat{\sigma}_s[t])$ ($x_i \in var((\sigma_s \circ_G \sigma_t)(f))$).*

Proof. We will prove by induction on the complexity of the term t. If $t \in X$, then $t = x_i$. So $\hat{\sigma}_s[t] = x_i$ and thus $x_i \in var(\hat{\sigma}_s[t])$. Let $t = f(t_1, t_2)$. Then $x_i \in$

$var(t_1)$ or $x_i \in var(t_2)$. Assume that $x_i \in var(t_1)$ and $x_i \in var(\hat{\sigma}_s[t_1])$. Consider $\hat{\sigma}_s[t] = \hat{\sigma}_s[f(t_1, t_2)] = S^2(s, \hat{\sigma}_s[t_1], \hat{\sigma}_s[t_2])$. Since $x_1 \in var(s)$ and $x_i \in var(\hat{\sigma}_s[t_1])$, thus $x_i \in var(\hat{\sigma}_s[t])$. By the same way, we can show that if $x_i \in var(t_2)$, then $x_i \in var(\hat{\sigma}_s[t])$. ∎

Lemma 4.2.3. *Let $s \in W_{(2)}(X) \backslash X$. If $x_1, x_2 \in var(s)$, then $x_1, x_2 \in var(\sigma_s^n(f))$ for all $n \in \mathbb{N}$.*

Proof. Let $s = f(s_1, s_2)$. For $n = 1$, $\sigma_s^1(f) = \sigma_s(f) = s$. So $x_1, x_2 \in var(\sigma_s^1(f))$. Assume that $x_1, x_2 \in var(\sigma_s^n(f))$. Consider $\sigma_s^{n+1}(f) = (\sigma_s^n \circ_G \sigma_s)(f) = \hat{\sigma_s^n}[\sigma_s(f)] = \hat{\sigma_s^n}[s] = \hat{\sigma_s^n}[f(s_1, s_2)] = S^2(\sigma_s^n(f), \hat{\sigma_s^n}[s_1], \hat{\sigma_s^n}[s_2])$. If $x_1, x_2 \in var(s_1)$, then by Lemma 4.2.2 we get $x_1, x_2 \in var(\hat{\sigma_s^n}[s_1])$. Since $x_1 \in var(\sigma_s^n(f))$ and $x_1, x_2 \in var(\hat{\sigma_s^n}[s_1])$ thus $x_1, x_2 \in var(\sigma_s^{n+1}(f))$. If $s_1 \in W_{(2)}^G(\{x_1\})$, then $x_2 \in var(s_2)$. By Lemma 4.2.2, we get $x_1 \in var(\hat{\sigma_s^n}[s_1])$ and $x_2 \in var(\hat{\sigma_s^n}[s_2])$. Since $x_1, x_2 \in var(\sigma_s^n(f))$, thus $x_1, x_2 \in var(\sigma_s^{n+1}(f))$. If $s_1 \in W_{(2)}^G(\{x_2\})$, then by the same proof of the case $s_1 \in W_{(2)}^G(\{x_1\})$ we get $x_1, x_2 \in var(\sigma_s^{n+1}(f))$. If $x_1, x_2 \notin var(s_1)$, then $x_1, x_2 \in var(s_2)$. By the same proof of the case $x_1, x_2 \in var(s_1)$, we get $x_1, x_2 \in var((\sigma_s)^{n+1}(f))$. ∎

Lemma 4.2.4. *Let $s \in W_{(2)}(X)$. If $leftmost(s) = x_1$, then $leftmost(\sigma_s^n(f)) = x_1$ for all $n \in \mathbb{N}$.*

Proof. It is clear for $s \in X$. Let $s = f(s_1, s_2)$. For $n = 1$, $\sigma_s^1(f) = \sigma_s(f) = s$. So $leftmost(\sigma_s^1(f)) = x_1$. Assume that $leftmost(\sigma_s^n(f)) = x_1$. Consider $\sigma_s^{n+1}(f) = (\sigma_s^n \circ_G \sigma_s)(f) = \hat{\sigma_s^n}[s] = \hat{\sigma_s^n}[f(s_1, s_2)] = S^2(\sigma_s^n(f), \hat{\sigma_s^n}[s_1], \hat{\sigma_s^n}[s_2])$. If $s_1 \in X$, then s_1 is the leftmost of s, so $s_1 = x_1$. Thus $\hat{\sigma_s^n}[s_1] = x_1$. Since $\sigma_s^{n+1}(f) = S^2(\sigma_s^n(f), \hat{\sigma_s^n}[s_1], \hat{\sigma_s^n}[s_2])$, $leftmost(\sigma_s^n(f)) = x_1$ and $\hat{\sigma_s^n}[s_1] = x_1$, thus $leftmost(\sigma_s^{n+1}(f)) = x_1$. Let $s_1 = f(s_3, s_4)$. Consider $\hat{\sigma_s^n}[s_1] = \hat{\sigma_s^n}[f(s_3, s_4)] = S^2(\sigma_s^n(f), \hat{\sigma_s^n}[s_3], \hat{\sigma_s^n}[s_4])$. If $s_3 \in X$, then s_3 is the leftmost of s, so $s_3 = x_1$. Thus $\hat{\sigma_s^n}[s_3] = x_1$. Since $\hat{\sigma_s^n}[s_1] = S^2(\sigma_s^n(f), \hat{\sigma_s^n}[s_3], \hat{\sigma_s^n}[s_4])$, $leftmost((\sigma_s^n)(f)) = x_1$ and $\hat{\sigma_s^n}[s_3] = x_1$, thus $leftmost(\hat{\sigma_s^n}[s_1]) = x_1$, which implies that $leftmost(\sigma_s^{n+1}(f)) = x_1$. This procedure stops after finitely many steps at $leftmost(s) = x_1$. ∎

Lemma 4.2.5. *Let $s \in W(\{x_1\})$. If $leftmost(s) = x_i$ where $i > 2$, then $x_1, x_2 \notin var(\sigma_s^2(f))$.*

Proof. Let $s = f(s_1, s_2)$. Consider $\sigma_s^2(f) = (\sigma_s \circ_G \sigma_s)(f) = \hat{\sigma}_s[s] = \hat{\sigma}_s[f(s_1, s_2)] = S^2(s, \hat{\sigma}_s[s_1], \hat{\sigma}_s[s_2])$. If $s_1 \in X$, then s_1 is the leftmost of s, so $s_1 = x_i$. Thus $\hat{\sigma}_s[s_1] = x_i$.

Since $s \in W(\{x_1\})$, $x_1, x_2 \notin var(\hat{\sigma}_s[s_1])$ and $\sigma_s^2(f) = S^2(s, \hat{\sigma}_s[s_1], \hat{\sigma}_s[s_2])$, thus $x_1, x_2 \notin var(\sigma_s^2(f))$. Let $s_1 = f(s_3, s_4)$. Consider $\hat{\sigma}_s[s_1] = \hat{\sigma}_s[f(s_3, s_4)] = S^2(s, \hat{\sigma}_s[s_3], \hat{\sigma}_s[s_4])$. If $s_3 \in X$, then s_3 is the leftmost of s, so $s_3 = x_i$. Thus $\hat{\sigma}_s[s_3] = x_i$. Since $s \in W(\{x_1\})$, $x_1, x_2 \notin var(\hat{\sigma}_s[s_3])$ and $\hat{\sigma}_s[s_1] = S^2(s, \hat{\sigma}_s[s_3], \hat{\sigma}_s[s_4])$, thus $x_1, x_2 \notin var(\hat{\sigma}_s[s_1])$, which implies that $x_1, x_2 \notin var(\sigma_s^2(f))$. This procedure stops after finitely many steps at $leftmost(s) = x_i$. ■

Lemma 4.2.6. *Let $s \in W_{(2)}(X)$. If $rightmost(s) = x_2$, then $rightmost(\sigma_s^n(f)) = x_2$ for all $n \in \mathbb{N}$.*

Proof. The proof is similar to the proof of Lemma 4.2.4. ■

Lemma 4.2.7. *Let $s \in W(\{x_2\})$. If $rightmost(s) = x_i$ where $i > 2$, then $x_1, x_2 \notin var(\sigma_s^2(f))$.*

Proof. The proof is similar to the proof of Lemma 4.2.5. ■

Note that $\{\sigma^n_{f(x_2,x_1)} | n \in \mathbb{N}\} = \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$, the order of $\sigma_{f(x_2,x_1)}$ is 2.

Proposition 4.2.8. *Let $s \in W_{(2)}(X)$, $x_1, x_2 \in var(s)$, σ_s not be idempotent and not be equal to $\sigma_{f(x_2,x_1)}$. Then the order of σ_s is infinite.*

Proof. Let $n \in \mathbb{N}$. Let w be the term for σ_s^n. By Lemma 4.2.3, we get $x_1, x_2 \in var(w)$. Then the equation $\sigma_s^{n+1} = \sigma_s^n \circ_G \sigma_s$ dose not fit any of E(1) to E(16), so by Lemma 4.2.1 we must have the term for σ_s^{n+1} is longer than w. This implies the order of σ_s is infinite. ■

Proposition 4.2.9. *Let $s \in W(\{x_1\})$ and σ_s not be idempotent. If $leftmost(s) = x_1$, then the order of σ_s is infinite.*

Proof. Let $n \in \mathbb{N}$. Let w be the term for σ_s^n. By Lemma 4.2.4, we get $leftmost(w) = x_1$. Then the equation $\sigma_s^{n+1} = \sigma_s^n \circ_G \sigma_s$ dose not fit any of E(1) to E(16), so by Lemma 4.2.1 we must have the term for σ_s^{n+1} is longer than w. This implies the order of σ_s is infinite. ■

Proposition 4.2.10. *Let $s \in W(\{x_1\})$ and σ_s not be idempotent. If $leftmost(s) = x_i$ where $i > 2$, then the order of σ_s is 2.*

Proof. Let w be the term for σ_s^2. By Lemma 4.2.5, we get $x_1, x_2 \notin var(w)$. This implies $\sigma_s^n = \sigma_s^2$ for all $n \in \mathbb{N}$ where $n \geq 2$. So the order of σ_s is 2. ■

Proposition 4.2.11. *Let $s \in W(\{x_2\})$ and σ_s not be idempotent. If $rightmost(s) = x_2$, then the order of σ_s is infinite.*

Proof. The proof is similar to the proof of Proposition 4.2.9. ∎

Proposition 4.2.12. *Let $s \in W(\{x_2\})$ and σ_s not be idempotent. If $rightmost(s) = x_i$ where $i > 2$, then the order of σ_s is 2.*

Proof. The proof is similar to the proof of Proposition 4.2.10. ∎

Then we have the main result:

Theorem 4.2.13. *The order of any generalized hypersubstitution of type $\tau = (2)$ is 1,2 or infinite.*

Proof. Let $\sigma_t \in Hyp_G(2)$. If σ_t is idempotent, then the order of σ_t is 1. If σ_t is not idempotent, then $x_1 \in var(t)$ or $x_2 \in var(t)$. Assume that $x_1, x_2 \in var(t)$. If $\sigma_t = \sigma_{f(x_2,x_1)}$, then the order of σ_t is 2. If $\sigma_t \neq \sigma_{f(x_2,x_1)}$, then by Proposition 4.2.8 we get the order of σ_t is infinite. Assume that $x_1 \in var(t)$ and $x_2 \notin var(t)$. If $leftmost(t) = x_1$, then by Proposition 4.2.9 we get the order of σ_t is infinite. If $leftmost(t) = x_i$ where $i > 2$, then by Proposition 4.2.10 we get the order of σ_t is 2. By the same way we can show that if $x_2 \in var(t)$ and $x_1 \notin var(t)$, then the order of σ_t is 2 or infinite. ∎

4.3 Regular Elements in $Hyp_G(2)$

Now, we characterize all regular elements of $Hyp_G(2)$.

Proposition 4.3.1. *For every $x_i \in X$, σ_{x_i} and σ_{id} are regular.*

Proof. Since every $\sigma_{x_i} \in P_G(2)$ and σ_{id} are idempotent, thus they are regular. ∎

Proposition 4.3.2. *$\sigma_{f(x_i,x_j)}$ is regular for every $x_i, x_j \in X$.*

Proof. Let $x_i, x_j \in X$. We consider into three cases.

Case 1: $i = 2, j \in \mathbb{N}$. We have

$$
\begin{aligned}
(\sigma_{f(x_2,x_j)} \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_2,x_j)})(f) &= \widehat{(\sigma_{f(x_2,x_j)} \circ_G \sigma_{f(x_2,x_1)})}[\sigma_{f(x_2,x_j)}(f)] \\
&= \widehat{(\sigma_{f(x_2,x_j)} \circ_G \sigma_{f(x_2,x_1)})}[f(x_2, x_j)] \\
&= \hat{\sigma}_{f(x_2,x_j)}[\hat{\sigma}_{f(x_2,x_1)}[f(x_2, x_j)]] \\
&= \hat{\sigma}_{f(x_2,x_j)}[S^2(f(x_2, x_1), x_2, x_j)] \\
&= \hat{\sigma}_{f(x_2,x_j)}[f(x_j, x_2)] \\
&= S^2(f(x_2, x_j), x_j, x_2) \\
&= f(x_2, x_j).
\end{aligned}
$$

Thus $\sigma_{f(x_2,x_j)} \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_2,x_j)} = \sigma_{f(x_2,x_j)}$.

Case 2: $i \neq 2, j = 1$. We have

$$
\begin{aligned}
(\sigma_{f(x_i,x_1)} \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_i,x_1)})(f) &= \widehat{(\sigma_{f(x_i,x_1)} \circ_G \sigma_{f(x_2,x_1)})}[\sigma_{f(x_i,x_1)}(f)] \\
&= \widehat{(\sigma_{f(x_i,x_1)} \circ_G \sigma_{f(x_2,x_1)})}[f(x_i, x_1)] \\
&= \hat{\sigma}_{f(x_i,x_1)}[\hat{\sigma}_{f(x_2,x_1)}[f(x_i, x_1)]] \\
&= \hat{\sigma}_{f(x_i,x_1)}[S^2(f(x_2, x_1), x_i, x_1)] \\
&= \hat{\sigma}_{f(x_i,x_1)}[f(x_1, x_i)] \\
&= S^2(f(x_i, x_1), x_1, x_i) \\
&= f(x_i, x_1).
\end{aligned}
$$

Thus $\sigma_{f(x_i,x_1)} \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_i,x_1)} = \sigma_{f(x_i,x_1)}$.

Case 3: $i \neq 2, j \neq 1$. We have $\sigma_{f(x_i,x_j)}$ is idempotent, thus it is regular.

Then $\sigma_{f(x_i,x_j)}$ is regular for all $x_i, x_j \in X$. ■

Proposition 4.3.3. *Let $t \in W_{(2)}(X) \setminus X$. Then the following statements hold:*

(i) *If $x_2 \notin var(t)$, then $\sigma_{f(t,x_1)}$, $\sigma_{f(x_1,t)}$ are regular.*

(ii) *If $x_1 \notin var(t)$, then $\sigma_{f(t,x_2)}$, $\sigma_{f(x_2,t)}$ are regular.*

Proof. (i) Let $x_2 \notin var(t)$. Then we have

$$\begin{aligned}
(\sigma_{f(t,x_1)} \circ_G \sigma_{f(x_2,x_2)} \circ_G \sigma_{f(t,x_1)})(f) &= (\sigma_{f(t,x_1)} \circ_G \sigma_{f(x_2,x_2)})\hat{}[\sigma_{f(t,x_1)}(f)] \\
&= (\sigma_{f(t,x_1)} \circ_G \sigma_{f(x_2,x_2)})\hat{}[f(t,x_1)] \\
&= \hat{\sigma}_{f(t,x_1)}[\hat{\sigma}_{f(x_2,x_2)}[f(t,x_1)]] \\
&= \hat{\sigma}_{f(t,x_1)}[S^2(f(x_2,x_2), \hat{\sigma}_{f(x_2,x_2)}[t], x_1)] \\
&= \hat{\sigma}_{f(t,x_1)}[f(x_1,x_1)] \\
&= S^2(f(t,x_1), x_1, x_1) \\
&= f(t,x_1) \;\; (x_2 \notin var(t)).
\end{aligned}$$

Thus $\sigma_{f(t,x_1)} \circ_G \sigma_{f(x_2,x_2)} \circ_G \sigma_{f(t,x_1)} = \sigma_{f(t,x_1)}$.

Since $\sigma_{f(x_1,t)}$ is idempotent, thus it is regular.

(ii) Let $x_1 \notin var(t)$. Since $\sigma_{f(t,x_2)}$ is idempotent, thus it is regular.

Consider

$$\begin{aligned}
(\sigma_{f(x_2,t)} \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_{f(x_2,t)})(f) &= (\sigma_{f(x_2,t)} \circ_G \sigma_{f(x_1,x_1)})\hat{}[\sigma_{f(x_2,t)}(f)] \\
&= (\sigma_{f(x_2,t)} \circ_G \sigma_{f(x_1,x_1)})\hat{}[f(x_2,t)] \\
&= \hat{\sigma}_{f(x_2,t)}[\hat{\sigma}_{f(x_1,x_1)}[f(x_2,t)]] \\
&= \hat{\sigma}_{f(x_2,t)}[S^2(f(x_1,x_1), x_2, \hat{\sigma}_{f(x_1,x_1)}[t])] \\
&= \hat{\sigma}_{f(x_2,t)}[f(x_2,x_2)] \\
&= S^2(f(x_2,t), x_2, x_2) \\
&= f(x_2,t) \;\; (x_1 \notin var(t)).
\end{aligned}$$

Thus $\sigma_{f(x_2,t)} \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_{f(x_2,t)} = \sigma_{f(x_2,t)}$. ∎

Proposition 4.3.4. *Every $\sigma_t \in G$ is regular.*

Proof. Since every $\sigma_t \in G$ is idempotent, thus it is regular. ∎

Proposition 4.3.5. *Let $t \in W_{(2)}(X) \setminus X$. Then the following statements hold:*

(i) *If $x_2 \in var(t)$, then $\sigma_{f(t,x_1)}, \sigma_{f(x_1,t)}$ are not regular.*

(ii) *If $x_1 \in var(t)$, then $\sigma_{f(t,x_2)}, \sigma_{f(x_2,t)}$ are not regular.*

Proof. (i) Let $x_2 \in var(t)$. We will show that $\sigma_{f(t,x_1)}, \sigma_{f(x_1,t)}$ are not regular. Suppose that $\sigma_{f(t,x_1)}$ is regular, thus there exists $\sigma_{t_1} \in W_{(2)}(X)$ such that $\sigma_{f(t,x_1)} \circ_G \sigma_{t_1} \circ_G \sigma_{f(t,x_1)} = \sigma_{f(t,x_1)}$. Thus $(\sigma_{f(t,x_1)} \circ_G \sigma_{t_1} \circ_G \sigma_{f(t,x_1)})(f) = \sigma_{f(t,x_1)}(f)$. We have $\hat{\sigma}_{f(t,x_1)}[\hat{\sigma}_{t_1}[f(t,x_1)]] = f(t,x_1)$. Put $s = \hat{\sigma}_{t_1}[f(t,x_1)]$. Then $\hat{\sigma}_{f(t,x_1)}[s] = f(t,x_1)$. We have $s \notin X$, thus $s = f(s_1,s_2)$ for some $s_1,s_2 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t,x_1)}[f(s_1,s_2)] = f(t,x_1)$. We have $S^2(f(t,x_1), \hat{\sigma}_{f(t,x_1)}[s_1], \hat{\sigma}_{f(t,x_1)}[s_2]) = f(t,x_1)$. Thus $\hat{\sigma}_{f(t,x_1)}[s_1] = x_1$ and since $x_2 \in var(t)$ thus $\hat{\sigma}_{f(t,x_1)}[s_2] = x_2$. We have $s_1 = x_1, s_2 = x_2$. Thus $s = f(x_1,x_2)$ and $\hat{\sigma}_{t_1}[f(t,x_1)] = f(x_1,x_2)$. By Lemma 4.1.4 (ii), we get $t_1 \notin X$, thus $t_1 = f(t_2,t_3)$ for some $t_2,t_3 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t_2,t_3)}[f(t,x_1)] = f(x_1,x_2)$. We have $S^2(f(t_2,t_3), \hat{\sigma}_{f(t_2,t_3)}[t], x_1) = f(x_1,x_2)$. Since $t \notin X$, thus $\hat{\sigma}_{f(t_2,t_3)}[t] \notin X$. From $S^2(f(t_2,t_3), \hat{\sigma}_{f(t_2,t_3)}[t], x_1) = f(x_1,x_2)$, thus $t_3 = x_1$ and $\hat{\sigma}_{f(t_2,t_3)}[t] = x_2$ which contradicts to $\hat{\sigma}_{f(t_2,t_3)}[t] \notin X$. Hence $\sigma_{f(t,x_1)}$ is not regular. Suppose that $\sigma_{f(x_1,t)}$ is regular, thus there exists $\sigma_{t_1} \in W_{(2)}(X)$ such that $\sigma_{f(x_1,t)} \circ_G \sigma_{t_1} \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)}$. Thus $(\sigma_{f(x_1,t)} \circ_G \sigma_{t_1} \circ_G \sigma_{f(x_1,t)})(f) = \sigma_{f(x_1,t)}(f)$. We have $\hat{\sigma}_{f(x_1,t)}[\hat{\sigma}_{t_1}[f(x_1,t)]] = f(x_1,t)$. Put $s = \hat{\sigma}_{t_1}[f(x_1,t)]$. Then $\hat{\sigma}_{f(x_1,t)}[s] = f(x_1,t)$. We have $s \notin X$, thus $s = f(s_1,s_2)$ for some $s_1,s_2 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(x_1,t)}[f(s_1,s_2)] = f(x_1,t)$. We have $S^2(f(x_1,t), \hat{\sigma}_{f(x_1,t)}[s_1], \hat{\sigma}_{f(x_1,t)}[s_2]) = f(x_1,t)$. Thus $\hat{\sigma}_{f(x_1,t)}[s_1] = x_1$ and since $x_2 \in var(t)$, thus $\hat{\sigma}_{f(x_1,t)}[s_2] = x_2$. We have $s_1 = x_1, s_2 = x_2$. Thus $s = f(x_1,x_2)$ and $\hat{\sigma}_{t_1}[f(x_1,t)] = f(x_1,x_2)$. By Lemma 4.1.4 (ii), we get $t_1 \notin X$ thus $t_1 = f(t_2,t_3)$ for some $t_2,t_3 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t_2,t_3)}[f(x_1,t)] = f(x_1,x_2)$. We have $S^2(f(t_2,t_3), x_1, \hat{\sigma}_{f(t_2,t_3)}[t]) = f(x_1,x_2)$. Since $t \notin X$, thus $\hat{\sigma}_{f(t_2,t_3)}[t] \notin X$. From $S^2(f(t_2,t_3), x_1, \hat{\sigma}_{f(t_2,t_3)}[t]) = f(x_1,x_2)$, thus $t_3 = x_2$ and $\hat{\sigma}_{f(t_2,t_3)}[t] = x_2$ which contradicts to $\hat{\sigma}_{f(t_2,t_3)}[t] \notin X$. Hence $\sigma_{f(x_1,t)}$ is not regular.

(ii) The proof is similar to (i). ■

Proposition 4.3.6. *For any $t \in W_{(2)}(X)\backslash X$. If $x_1 \in var(t)$ or $x_2 \in var(t)$, then $\sigma_{f(t,x_i)}$ and $\sigma_{f(x_i,t)}$ where $i > 2$, are not regular.*

Proof. Let $x_1 \in var(t)$ or $x_2 \in var(t)$ and let $i \in \mathbb{N}$ with $i > 2$. We will show that $\sigma_{f(t,x_i)}$ and $\sigma_{f(x_i,t)}$ are not regular.

Case 1: $x_1 \in var(t)$. Suppose that $\sigma_{f(t,x_i)}$ is regular, thus there exists $\sigma_{t_1} \in W_{(2)}(X)$ such that $\sigma_{f(t,x_i)} \circ_G \sigma_{t_1} \circ_G \sigma_{f(t,x_i)} = \sigma_{f(t,x_i)}$. Thus $(\sigma_{f(t,x_i)} \circ_G \sigma_{t_1} \circ_G \sigma_{f(t,x_i)})(f) = \sigma_{f(t,x_i)}(f)$. We have $\hat{\sigma}_{f(t,x_i)}[\hat{\sigma}_{t_1}[f(t,x_i)]] = f(t,x_i)$. Put $s = \hat{\sigma}_{t_1}[f(t,x_i)]$. Then $\hat{\sigma}_{f(t,x_i)}[s] = f(t,x_i)$. We have $s \notin X$, thus $s = f(s_1,s_2)$ for some $s_1,s_2 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t,x_i)}[f(s_1,s_2)] = f(t,x_i)$. We have $S^2(f(t,x_i), \hat{\sigma}_{f(t,x_i)}[s_1], \hat{\sigma}_{f(t,x_i)}[s_2]) = f(t,x_i)$. Since $x_1 \in var(t)$, thus

$\hat{\sigma}_{f(t,x_i)}[s_1] = x_1$. We have $s_1 = x_1$. Thus $s = f(x_1, s_2)$ and $\hat{\sigma}_{t_1}[f(t, x_i)] = f(x_1, s_2)$. By Lemma 4.1.4 (ii), we get $t_1 \notin X$ thus $t_1 = f(t_2, t_3)$ for some $t_2, t_3 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t_2,t_3)}[f(t, x_i)] = f(x_1, s_2)$. We have $S^2(f(t_2, t_3), \hat{\sigma}_{f(t_2,t_3)}[t], x_i) = f(x_1, s_2)$. Since $t \notin X$, thus $\hat{\sigma}_{f(t_2,t_3)}[t] \notin X$. From $S^2(f(t_2, t_3), \hat{\sigma}_{f(t_2,t_3)}[t], x_i) = f(x_1, s_2)$, thus $t_2 = x_1$ and $\hat{\sigma}_{f(t_2,t_3)}[t] = x_1$ which contradicts to $\hat{\sigma}_{f(t_2,t_3)}[t] \notin X$. Hence $\sigma_{f(t,x_i)}$ is not regular. For $\sigma_{f(x_i,t)}$ is not regular we can prove in the similar way.

Case 2: $x_2 \in var(t)$. Suppose that $\sigma_{f(t,x_i)}$ is regular, thus there exists $\sigma_{t_1} \in W_{(2)}(X)$ such that $\sigma_{f(t,x_i)} \circ_G \sigma_{t_1} \circ_G \sigma_{f(t,x_i)} = \sigma_{f(t,x_i)}$. Thus $(\sigma_{f(t,x_i)} \circ_G \sigma_{t_1} \circ_G \sigma_{f(t,x_i)})(f) = \sigma_{f(t,x_i)}(f)$. We have $\hat{\sigma}_{f(t,x_i)}[\hat{\sigma}_{t_1}[f(t, x_i)]] = f(t, x_i)$. Put $s = \hat{\sigma}_{t_1}[f(t, x_i)]$. Then $\hat{\sigma}_{f(t,x_i)}[s] = f(t, x_i)$. We have $s \notin X$, thus $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t,x_i)}[f(s_1, s_2)] = f(t, x_i)$. We have $S^2(f(t, x_i), \hat{\sigma}_{f(t,x_i)}[s_1], \hat{\sigma}_{f(t,x_i)}[s_2]) = f(t, x_i)$. Since $x_2 \in var(t)$, thus $\hat{\sigma}_{f(t,x_i)}[s_2] = x_2$. We have $s_2 = x_2$. Thus $s = f(s_1, x_2)$ and $\hat{\sigma}_{t_1}[f(t, x_i)] = f(s_1, x_2)$. By Lemma 4.1.4 (ii), we get $t_1 \notin X$ thus $t_1 = f(t_2, t_3)$ for some $t_2, t_3 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t_2,t_3)}[f(t, x_i)] = f(s_1, x_2)$. We have $S^2(f(t_2, t_3), \hat{\sigma}_{f(t_2,t_3)}[t], x_i) = f(s_1, x_2)$. Since $t \notin X$, thus $\hat{\sigma}_{f(t_2,t_3)}[t] \notin X$. From $S^2(f(t_2, t_3), \hat{\sigma}_{f(t_2,t_3)}[t], x_i) = f(s_1, x_2)$, thus $t_3 = x_1$ and $\hat{\sigma}_{f(t_2,t_3)}[t] = x_2$ which contradicts to $\hat{\sigma}_{f(t_2,t_3)}[t] \notin X$. Hence $\sigma_{f(t,x_i)}$ is not regular. For $\sigma_{f(x_i,t)}$ is not regular we can prove in the similar way. ■

Proposition 4.3.7. *If $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \backslash X$ and $x_1 \in var(t_1) \cup var(t_2)$ or $x_2 \in var(t_1) \cup var(t_2)$, then σ_t is not regular.*

Proof. Let $t = f(t_1, t_2)$ where $t_1, t_2 \in W_{(2)}(X) \backslash X$ and $x_1 \in var(t_1) \cup var(t_2)$ or $x_2 \in var(t_1) \cup var(t_2)$. Then we will show that σ_t is not regular.

Case 1: $x_1 \in var(t_1) \cup var(t_2)$. Suppose that $\sigma_t = \sigma_{f(t_1,t_2)}$ is regular, thus there exists $\sigma_u \in W_{(2)}(X)$ such that $\sigma_{f(t_1,t_2)} \circ_G \sigma_u \circ_G \sigma_{f(t_1,t_2)} = \sigma_{f(t_1,t_2)}$. Thus $(\sigma_{f(t_1,t_2)} \circ_G \sigma_u \circ_G \sigma_{f(t_1,t_2)})(f) = \sigma_{f(t_1,t_2)}(f)$. We have $\hat{\sigma}_{f(t_1,t_2)}[\hat{\sigma}_u[f(t_1, t_2)]] = f(t_1, t_2)$. Put $s = \hat{\sigma}_u[f(t_1, t_2)]$. Then $\hat{\sigma}_{f(t_1,t_2)}[s] = f(t_1, t_2)$. We have $s \notin X$, thus $s = f(s_1, s_2)$ for some $s_1, s_2 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t_1,t_2)}[f(s_1, s_2)] = f(t_1, t_2)$. We have $S^2(f(t_1, t_2), \hat{\sigma}_{f(t_1,t_2)}[s_1], \hat{\sigma}_{f(t_1,t_2)}[s_2]) = f(t_1, t_2)$. Since $x_1 \in var(t_1) \cup var(t_2)$, thus $\hat{\sigma}_{f(t_1,t_2)}[s_1] = x_1$. We have $s_1 = x_1$. Thus $s = f(x_1, s_2)$ and $\hat{\sigma}_u[f(t_1, t_2)] = f(x_1, s_2)$. By Lemma 4.1.4 (ii), we get $u \notin X$ thus $u = f(t_3, t_4)$ for some $t_3, t_4 \in W_{(2)}(X)$. Thus $\hat{\sigma}_{f(t_3,t_4)}[f(t_1, t_2)] = f(x_1, s_2)$. We have $S^2(f(t_3, t_4), \hat{\sigma}_{f(t_3,t_4)}[t_1], \hat{\sigma}_{f(t_3,t_4)}[t_2]) = f(x_1, s_2)$. Since $t_1, t_2 \notin X$, thus $\hat{\sigma}_{f(t_3,t_4)}[t_1], \hat{\sigma}_{f(t_3,t_4)}[t_2] \notin X$. From $S^2(f(t_3, t_4), \hat{\sigma}_{f(t_3,t_4)}[t_1], \hat{\sigma}_{f(t_3,t_4)}[t_2]) = f(x_1, s_2)$, thus $t_3 = x_1$ or $t_3 = x_2$ and this implies that $\hat{\sigma}_{f(t_3,t_4)}[t_1] = x_1$ or $\hat{\sigma}_{f(t_3,t_4)}[t_2] = x_1$, which contradicts to $\hat{\sigma}_{f(t_3,t_4)}[t_1], \hat{\sigma}_{f(t_3,t_4)}[t_2] \notin X$. Hence $\sigma_{f(t_1,t_2)}$ is not regular.

Case 2: $x_2 \in var(t_1) \cup var(t_2)$. The proof is similar to Case 1. ■

Then we have the main result:

Theorem 4.3.8. *$P_G(2) \cup E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}} \cup G \cup \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$ is the set of all regular elements in $Hyp_G(2)$.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.1.8. ■

4.4 Green's Relations on $Hyp_G(2)$

In this section, we study Green's relations on $Hyp_G(2)$.

Proposition 4.4.1. *For any $\sigma_t \in Hyp_G(2) \backslash P_G(2)$, we have $\sigma_t \mathcal{R} \sigma_{\bar{t}}$, $\sigma_t \mathcal{L} \sigma_{t'}$ and $\sigma_t \mathcal{D} \sigma_{\bar{t}} \mathcal{D} \sigma_{t'} \mathcal{D} \sigma_{\overline{t'}}$.*

Proof. Let $\sigma_t \in Hyp_G(2) \setminus P_G(2)$. Then $\sigma_{\bar{t}} \circ_G \sigma_{f(x_2,x_1)} = \sigma_t$, $\sigma_t \circ_G \sigma_{f(x_2,x_1)} = \sigma_{\bar{t}}$, $\sigma_{f(x_2,x_1)} \circ_G \sigma_{t'} = \sigma_t$ and $\sigma_{f(x_2,x_1)} \circ_G \sigma_t = \sigma_{t'}$. So $\sigma_t \mathcal{R} \sigma_{\bar{t}}$ and $\sigma_t \mathcal{L} \sigma_{t'}$. Therefore $\sigma_t \mathcal{D} \sigma_{\bar{t}} \mathcal{D} \sigma_{t'} \mathcal{D} \sigma_{\overline{t'}}$. ■

Proposition 4.4.2. *Any $\sigma_{x_i} \in P_G(2)$ is $\mathcal{L}$-related only to itself, but is $\mathcal{R}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $P_G(2)$, and not related to any other generalized hypersubstitutions. Moreover, the set $P_G(2)$ forms an $\mathcal{R}$-, $\mathcal{D}$- and $\mathcal{J}$- class.*

Proof. By Lemma 4.1.4 (i), we get for any $\sigma_{x_i} \in P_G(2)$, $\sigma \circ_G \sigma_{x_i} = \sigma_{x_i}$ for all $\sigma \in Hyp_G(2)$. This shows that any $\sigma_{x_i} \in P_G(2)$ can be $\mathcal{L}$-related only to itself. Since $\sigma_{x_i} \circ_G \sigma_{x_j} = \sigma_{x_j}$ for all $\sigma_{x_i}, \sigma_{x_j} \in P_G(2)$, so any two elements in $P_G(2)$ are $\mathcal{R}$-related. From $\mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$, thus any two elements in $P_G(2)$ are $\mathcal{D}-$ and $\mathcal{J}-$related. Moreover by Lemma 4.1.4 (i),(ii), we get $\sigma_s \circ_G \sigma_{x_i} \circ_G \sigma_t \in P_G(2)$ for all $\sigma_s, \sigma_t \in Hyp_G(2), \sigma_{x_i} \in P_G(2)$. This implies if $\sigma \notin P_G(2)$, then σ cannot be $\mathcal{J}$-related to every element in $P_G(2)$. So $P_G(2)$ is the $\mathcal{J}$-class of its elements. Since any two elements in $P_G(2)$ are $\mathcal{R}-$ and $\mathcal{D}-$ related, $\mathcal{R} \subseteq \mathcal{J}, \mathcal{D} \subseteq \mathcal{J}$ and $P_G(2)$ is the $\mathcal{J}$-class of its elements, thus $P_G(2)$ forms an $\mathcal{R}$-, $\mathcal{D}$-class. ■

Lemma 4.4.3. *Let $\sigma_s, \sigma_t \in Hyp_G(2)$. Then the following statements hold:*

(i) *If $\sigma_s \circ_G \sigma_t = \sigma_{id}$, then either $\sigma_s = \sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_t = \sigma_{f(x_2,x_1)}$.*

(ii) *If* $\sigma_s \circ_G \sigma_t = \sigma_{f(x_2,x_1)}$, *then either* $(\sigma_s = \sigma_{id}, \sigma_t = \sigma_{f(x_2,x_1)})$ *or* $(\sigma_s = \sigma_{f(x_2,x_1)}, \sigma_t = \sigma_{id})$.

Proof. (i) Assume that $\sigma_s \circ_G \sigma_t = \sigma_{id}$. Since $f(x_1, x_2) \notin X$, thus by Lemma 4.1.4 (i),(ii) we get $s, t \notin X$ and thus $s = f(a, b), t = f(c, d)$ for some $a, b, c, d \in W_{(2)}(X)$. From $\sigma_s \circ_G \sigma_t = \sigma_{id}$, thus $S^2(f(a, b), \hat{\sigma}_{f(a,b)}[c], \hat{\sigma}_{f(a,b)}[d]) = f(x_1, x_2)$. So $(a = c = x_1$ or $a = x_2, d = x_1)$ and $(b = d = x_2$ or $b = x_1, c = x_2)$. This implies $\sigma_s = \sigma_t = \sigma_{id}$ or $\sigma_s = \sigma_t = \sigma_{f(x_2,x_1)}$.

(ii) The proof of (ii) is similar to the proof of (i). ■

Proposition 4.4.4. *All of* $\mathcal{R}$-, $\mathcal{L}$- *and* $\mathcal{D}$-*class of* σ_{id} *are equal to* $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

Proof. By Proposition 4.4.1, we get σ_{id} and $\sigma_{f(x_2,x_1)}$ are $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{D}$-related. This implies the $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{D}$-class of σ_{id} contain at least $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$. Let $\sigma_t \in Hyp_G(2)$ where $\sigma_t \mathcal{D} \sigma_{id}$. So $\sigma_t \mathcal{L} \sigma_s$ and $\sigma_s \mathcal{R} \sigma_{id}$ for some $\sigma_s \in Hyp_G(2)$. Then there exist $\sigma_u, \sigma_v, \sigma_p, \sigma_q \in Hyp_G(2)$ such that $\sigma_t = \sigma_p \circ_G \sigma_s$, $\sigma_s = \sigma_q \circ_G \sigma_t$, $\sigma_s = \sigma_{id} \circ_G \sigma_u$ and $\sigma_{id} = \sigma_s \circ_G \sigma_v$. From $\sigma_{id} = \sigma_s \circ_G \sigma_v$, thus by Lemma 4.4.3 (i) we get $\sigma_s = \sigma_{id}$ or $\sigma_s = \sigma_{f(x_2,x_1)}$. From $\sigma_s = \sigma_{id}$ or $\sigma_s = \sigma_{f(x_2,x_1)}$ and $\sigma_s = \sigma_q \circ_G \sigma_t$, thus by Lemma 4.4.3 we get $\sigma_t = \sigma_{id}$ or $\sigma_t = \sigma_{f(x_2,x_1)}$. So the $\mathcal{D}$-class of σ_{id} is equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$. From $\mathcal{R} \subseteq \mathcal{D}, \mathcal{L} \subseteq \mathcal{D}$, thus the $\mathcal{R}$- and the $\mathcal{L}$-class of σ_{id} are equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$. ■

Proposition 4.4.5. $(\sigma_{id})_i = Hyp_G(2) = (\sigma_{f(x_2,x_1)})_i$, *and if* $\sigma \in Hyp_G(2)$ *and* $(\sigma)_i = Hyp_G(2)$, *then* σ *is one of* σ_{id} *or* $\sigma_{f(x_2,x_1)}$. *Moreover, the* $\mathcal{J}$-*class of* σ_{id} *is equal to its* $\mathcal{D}$-*class,* $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

Proof. Since σ_{id} is the identity element, thus $(\sigma_{id})_i = Hyp_G(2)$. Let $\sigma \in Hyp_G(2)$. Then $\sigma \circ_G \sigma_{f(x_2,x_1)} \circ_G \sigma_{f(x_2,x_1)} = \sigma$. So $(\sigma_{id})_i = Hyp_G(2) = (\sigma_{f(x_2,x_1)})_i$. This implies $\sigma_{id} \mathcal{J} \sigma_{f(x_2,x_1)}$. Assume that $(\sigma)_i = Hyp_G(2)$. Then $\sigma \mathcal{J} \sigma_{id}$ and thus there exist $\delta, \rho \in Hyp_G(2)$ such that $\delta \circ_G \sigma \circ_G \rho = \sigma_{id}$. By Lemma 4.4.3 (i), we get $\sigma \circ_G \rho = \sigma_{id}$ or $\sigma \circ_G \rho = \sigma_{f(x_2,x_1)}$. Again by Lemma 4.4.3, we get $\sigma = \sigma_{id}$ or $\sigma = \sigma_{f(x_2,x_1)}$. ■

Lemma 4.4.6. *Let* $u \in W_{(2)}(X)$, $\sigma_t \in Hyp_G(2)$ *and* $x = x_1$ *or* $x = x_2$. *If* $x \notin var(u)$, *then* $x \notin var(\hat{\sigma}_t[u])$ (x *is not a variable occurring in the term* $(\sigma_t \circ_G \sigma_u)(f)$).

Proof. We will prove by induction on the complexity of the term u. If $u \in X$, then $\hat{\sigma}_t[u] = u$ and so $x \notin var(\hat{\sigma}_t[u])$. Assume that $u = f(u_1, u_2)$ and $x \notin var(\hat{\sigma}_t[u_1])$,

$x \notin var(\hat{\sigma}_t[u_2])$. Since $x \notin var(\hat{\sigma}_t[u_1])$, $x \notin var(\hat{\sigma}_t[u_2])$ and $\hat{\sigma}_t[u] = \hat{\sigma}_t[f(u_1, u_2)] = S^2(t, \hat{\sigma}_t[u_1], \hat{\sigma}_t[u_2])$, thus $x \notin var(\hat{\sigma}_t[u])$. ■

Proposition 4.4.7. *Any $\sigma_t \in G$ is $\mathcal{R}$-related only to itself, but is $\mathcal{L}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of G, and not related to any other generalized hypersubstitutions. Moreover, the set G forms an $\mathcal{L}$-, $\mathcal{D}$- and $\mathcal{J}$- class.*

Proof. Let $\sigma_t \in G$. Assume that $\sigma_s \in Hyp_G(2)$ where $\sigma_s \mathcal{R} \sigma_t$. By Proposition 4.4.2,we get $s \notin X$. Then there exists $\sigma_p \in Hyp_G(2)$ such that $\sigma_s = \sigma_t \circ_G \sigma_p$. Since $s \notin X$, thus by Lemma 4.1.4 (i) we get $p \notin X$. Since $\sigma_t \in G$ and $p \notin X$, thus by Lemma 4.1.4 (iii) we get $\sigma_t \circ_G \sigma_p = \sigma_t$. So $\sigma_s = \sigma_t$. Thus σ_t is $\mathcal{R}$-related only to itself. Let $\sigma_s, \sigma_t \in G$. By Lemma 4.1.4 (iii), we get $\sigma_s \circ_G \sigma_t = \sigma_s$ and $\sigma_t \circ_G \sigma_s = \sigma_t$. Thus $\sigma_s \mathcal{L} \sigma_t$. So any two elements in G are $\mathcal{L}$-related. Since $\mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$, thus any two elements in G are $\mathcal{D}-$ and $\mathcal{J}-$ related. Assume that $\sigma_t \in G$ and $\sigma_s \in Hyp_G(2)$ where $\sigma_s \mathcal{J} \sigma_t$. By Proposition 4.4.2, we get $s \notin X$. Then there exist $\sigma_p, \sigma_q \in Hyp_G(2)$ such that $\sigma_p \circ_G \sigma_t \circ_G \sigma_q = \sigma_s$. Since $s \notin X$, thus by Lemma 4.1.4 (i),(ii) we get $p, q \notin X$. Since $\sigma_t \in G$ and $q \notin X$, thus by Lemma 4.1.4 (iii) we get $\sigma_t \circ_G \sigma_q = \sigma_t$. Since $x_1, x_2 \notin var(t)$, thus by Lemma 4.4.6 we get x_1, x_2 are not variables occurring in the term $(\sigma_p \circ_G \sigma_t)(f) = (\sigma_p \circ_G \sigma_t \circ_G \sigma_q)(f)$. Thus $x_1, x_2 \notin var(s)$ and so $\sigma_s \in G$. So G is the $\mathcal{J}$-class of its elements. Since any two elements in G are $\mathcal{L}-$ and $\mathcal{D}-$ related, $\mathcal{L} \subseteq \mathcal{J}, \mathcal{D} \subseteq \mathcal{J}$ and G is the $\mathcal{J}$-class of its elements, thus G forms an $\mathcal{L}$-, $\mathcal{D}$-class. ■

Theorem 4.4.8. *Let $\tau = (n_i)_{i \in I}$ be a type and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then $\sigma_1 \mathcal{R} \sigma_2$ if and only if $Im\hat{\sigma}_1 = Im\hat{\sigma}_2$.*

Proof. Assume that $\sigma_1 \mathcal{R} \sigma_2$. Then $\sigma_1 = \sigma_2 \circ_G \sigma_3$ and $\sigma_2 = \sigma_1 \circ_G \sigma_4$ for some $\sigma_3, \sigma_4 \in Hyp_G(\tau)$. By Proposition 2.2.10 (ii), we get $\hat{\sigma}_1 = (\sigma_2 \circ_G \sigma_3)\hat{} = (\hat{\sigma}_2 \circ \sigma_3)\hat{} = \hat{\sigma}_2 \circ \hat{\sigma}_3$ and $\hat{\sigma}_2 = (\sigma_1 \circ_G \sigma_4)\hat{} = (\hat{\sigma}_1 \circ \sigma_4)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_4$. Thus $Im\hat{\sigma}_1 = \hat{\sigma}_1[W_\tau(X)] = (\hat{\sigma}_2 \circ \hat{\sigma}_3)[W_\tau(X)] = \hat{\sigma}_2[\hat{\sigma}_3[W_\tau(X)]] \subseteq \hat{\sigma}_2[W_\tau(X)] = Im\hat{\sigma}_2$. By the same way we can show that $Im\hat{\sigma}_2 \subseteq Im\hat{\sigma}_1$. Conversely, assume that $Im\hat{\sigma_1} = Im\hat{\sigma_2}$. For each $i \in I$, we have $\sigma_1(f_i) = S^{n_i}(\sigma_1(f_i), x_1, \ldots, x_{n_i}) = \hat{\sigma}_1[f_i(x_1, \ldots, x_{n_i})] \in Im\hat{\sigma}_1 = Im\hat{\sigma}_2$. So $\sigma_1(f_i) = \hat{\sigma}_2[t_i]$ for some $t_i \in W_\tau(X)$. We define $\gamma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$ by $\gamma(f_i) = t_i$ for all $i \in I$. Let $i \in I$. Then $(\sigma_2 \circ_G \gamma)(f_i) = \hat{\sigma}_2[\gamma(f_i)] = \hat{\sigma}_2[t_i] = \sigma_1(f_i)$. So $\sigma_1 = \sigma_2 \circ_G \gamma$. By the same way we can show that $\sigma_2 = \sigma_1 \circ_G \beta$ for some $\beta \in W_\tau(X)$. ■

Theorem 4.4.9. *For any $\sigma_s, \sigma_t \in Hyp_G(2)$, $\sigma_s \mathcal{R} \sigma_t$ if and only if the following conditions hold:*

(i) *If $s \in X$, then $t \in X$.*

(ii) *If $s \notin X$, then $s = t$ or $s = \bar{t}$.*

Proof. Assume that $\sigma_s \mathcal{R} \sigma_t$. If $s \in X$, then by Proposition 4.4.2 we get $t \in X$. Let $s \notin X$. Then there exist $\sigma_u, \sigma_v \in Hyp_G(2)$ such that $\sigma_s = \sigma_t \circ_G \sigma_u$ and $\sigma_t = \sigma_s \circ_G \sigma_v$. By Lemma 4.1.4 (i), (ii), we get $t, u, v \notin X$. Then $u = f(u_1, u_2)$ and $v = f(v_1, v_2)$ for some $u_1, u_2, v_1, v_2 \in W_{(2)}(X)$. Then we have two equations

$$s = S^2(t, \hat{\sigma}_t[u_1], \hat{\sigma}_t[u_2]) \qquad (1)$$

$$t = S^2(s, \hat{\sigma}_s[v_1], \hat{\sigma}_s[v_2]) \qquad (2).$$

From (1) and (2), we get $vb(s) = vb(t)$. We consider into four cases:

Case 1: $t \in W^G$. From (1), we get $s = t$.

Case 2: $t \in W^G_{(2)}(\{x_1, x_2\})$. Suppose that $u_1 \notin X$ or $u_2 \notin X$. Then $\hat{\sigma}_t[u_1] \notin X$ or $\hat{\sigma}_t[u_2] \notin X$. From (1) and $x_1, x_2 \in var(t)$, thus $vb(s) > vb(t)$ and it is a contradiction. So $u_1, u_2 \in X$. Suppose that $u_1 = u_2 = x_1$. Then $\hat{\sigma}_t[u_1] = \hat{\sigma}_t[u_2] = x_1$. From (1), we get $s \in W(\{x_1\})$. Suppose that $v_1 \notin X$. Then $\hat{\sigma}_s[v_1] \notin X$. From (2) and $x_1 \in var(s)$, thus $vb(t) > vb(s)$ and it is a contradiction. So $v_1 \in X$ and thus $\hat{\sigma}_s[v_1] = v_1$. Since $s \in W(\{x_1\})$ and $\hat{\sigma}_s[v_1] = v_1$, thus from (2) we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$ which contradicts to $t \in W^G_{(2)}(\{x_1, x_2\})$. If $u_1 = x_1, u_2 = x_2$, then $\hat{\sigma}_t[u_1] = x_1, \hat{\sigma}_t[u_2] = x_2$. From (1), we get $s = t$. If $u_1 = x_1, u_2 = x_i$ where $i > 2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. If $u_1 = x_2, u_2 = x_1$, then $\hat{\sigma}_t[u_1] = x_2, \hat{\sigma}_t[u_2] = x_1$. From (1), we get $s = \bar{t}$. If $u_1 = x_2, u_2 = x_2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. If $u_1 = x_2, u_2 = x_i$ where $i > 2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. If $u_1 = x_i, u_2 = x_1$ where $i > 2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. If $u_1 = x_i, u_2 = x_2$ where $i > 2$, then by the same proof as the case $u_1 = u_2 = x_1$ we get $x_1 \notin var(t)$ or $x_2 \notin var(t)$. Suppose that $u_1 = x_i, u_2 = x_j$ where $i, j > 2$. Then $\hat{\sigma}_t[u_1] = x_i, \hat{\sigma}_t[u_2] = x_j$. From (1), we get $s \in W^G$. Since $x_1, x_2 \notin var(s)$, thus from (2) we get $s = t$. So $x_1, x_2 \notin var(t)$ and it is a contradiction.

Case 3: $t \in W(\{x_1\})$. Suppose that $u_1 \notin X$. Then $\hat{\sigma}_t[u_1] \notin X$. From (1),

$x_1 \in var(t)$ and $\hat{\sigma}_t[u_1] \notin X$, thus $vb(s) > vb(t)$ and it is a contradiction. So $u_1 \in X$ and thus $\hat{\sigma}_s[u_1] = u_1$. If $u_1 = x_1$, then by (1) we get $s = t$. If $u_1 = x_2$, then by (1) we get $s = \bar{t}$. Suppose that $u_1 = x_i$ where $i > 2$. From (1), we get $s \in W^G$. Since $x_1, x_2 \notin var(s)$, thus from (2) we get $s = t$. So $x_1 \notin var(t)$ and it is a contradiction.

Case 4: $t \in W(\{x_2\})$. By the same proof as the case $t \in W(\{x_1\})$ we get $s = t$ or $s = \bar{t}$.

Conversely, assume that the conditions hold. By Proposition 4.4.1 and Proposition 4.4.2, we get $\sigma_s \mathcal{R} \sigma_t$. ■

Lemma 4.4.10. *$E^G_{x_1}$ is a left zero band.*

Proof. Let $\sigma_{f(x_1,s)}, \sigma_{f(x_1,t)} \in E^G_{x_1}$. Since $x_2 \notin var(s)$, thus $(\sigma_{f(x_1,s)} \circ_G \sigma_{f(x_1,t)})(f) = S^2(f(x_1,s), x_1, \hat{\sigma}_{f(x_1,s)}[t]) = f(x_1,s)$. So $\sigma_{f(x_1,s)} \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,s)}$. So $E^G_{x_1}$ is a left zero band. ■

Proposition 4.4.11. *The $\mathcal{L}$-class of the element $\sigma_{f(x_1,x_1)}$ is precisely the set $E^G_{x_1} \cup \overline{E^G_{x_2}}$.*

Proof. For any two idempotent elements e and f in a semigroup S, $e\mathcal{L}f$ if and only if $ef = e$ and $fe = f$. Since $E^G_{x_1}$ is a left zero band, it follows that $\sigma_{f(x_1,x_1)}$ is $\mathcal{L}$-related to any element of $E^G_{x_1}$. By Proposition 4.4.1, we get $\sigma_{f(x_1,x_1)}$ is $\mathcal{L}$-related to any element of $(E^G_{x_1})' = \overline{E^G_{x_2}}$. Thus the $\mathcal{L}$-class of $\sigma_{f(x_1,x_1)}$ contains at least $E^G_{x_1} \cup \overline{E^G_{x_2}}$. For the opposite inclusion, assume that $\sigma_t \in Hyp_G(2)$ where $\sigma_t \mathcal{L} \sigma_{f(x_1,x_1)}$. By Proposition 4.4.2, we get $t \notin X$. Then $t = f(u,v)$ for some $u, v \in W_{(2)}(X)$. From $\sigma_t \mathcal{L} \sigma_{f(x_1,x_1)}$, then there exist $\sigma_p, \sigma_q \in Hyp_G(2)$ such that $\sigma_p \circ_G \sigma_{f(x_1,x_1)} = \sigma_t$ and $\sigma_q \circ_G \sigma_t = \sigma_{f(x_1,x_1)}$. Since $t, f(x_1,x_1) \notin X$, thus by Lemma 4.1.4 (ii) we get $p, q \notin X$. Then there exist $a, b, c, d \in W_{(2)}(X)$ such that $p = f(a,b)$ and $q = f(c,d)$. Thus we have $\sigma_{f(a,b)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{f(u,v)}$ and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(x_1,x_1)}$. From $\sigma_{f(a,b)} \circ_G \sigma_{f(x_1,x_1)} = \sigma_{f(u,v)}$, thus by Lemma 4.4.6 we get $x_2 \notin var(f(u,v))$. From $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(x_1,x_1)}$, thus $S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v]) = f(x_1,x_1)$. Suppose that $u, v \neq x_1$. Thus $\hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v] \neq x_1$. This implies $S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v]) \neq f(x_1,x_1)$, which is a contradiction. So $u = x_1$ or $v = x_1$. Since $x_2 \notin var(f(u,v))$ and $u = x_1$ or $v = x_1$, thus $\sigma_t = \sigma_{f(u,v)} \in E^G_{x_1} \cup \overline{E^G_{x_2}}$. ■

Corollary 4.4.12. *The $\mathcal{D}$-class of the element $\sigma_{f(x_1,x_1)}$ is precisely the set $E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$.*

Proof. Assume that $\sigma_t \in Hyp_G(2)$ where $\sigma_t \mathcal{D} \sigma_{f(x_1,x_1)}$. Then there exists $\sigma_s \in Hyp_G(2)$ such that $\sigma_t \mathcal{R} \sigma_s$ and $\sigma_s \mathcal{L} \sigma_{f(x_1,x_1)}$. Since $\sigma_t \mathcal{R} \sigma_s$, thus by Theorem 4.4.9 we get $\sigma_t = \sigma_s$ or $\sigma_t = \sigma_{\overline{s}}$. Since $\sigma_s \mathcal{L} \sigma_{f(x_1,x_1)}$, thus by Proposition 4.4.11 we get $\sigma_s \in E^G_{x_1} \cup \overline{E^G_{x_2}}$. If $\sigma_s \in E^G_{x_1}$, then $\sigma_t \in E^G_{x_1} \cup \overline{E^G_{x_1}} \subseteq E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$. If $\sigma_s \in \overline{E^G_{x_2}}$, then $\sigma_t \in E^G_{x_2} \cup \overline{E^G_{x_2}} \subseteq E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$. For the opposite inclusion, assume that $\sigma_t \in E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$. If $\sigma_t \in E^G_{x_1} \cup \overline{E^G_{x_2}}$, then by Proposition 4.4.11 we get $\sigma_t \mathcal{L} \sigma_{f(x_1,x_1)}$. Since $\mathcal{L} \subseteq \mathcal{D}$, thus $\sigma_t \mathcal{D} \sigma_{f(x_1,x_1)}$. If $\sigma_t \in E^G_{x_2} \cup \overline{E^G_{x_1}}$, then $\sigma_{\overline{t}} \in E^G_{x_1} \cup \overline{E^G_{x_2}}$. By Proposition 4.4.11, we get $\sigma_{\overline{t}} \mathcal{L} \sigma_{f(x_1,x_1)}$. By Theorem 4.4.9, we get $\sigma_t \mathcal{R} \sigma_{\overline{t}}$. So $\sigma_t \mathcal{D} \sigma_{f(x_1,x_1)}$. ∎

Lemma 4.4.13. *Let $\sigma_{f(c,d)} \in Hyp_G(2) \setminus \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$ and $u \in W_{(2)}(X) \setminus X$. If $\sigma_{f(c,d)} \in E^G(\{x_1, x_2\})$, then the term w corresponding to the composition $\sigma_{f(c,d)} \circ_G \sigma_u$ is longer than u.*

Proof. We will prove by induction on the complexity of the term u. Since $x_1, x_2 \in var(f(c,d))$ and $f(c,d) \neq f(x_1,x_2), f(x_2,x_1)$, thus $c \notin X$ or $d \notin X$ and $vb(f(c,d)) \geq 3$. Let $vb(u) = 2$. Then $u = f(x_i,x_j)$ for some $x_i, x_j \in X$. So $vb(w) = vb((\sigma_{f(c,d)} \circ_G \sigma_u)(f)) = vb((\sigma_{f(c,d)} \circ_G \sigma_{f(x_i,x_j)})(f)) = vb(S^2(f(c,d), x_i, x_j)) \geq 3 > vb(u)$. Let $u = f(s,t)$ where $s \in X$ and $t \notin X$. Then $\hat{\sigma}_{f(c,d)}[s] = s \in X$. Assume that $vb(\hat{\sigma}_{f(c,d)}[t]) > vb(t)$. Since $x_1, x_2 \in var(f(c,d))$ and $vb(\hat{\sigma}_{f(c,d)}[t]) > vb(t)$, thus $vb(w) = vb((\sigma_{f(c,d)} \circ_G \sigma_u)(f)) = vb((\sigma_{f(c,d)} \circ_G \sigma_{f(s,t)})(f)) = vb(S^2(f(c,d), s, \hat{\sigma}_{f(c,d)}[t])) > vb(f(s,t)) = vb(u)$. Let $u = f(s,t)$ where $s, t \notin X$. Assume that $vb(\hat{\sigma}_{f(c,d)}[s]) > vb(s)$ and $vb(\hat{\sigma}_{f(c,d)}[t]) > vb(t)$. Since $x_1, x_2 \in var(f(c,d))$ and $vb(\hat{\sigma}_{f(c,d)}[s]) > vb(s)$, $vb(\hat{\sigma}_{f(c,d)}[t]) > vb(t)$, thus $vb(w) = vb((\sigma_{f(c,d)} \circ_G \sigma_u)(f)) = vb((\sigma_{f(c,d)} \circ_G \sigma_{f(s,t)})(f)) = vb(S^2(f(c,d), \hat{\sigma}_{f(c,d)}[s], \hat{\sigma}_{f(c,d)}[t])) > vb(f(s,t)) = vb(u)$. ∎

Lemma 4.4.14. *If $f(c,d) \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$ ($x_1 \notin var(f(c,d))$ or $x_2 \notin var(f(c,d))$), then for any $u, v \in W_{(2)}(X)$ the term w corresponding to $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)}$ is in $W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$.*

Proof. We will prove by induction on the complexity of the term u. Assume that $f(c,d) \in W(\{x_1\})$. We have to consider the letters used in the term $w = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v])$. If $u \in X$, then $\hat{\sigma}_{f(c,d)}[u] = u \in X$. Since $f(c,d) \in W(\{x_1\})$, $\hat{\sigma}_{f(c,d)}[u] \in X$ and $w = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v])$, thus $w \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. Let $u = f(p,q)$ and $\hat{\sigma}_{f(c,d)}[p] \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. So $\hat{\sigma}_{f(c,d)}[u] =$

$S^2(f(c,d), \hat{\sigma}_{f(c,d)}[p], \hat{\sigma}_{f(c,d)}[q]) \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. Since $f(c,d) \in W(\{x_1\})$, $\hat{\sigma}_{f(c,d)}[u] \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$ and $w = S^2(f(c,d), \hat{\sigma}_{f(c,d)}[u], \hat{\sigma}_{f(c,d)}[v])$, thus $w \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. By the same way we can show that if $f(c,d) \in W(\{x_2\})$, then $w \in W(\{x_1\}) \cup W(\{x_2\}) \cup W^G$. If $f(c,d) \in W^G$, then $w = f(c,d) \in W^G$. ■

Proposition 4.4.15. *The following statements hold:*

(i) $(\sigma_{f(x_1,x_1)})_i = I := \{\sigma_t \in Hyp_G(2) | t \in W^G_{(2)}(\{x_1\}) \cup W^G_{(2)}(\{x_2\})$ *or* $x_1, x_2 \notin var(t)\}$.

(ii) *If* $\sigma \in I$ *where* $\sigma \notin E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$, *then* $(\sigma)_i \subsetneq I$.

(iii) *The* $\mathcal{J}$*-class of* $\sigma_{f(x_1,x_1)}$ *is equal to its* $\mathcal{D}$*-class,* $E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$.

Proof. (i) Assume that $\sigma_s \in (\sigma_{f(x_1,x_1)})_i$. Then there exist $\delta, \rho \in Hyp_G(2)$ such that $\delta \circ_G \sigma_{f(x_1,x_1)} \circ_G \rho = \sigma_s$. If δ or $\rho \in P_G(2)$, then by Lemma 4.1.4 (i), (ii) we get $\sigma_s = \delta \circ_G \sigma_{f(x_1,x_1)} \circ_G \rho \in P_G(2) \subseteq I$. Assume that $\delta, \rho \notin P_G(2)$. By Lemma 4.4.14, we get $\sigma_{f(x_1,x_1)} \circ_G \rho \in I$. By Lemma 4.4.6, we get $\sigma_s = \delta \circ_G (\sigma_{f(x_1,x_1)} \circ_G \rho) \in I$. For the opposite inclusion, suppose that $\sigma_s \in I$. If $\sigma_s \in P_G(2)$, then by Lemma 4.1.4 (i) we get $\sigma_s = \sigma_{f(x_1,x_1)} \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_s \in (\sigma_{f(x_1,x_1)})_i$. Let $\sigma_s \notin P_G(2)$. If $x_1, x_2 \notin var(s)$, then by Lemma 4.1.4 (iii) we get $\sigma_s = \sigma_s \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_s \in (\sigma_{f(x_1,x_1)})_i$. If $s \in W(\{x_1\})$, then $\sigma_s = \sigma_s \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_{f(x_1,x_1)} \in (\sigma_{f(x_1,x_1)})_i$. If $s \in W(\{x_2\})$, then $\sigma_s = \sigma_s \circ_G \sigma_{f(x_1,x_1)} \circ_G \sigma_{f(x_2,x_2)} \in (\sigma_{f(x_1,x_1)})_i$.

(ii) Assume that $\sigma \in I$ where $\sigma \notin E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$. If $\sigma \in P_G(2)$, then $(\sigma)_i = Hyp_G(2)\sigma Hyp_G(2) = P_G(2) \subsetneq I$. Assume that $\sigma \notin P_G(2)$ and $\sigma = \sigma_{f(u,v)}$. Let $f(u,v) \in W(\{x_1\}) \cup W(\{x_2\})$. Suppose that $u, v \in X$. Since $f(u,v) \in W(\{x_1\}) \cup W(\{x_2\})$, thus $\sigma_{f(u,v)} \in E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$ and it is a contradiction. Suppose that $u \in X$ and $v \notin X$. If $u = x_1$ or $u = x_2$, then $\sigma_{f(u,v)} \in E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$ and it is a contradiction. So $u = x_i$ for some $i > 2$. Suppose that $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$. Since $f(x_1,x_1) \notin X$ and $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$, thus there exist $p, q, r, s \in W_{(2)}(X)$ such that $\sigma_{f(p,q)} \circ_G \sigma_{f(x_i,v)} \circ_G \sigma_{f(r,s)} = \sigma_{f(x_1,x_1)}$. Let w be the term $(\sigma_{f(x_i,v)} \circ_G \sigma_{f(r,s)})(f)$. So $w = f(x_i, k)$ for some $k \in W_{(2)}(X) \setminus X$. Then we have $\sigma_{f(p,q)} \circ_G \sigma_{f(x_i,k)} = \sigma_{f(x_1,x_1)}$. This implies $f(p,q) = f(x_2,x_2)$. Consider $(\sigma_{f(x_2,x_2)} \circ_G \sigma_{f(x_i,k)})(f) = S^2(f(x_2,x_2), x_i, \hat{\sigma}_{f(x_2,x_2)}[k]) = f(\hat{\sigma}_{f(x_2,x_2)}[k], \hat{\sigma}_{f(x_2,x_2)}[k]) \neq f(x_1,x_1)$, which is a contradiction. So $(\sigma)_i \subsetneq I$. By the same way we can show that if $u \notin X$ and $v \in X$, then $(\sigma)_i \subsetneq I$. Suppose that $u, v \notin X$. Then $vb(f(u,v)) \geq 4$. Suppose that $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$. Since $f(x_1,x_1) \notin X$ and

$\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$, thus there exist $p, q, r, s \in W_{(2)}(X)$ such that $\sigma_{f(p,q)} \circ_G \sigma_{f(u,v)} \circ_G \sigma_{f(r,s)} = \sigma_{f(x_1,x_1)}$. Let w be the term $(\sigma_{f(u,v)} \circ_G \sigma_{f(r,s)})(f)$. Then $vb(w) \geq 4$. By Lemma 4.1.4 (iii), we get $x_1 \in var(f(p,q))$ or $x_2 \in var(f(p,q))$. Suppose that $f(p,q) \in W_{(2)}^G(\{x_1, x_2\})$. If $f(p,q) = f(x_1, x_2)$ or $f(p,q) = f(x_2, x_1)$, then $\sigma_w = \sigma_{f(x_1,x_1)}$ or $\sigma_{w'} = \sigma_{f(x_1,x_1)}$ and it is a contradiction. Suppose that $f(p,q) \neq f(x_1,x_2), f(x_2,x_1)$. By Lemma 4.4.13, we get $vb(f(x_1,x_1)) > vb(w)$, which is a contradiction. Suppose that $f(p,q) \in W(\{x_1\}) \cup W(\{x_2\})$. Then the equation $\sigma_{f(p,q)} \circ_G \sigma_w = \sigma_{f(x_1,x_1)}$ does not fit any of E(1) to E(16), so by Lemma 4.2.1 we must have $f(x_1,x_1)$ is longer than $f(p,q)$ and it is a contradiction. So $(\sigma)_i \subsetneq I$. Let $f(u,v) \in W^G$. Suppose that $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$. Since $f(x_1,x_1) \notin X$ and $\sigma_{f(x_1,x_1)} \in (\sigma_{f(u,v)})_i$, thus there exist $p, q, r, s \in W_{(2)}(X)$ such that $\sigma_{f(p,q)} \circ_G \sigma_{f(u,v)} \circ_G \sigma_{f(r,s)} = \sigma_{f(x_1,x_1)}$. By Lemma 4.1.4 (iii), we get $\sigma_{f(u,v)} \circ_G \sigma_{f(r,s)} = \sigma_{f(u,v)}$. By Lemma 4.4.6, we get x_1, x_2 are not variables occurring in the term $(\sigma_{f(p,q)} \circ_G \sigma_{f(u,v)})(f) = (\sigma_{f(p,q)} \circ_G \sigma_{f(u,v)} \circ_G \sigma_{f(r,s)})(f)$, which is a contradiction. So $(\sigma)_i \subsetneq I$.

(iii) Since $\mathcal{D} \subseteq \mathcal{J}$, thus we must have $E_{x_1}^G \cup E_{x_2}^G \cup \overline{E_{x_1}^G} \cup \overline{E_{x_2}^G}$ contained in the $\mathcal{J}$-class of $\sigma_{f(x_1,x_1)}$. Assume that $\sigma \in Hyp_G(2)$ where $\sigma \mathcal{J} \sigma_{f(x_1,x_1)}$. Then $(\sigma)_i = (\sigma_{f(x_1,x_1)})_i = I$. So $\sigma \in I$. By (ii), we get $\sigma \in E_{x_1}^G \cup E_{x_2}^G \cup \overline{E_{x_1}^G} \cup \overline{E_{x_2}^G}$. ■

Proposition 4.4.16. *For any $\sigma_t \in E^G(\{x_1, x_2\})$, the elements which are $\mathcal{L}$-related to σ_t are only σ_t itself and $\sigma_{t'}$.*

Proof. Let $t = f(u,v)$. Assume that $\sigma_s \in Hyp_G(2)$ where $\sigma_s \mathcal{L} \sigma_t$. By Proposition 4.4.2, we get $s \notin X$. Then $s = f(a,b)$ for some $a, b \in W_{(2)}(X)$. Since $s, t \notin X$ and $\sigma_s \mathcal{L} \sigma_t$, thus there exist $c, d, e, g \in W_{(2)}(X)$ such that $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(a,b)}$ and $\sigma_{f(e,g)} \circ_G \sigma_{f(a,b)} = \sigma_{f(u,v)}$. Since $x_1, x_2 \in var(f(u,v))$, then by Lemma 4.4.14 and $\sigma_{f(e,g)} \circ_G \sigma_{f(a,b)} = \sigma_{f(u,v)}$ we get $x_1, x_2 \in var(f(e,g))$. Since $x_1, x_2 \in var(f(u,v))$, then by Lemma 4.4.6 and $\sigma_{f(e,g)} \circ_G \sigma_{f(a,b)} = \sigma_{f(u,v)}$ we get $x_1, x_2 \in var(f(a,b))$. Since $x_1, x_2 \in var(f(a,b))$, thus by Lemma 4.4.14 and $\sigma_{f(c,d)} \circ_G \sigma_{f(u,v)} = \sigma_{f(a,b)}$ we get $x_1, x_2 \in var(f(c,d))$. Suppose that $f(c,d), f(e,g) \notin \{f(x_1,x_2), f(x_2,x_1)\}$. Since $x_1, x_2 \in var(f(e,g))$ and $x_1, x_2 \in var(f(c,d))$, thus by Proposition 4.4.13 we get $vb(f(a,b)) > vb(f(u,v))$ and $vb(f(u,v)) > vb(f(a,b))$, which is a contradiction. So $f(c,d) \in \{f(x_1,x_2), f(x_2,x_1)\}$ or $f(e,g) \in \{f(x_1,x_2), f(x_2,x_1)\}$. This implies $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{t'}$. ■

Corollary 4.4.17. *For* $\sigma_t \in E^G(\{x_1, x_2\})$, $D_{\sigma_t} = \{\sigma_t, \sigma_{t'}, \sigma_{\bar{t}}, \sigma_{\overline{t'}}\}$.

Proof. By Theorem 4.4.9 and Proposition 4.4.16. ∎

Proposition 4.4.18. *For* $\sigma_t \in E^G(\{x_1, x_2\})$, *the* $\mathcal{J}$*-class of* σ_t *is equal to its* $\mathcal{D}$*-class,* $\{\sigma_t, \sigma_{t'}, \sigma_{\bar{t}}, \sigma_{\overline{t'}}\}$.

Proof. If $\sigma_t = \sigma_{id}$ or $\sigma_t = \sigma_{f(x_2,x_1)}$, then by Proposition 4.4.5 we get $D_{\sigma_{id}} = J_{\sigma_{id}}$. Let $\sigma_t \neq \sigma_{id}, \sigma_{f(x_2,x_1)}$ and $\sigma_s \in Hyp_G(2)$ where $\sigma_s \mathcal{J} \sigma_t$. By Proposition 4.4.2, we get $s \notin X$. Then there exist $\sigma_u, \sigma_v, \sigma_p, \sigma_q \in Hyp_G(2)$ such that $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$. This implies $\sigma_p \circ_G \sigma_u \circ_G \sigma_t \circ_G \sigma_v \circ_G \sigma_q = \sigma_t$. Since $t \notin X$, thus by Lemma 4.1.4 (i),(ii) we get $u, v, p, q \notin X$. Since $t \in W^G_{(2)}(\{x_1, x_2\})$, thus by Lemma 4.4.6 and Lemma 4.4.14 we get $u, v, p, q \in W^G_{(2)}(\{x_1, x_2\})$ and terms corresponding to the intermediate products are in $W^G_{(2)}(\{x_1, x_2\})$. We consider into three cases.

Case 1: $\sigma_p \circ_G \sigma_u = \sigma_{id}$. Then by Lemma 4.4.3, we get $\sigma_p = \sigma_u = \sigma_{id}$ or $\sigma_p = \sigma_u = \sigma_{f(x_2,x_1)}$. If $\sigma_p = \sigma_u = \sigma_{id}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_s \circ_G \sigma_q = \sigma_t$. So $\sigma_s \mathcal{R} \sigma_t$. By Theorem 4.4.9, we get $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{\bar{t}}$. If $\sigma_p = \sigma_u = \sigma_{f(x_2,x_1)}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_{t'} \circ_G \sigma_v = \sigma_s$ and $\sigma_s \circ_G \sigma_q = \sigma_{t'}$. So $\sigma_s \mathcal{R} \sigma_{t'}$. By Theorem 4.4.9, we get $\sigma_s = \sigma_{t'}$ or $\sigma_s = \sigma_{\overline{t'}}$.

Case 2: $\sigma_p \circ_G \sigma_u = \sigma_{f(x_2,x_1)}$. Then by Lemma 4.4.3, we get $\sigma_p = \sigma_{id}, \sigma_u = \sigma_{f(x_2,x_1)}$ or $\sigma_p = \sigma_{f(x_2,x_1)}, \sigma_u = \sigma_{id}$. Then $\sigma_t = \sigma_p \circ_G \sigma_u \circ_G \sigma_t \circ_G \sigma_v \circ_G \sigma_q = \sigma_{f(x_2,x_1)} \circ_G \sigma_t \circ_G \sigma_v \circ_G \sigma_q = \sigma_{t'} \circ_G (\sigma_v \circ_G \sigma_q)$. By Lemma 4.2.1, we get t is longer than t', unless the product $\sigma_{t'} \circ_G (\sigma_v \circ_G \sigma_q)$ fits one of $E(1)$ to $E(16)$. But $vb(t) = vb(t')$, thus the product $\sigma_{t'} \circ_G (\sigma_v \circ_G \sigma_q)$ fits one of $E(1)$ to $E(16)$. We see that the cases $E(1) - E(3)$, $E(5)$, $E(7) - E(16)$ are impossible. Assume that $E(4)$ holds. We have $\sigma_v \circ_G \sigma_q = \sigma_{id}$. By Lemma 4.4.3, we get $\sigma_v = \sigma_q = \sigma_{id}$ or $\sigma_v = \sigma_q = \sigma_{f(x_2,x_1)}$. If $\sigma_v = \sigma_q = \sigma_{id}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_u \circ_G \sigma_t = \sigma_s$ and $\sigma_p \circ_G \sigma_s = \sigma_t$. So $\sigma_s \mathcal{L} \sigma_t$. By Proposition 4.4.16, we get $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{t'}$. If $\sigma_v = \sigma_q = \sigma_{f(x_2,x_1)}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_u \circ_G \sigma_t \circ_G \sigma_{f(x_2,x_1)} = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_{f(x_2,x_1)} = \sigma_t$. This implies $\sigma_u \circ_G \sigma_{\bar{t}} = \sigma_s$ and $\sigma_p \circ_G \sigma_s = \sigma_{\bar{t}}$. So $\sigma_s \mathcal{L} \sigma_{\bar{t}}$. By Proposition 4.4.16, we get $\sigma_s = \sigma_{\bar{t}}$ or $\sigma_s = \sigma_{\bar{t}'} = \sigma_{\overline{t'}}$. Assume that $E(6)$ holds. We have $\sigma_v \circ_G \sigma_q = \sigma_{f(x_2,x_1)}$. By Lemma 4.4.3, we get $\sigma_q = \sigma_{id}$ or $\sigma_q = \sigma_{f(x_2,x_1)}$. If $\sigma_p = \sigma_q = \sigma_{f(x_1,x_2)}$, then from $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_s = \sigma_t$. If $\sigma_p = \sigma_q = \sigma_{f(x_2,x_1)}$, then from $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_s = \sigma_{\overline{t'}}$. If $\sigma_p = \sigma_{id}, \sigma_q = \sigma_{f(x_2,x_1)}$, then from $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_s = \sigma_{\bar{t}}$. If $\sigma_p = \sigma_{f(x_2,x_1)}, \sigma_q = \sigma_{id}$, then from $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_s = \sigma_{t'}$.

Case 3: $\sigma_p \circ_G \sigma_u \neq \sigma_{id}, \sigma_{f(x_2,x_1)}$. By Lemma 4.4.13, we get t is longer than the term $w = (\sigma_t \circ_G \sigma_v \circ_G \sigma_q)(f)$. By Lemma 4.2.1 , we get w is longer than t, unless the product $\sigma_t \circ_G (\sigma_v \circ_G \sigma_q)$ fits one of $E(1)$ to $E(16)$. But the case w is longer than t is impossible. We see that the cases $E(1) - E(3)$, $E(5)$, $E(7) - E(16)$ are impossible. Assume that $E(4)$ holds. We must have $\sigma_v \circ_G \sigma_q = \sigma_{id}$. By Lemma 4.4.3, we get $\sigma_v = \sigma_q = \sigma_{id}$ or $\sigma_v = \sigma_q = \sigma_{f(x_2,x_1)}$. If $\sigma_v = \sigma_q = \sigma_{id}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_u \circ_G \sigma_t = \sigma_s$ and $\sigma_p \circ_G \sigma_s = \sigma_t$. So $\sigma_s \mathcal{L} \sigma_t$. By Proposition 4.4.16, we get $\sigma_s = \sigma_t$ or $\sigma_s = \sigma_{t'}$. If $\sigma_v = \sigma_q = \sigma_{f(x_2,x_1)}$, then from $\sigma_u \circ_G \sigma_t \circ_G \sigma_v = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_q = \sigma_t$ we get $\sigma_u \circ_G \sigma_t \circ_G \sigma_{f(x_2,x_1)} = \sigma_s$ and $\sigma_p \circ_G \sigma_s \circ_G \sigma_{f(x_2,x_1)} = \sigma_t$. This implies $\sigma_u \circ_G \sigma_{\bar{t}} = \sigma_s$ and $\sigma_p \circ_G \sigma_s = \sigma_{\bar{t}}$. So $\sigma_s \mathcal{L} \sigma_{\bar{t}}$. By Proposition 4.4.16, we get $\sigma_s = \sigma_{\bar{t}}$ or $\sigma_s = \sigma_{\bar{t}'} = \sigma_{\overline{t'}}$. Assume that $E(6)$ holds. We must have $\sigma_v \circ_G \sigma_q = \sigma_{f(x_2,x_1)}$. Then $\sigma_t = \sigma_p \circ_G \sigma_u \circ_G \sigma_t \circ_G \sigma_v \circ_G \sigma_q = \sigma_p \circ_G \sigma_u \circ_G \sigma_t \circ_G \sigma_{f(x_2,x_1)} = (\sigma_p \circ_G \sigma_u) \circ_G \sigma_{\bar{t}}$. Since $\sigma_p \circ_G \sigma_u \neq \sigma_{id}, \sigma_{f(x_2,x_1)}$, thus by Lemma 4.4.13 we get t is longer than $\bar{t}$ and it is a contradiction. ■

Proposition 4.4.19. *Let $t \in W_{(2)}(X) \setminus X$ and $x_1 \in var(t)$ or $x_2 \in var(t)$. Then the following statements are equivalent:*

(i) *σ_t has an $\mathcal{H}$-class of size 2.*

(ii) $t' = \bar{t}$.

(iii) *$t = f(u, v)$ for some $u, v \in W_{(2)}(X)$ with $v = \overline{u'}$.*

Proof. (i)$\Longrightarrow$(ii) Assume that (i) holds. By Theorem 4.4.9, we get $R_{\sigma_t} = \{\sigma_t, \sigma_{\bar{t}}\}$. Since $H_{\sigma_t} \subseteq R_{\sigma_t}$ and $|H_{\sigma_t}| = 2$, thus $H_{\sigma_t} = \{\sigma_t, \sigma_{\bar{t}}\}$. So $\sigma_t \mathcal{L} \sigma_{\bar{t}}$. By Proposition 4.4.1, we get $\sigma_t \mathcal{L} \sigma_{t'}$. So $\sigma_{\bar{t}} \mathcal{L} \sigma_{t'}$. If $t \in W^G_{(2)}(\{x_1, x_2\})$, then by Proposition 4.4.16, we get $t' = \bar{t}$. If $t \in W(\{x_1\})$, then by Lemma 4.4.6, we get x_2 is not a variable occurring in the term $(\sigma \circ_G \sigma_t)(f)$ for all $\sigma \in Hyp_G(2)$. So $\sigma \circ_G \sigma_t \neq \sigma_{\bar{t}}$ for all $\sigma \in Hyp_G(2)$. Thus it is impossible that $\sigma_{\bar{t}}$ is $\mathcal{L}$-related to σ_t. By the same way we can show that if $t \in W(\{x_2\})$, then σ_t and $\sigma_{\bar{t}}$ are not related.

(ii)$\Longrightarrow$ (i) Assume that $t' = \bar{t}$. By Proposition 4.4.1, we get $\sigma_t \mathcal{L} \sigma_{\bar{t}}$. So $R_{\sigma_t} = \{\sigma_t, \sigma_{\bar{t}}\} \subseteq L_{\sigma_t}$. Thus $H_{\sigma_t} = L_{\sigma_t} \cap R_{\sigma_t} = R_{\sigma_t} = \{\sigma_t, \sigma_{\bar{t}}\}$. So $|H_{\sigma_t}| = 2$.

(ii)$\Longrightarrow$(iii) Assume that $t = f(u, v)$ for some $u, v \in W_{(2)}(X)$ with $t' = \bar{t}$. So $\overline{f(u,v)} = f(u,v)'$

$$\Rightarrow \quad f(\overline{u}, \overline{v}) = f(v', u')$$

$$\Rightarrow \quad \overline{u} = v'$$

$$\Rightarrow \quad v = (v')' = \overline{u}' = \overline{u'}.$$

(iii)$\Longrightarrow$(ii) Assume that $t = f(u, v)$ for some $u, v \in W_{(2)}(X)$ with $v = \overline{u'}$. So $t' = f(u,v)' = f(u, \overline{u'})' = f(\overline{u'}', u') = f(\overline{u}, u') = \overline{\overline{f(\overline{u}, u')}} = \overline{f(\overline{\overline{u}}, \overline{u'})} = \overline{f(u,v)} = \bar{t}$. ■

4.5 Natural Partial Ordering on the Set of All Idempotent Elements of $Hyp_G(2)$

In this section, we characterize all primitive idempotent elements of $Hyp_G(2)$ and characterize the natural partial ordering on the set of all idempotent elements of $Hyp_G(2)$.

Proposition 4.5.1. *For all $x_i \in X$, σ_{x_i} is primitive.*

Proof. Let σ_t be an idempotent element with $\sigma_t \leq \sigma_{x_i}$. Then $\sigma_t \circ_G \sigma_{x_i} = \sigma_{x_i} \circ_G \sigma_t = \sigma_t$. By Lemma 4.1.4 (i), we get $\sigma_t \circ_G \sigma_{x_i} = \sigma_{x_i}$. So $\sigma_t = \sigma_{x_i}$. ■

Proposition 4.5.2. *Let σ_t be an idempotent element with $t \notin X$. Then σ_t is not primitive.*

Proof. By Lemma 4.1.4 (i), we get $\sigma_t \circ_G \sigma_{x_3} = \sigma_{x_3}$. It is clear that $\sigma_{x_3} \circ_G \sigma_t = \sigma_{x_3}$. So $\sigma_{x_3} \leq \sigma_t$ and thus σ_t is not primitive. ■

By the previous two propositions, we get $P_G(2)$ is the set of all primitive idempotent elements.

Lemma 4.5.3. *Let $\sigma_t \in Hyp_G(2)$. Then $\sigma_{x_1} \circ_G \sigma_t = \sigma_{leftmost(t)}$ $(\hat{\sigma}_{x_1}[t] = leftmost(t))$ and $\sigma_{x_2} \circ_G \sigma_t = \sigma_{rightmost(t)}$ $(\hat{\sigma}_{x_2}[t] = rightmost(t))$.*

Proof. We will show that $\sigma_{x_1} \circ_G \sigma_t = \sigma_{leftmost(t)}$. To do this we will prove by induction on the complexity of the term t. If $t \in X$, then $leftmost(t) = t$ and $\sigma_{x_1} \circ_G \sigma_t = \sigma_t = \sigma_{leftmost(t)}$. Assume that $t = f(t_1, t_2)$ and $\sigma_{x_1} \circ_G \sigma_{t_1} = \sigma_{leftmost(t_1)}$ i.e. $\hat{\sigma}_{x_1}[t_1] = leftmost(t_1)$. Consider $(\sigma_{x_1} \circ_G \sigma_t)(f) = (\sigma_{x_1} \circ_G \sigma_{f(t_1,t_2)})(f) = S^2(x_1, \hat{\sigma}_{x_1}[t_1], \hat{\sigma}_{x_1}[t_2]) = \hat{\sigma}_{x_1}[t_1] = leftmost(t_1) = leftmost(t)$. So $\sigma_{x_1} \circ_G \sigma_t = \sigma_{leftmost(t)}$. By the same way we can show that $\sigma_{x_2} \circ_G \sigma_t = \sigma_{rightmost(t)}$. ∎

Proposition 4.5.4. *Let σ_t be an idempotent element. Then $\sigma_{x_1} \leq \sigma_t$ if and only if $leftmost(t) = x_1$.*

Proof. Assume that $\sigma_{x_1} \leq \sigma_t$. Then $\sigma_{x_1} \circ_G \sigma_t = \sigma_t \circ_G \sigma_{x_1} = \sigma_{x_1}$. By Lemma 4.5.3, $\sigma_{x_1} \circ_G \sigma_t = \sigma_{leftmost(t)}$. So $leftmost(t) = x_1$.

The proof of the converse direction is straightforward. ∎

Proposition 4.5.5. *Let σ_t be an idempotent element. Then $\sigma_{x_2} \leq \sigma_t$ if and only if $rightmost(t) = x_2$.*

Proof. The proof is similar to the proof of Proposition 4.5.4. ∎

Proposition 4.5.6. *Let $x_i \in X$ where $i > 2$ and σ_t be an idempotent element. Then $\sigma_{x_i} \leq \sigma_t$ if and only if $t = x_i$ or $t \notin X$.*

Proof. Assume that $\sigma_{x_i} \leq \sigma_t$. Then $\sigma_{x_i} \circ_G \sigma_t = \sigma_t \circ_G \sigma_{x_i} = \sigma_{x_i}$. Suppose that $t \in X$. If $t \neq x_i$, then $\sigma_{x_i} \circ_G \sigma_t = \sigma_t \neq \sigma_{x_i}$ and it is a contradiction. So $t = x_i$.

The proof of the converse direction is straightforward. ∎

Proposition 4.5.7. *Let $t \in W_{(2)}(X)$ with $x_2 \notin var(t)$ and σ_s be an idempotent element. Then $\sigma_{f(x_1,t)} \leq \sigma_s$ if and only if $s = f(x_1, x_2)$ or $s = f(x_1, t)$.*

Proof. Assume that $\sigma_{f(x_1,t)} \leq \sigma_s$. Then $\sigma_{f(x_1,t)} \circ_G \sigma_s = \sigma_s \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)}$. By Lemma 4.1.4 (i),(ii), we get $s \notin X$. Let $s = f(s_1, s_2)$. Suppose that $s \neq f(x_1, x_2)$. From $\sigma_{f(x_1,t)} \circ_G \sigma_s = \sigma_{f(x_1,t)}$, thus $f(x_1, t) = S^2(f(x_1, t), \hat{\sigma}_{f(x_1,t)}[s_1], \hat{\sigma}_{f(x_1,t)}[s_2])$. Hence $\hat{\sigma}_{f(x_1,t)}[s_1] = x_1$ and then $s_1 = x_1$. Since σ_s is an idempotent element and $f(x_1, x_2) \neq s = f(x_1, s_2)$, thus $x_2 \notin var(s_2)$. From $\sigma_s \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)}$, thus $f(x_1, t) = S^2(f(x_1, s_2), x_1, \hat{\sigma}_s[t])$. From $x_2 \notin var(s_2)$ and $f(x_1, t) = S^2(f(x_1, s_2), x_1, \hat{\sigma}_s[t])$, thus $s_2 = t$.

The proof of the converse direction is straightforward. ∎

Proposition 4.5.8. *Let $t \in W_{(2)}(X)$ with $x_1 \notin var(t)$ and σ_s be an idempotent element. Then $\sigma_{f(t,x_2)} \leq \sigma_s$ if and only if $s = f(x_1, x_2)$ or $s = f(t, x_2)$.*

Proof. The proof is similar to the proof of Proposition 4.5.7. ■

Now, we assume that for an arbitrary term t of type $\tau = (2)$, we define two semigroup words $Lp(t)$ and $Rp(t)$ over the alphabet $\{f\}$ inductively as follows :

(i) If $t = f(x_i, t_2)$ where $t_2 \in W_{(2)}(X)$, $x_i \in X$, then $Lp(t) := f$.

(ii) If $t = f(t_1, x_i)$ where $t_1 \in W_{(2)}(X)$, $x_i \in X$, then $Rp(t) := f$.

(iii) if $t = f(t_1, t_2)$ where $t_1 \in W_{(2)}(X) \setminus X$, then $Lp(t) := f(Lp(t_1))$.

(iv) If $t = f(t_1, t_2)$ where $t_2 \in W_{(2)}(X) \setminus X$, then $Rp(t) := f(Rp(t_2))$.

We denote the number of symbols occurring in the semigroup word $Lp(t)$ ($Rp(t)$) by $length(Lp(t))$ ($length(Rp(t))$).

As an example, let $t, t_1, t_2 \in W_{(2)}(X)$ where $t_1 = f(x_1, f(x_3, x_4))$, $t_2 = f(f(x_1, x_2), f(x_1, x_5))$ and $t = f(t_1, t_2)$, then $Lp(t_1) = f$, $Rp(t_1) = ff$, $Lp(t_2) = ff$, $Rp(t_2) = ff$, $Lp(t) = ff$, $Rp(t) = fff$, $length(Lp(t)) = 2$ and $length(Rp(t)) = 3$.

For any term $t \in W_{(2)}(X)$ with $x_1 \notin var(t)$ or $x_2 \notin var(t)$. Then we define

(i) $t^1 := t$.

(ii) $t^n := S^2(t, t^{n-1}, t^{n-1})$ if $n > 1$.

(iii) $t^n_{x_i} := S^2(t^n, x_i, x_i)$ if $x_i \in X$, $n \in \mathbb{N}$.

Proposition 4.5.9. *Let $t \in W_{(2)}(X)$ with $x_2 \notin var(t)$ and σ_s be an idempotent element with $f(x_1, t) \neq s \notin X$. Then $\sigma_s \leq \sigma_{f(x_1,t)}$ if and only if $s = f(x_1, t)^{length(Lp(s))}_{x_i}$ where $x_i = leftmost(s)$ with $i > 2$.*

Proof. Assume that $\sigma_s \leq \sigma_{f(x_1,t)}$. Then $\sigma_s \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)} \circ_G \sigma_s = \sigma_s$. Let $s = f(s_1, s_2)$. So we have two equations

$$S^2(f(x_1, t), \hat{\sigma}_{f(x_1,t)}[s_1], \hat{\sigma}_{f(x_1,t)}[s_2]) = f(s_1, s_2) \quad (1)$$

$$S^2(f(s_1, s_2), x_1, \hat{\sigma}_s[t]) = f(s_1, s_2) \quad (2).$$

It is clear that $\hat{\sigma}_s[t] \neq x_2$. If $s_1 = x_1$, then $\hat{\sigma}_{f(x_1,t)}[s_1] = x_1$. By (1), we get $f(x_1, t) = f(s_1, s_2)$ and it is a contradiction. If $s_1 = x_2$, thus $s_2 = x_2$ since σ_s is an idempotent element. By (2), we get $\hat{\sigma}_s[t] = x_2$ and it is a contradiction. If $s_1 = x_i$ where $i > 2$, then $\hat{\sigma}_{f(x_1,t)}[s_1] = x_i$, $leftmost(s) = x_i$ and $length(Lp(s)) = 1$. By (1), we get $f(x_1,t)^1_{x_i} = f(s_1, s_2)$. Let $s_1 = f(s_3, s_4)$. Then $\hat{\sigma}_{f(x_1,t)}[s_1] = S^2(f(x_1,t), \hat{\sigma}_{f(x_1,t)}[s_3], \hat{\sigma}_{f(x_1,t)}[s_4])$. If $s_3 = x_1$, then $\hat{\sigma}_{f(x_1,t)}[s_3] = x_1$. From $\hat{\sigma}_{f(x_1,t)}[s_1] = S^2(f(x_1,t), \hat{\sigma}_{f(x_1,t)}[s_3], \hat{\sigma}_{f(x_1,t)}[s_4])$, thus $\hat{\sigma}_{f(x_1,t)}[s_1] = f(x_1,t)$. From (1), we get $s_1 \notin X$ and $x_1 \in var(s)$, which contradicts to σ_s is an idempotent element. If $s_3 = x_2$, then $\hat{\sigma}_{f(x_1,t)}[s_3] = x_2$. From $\hat{\sigma}_{f(x_1,t)}[s_1] = S^2(f(x_1,t), \hat{\sigma}_{f(x_1,t)}[s_3], \hat{\sigma}_{f(x_1,t)}[s_4])$ and (1), we get $x_2 \in var(s)$. Since σ_s is an idempotent element, thus $s_2 = x_2$. By (2), we get $\hat{\sigma}_s[t] = x_2$ and it is a contradiction. If $s_3 = x_i$ where $i > 2$, then $\hat{\sigma}_{f(x_1,t)}[s_3] = x_i$, $leftmost(s) = x_i$ and $length(Lp(s)) = 2$. From $\hat{\sigma}_{f(x_1,t)}[s_1] = S^2(f(x_1,t), \hat{\sigma}_{f(x_1,t)}[s_3], \hat{\sigma}_{f(x_1,t)}[s_4])$ and (1), we get $f(x_1,t)^2_{x_i} = f(s_1, s_2)$. This procedure stops with a variable and then we have $f(x_1,t)^{length(Lp(s))}_{x_i} = f(s_1, s_2)$ where $leftmost(s) = x_i$. Conversely, assume that the condition holds. We will show that $\sigma_s \leq \sigma_{f(x_1,t)}$. To do this we will prove by induction on $length(Lp(s))$. If $length(Lp(s)) = 1$, then $s = f(x_1,t)^1_{x_i}$. By Lemma 4.1.4 (iii), we get $\sigma_s \circ_G \sigma_{f(x_1,t)} = \sigma_s$. Consider $(\sigma_{f(x_1,t)} \circ_G \sigma_s)(f) = (\sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,t)^1_{x_i}})(f) = S^2(f(x_1,t), x_i, x_i) = f(x_1,t)^1_{x_i}$. So $\sigma_{f(x_1,t)} \circ_G \sigma_s = \sigma_s$. Assume that $length(Lp(s)) = k$ and $\sigma_{f(x_1,t)^k_{x_i}} \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,t)^k_{x_i}} = \sigma_{f(x_1,t)^k_{x_i}}$. Then $\hat{\sigma}_{f(x_1,t)}[f(x_1,t)^k_{x_i}] = f(x_1,t)^k_{x_i}$. By Lemma 4.1.4 (iii), we get $\sigma_{f(x_1,t)^{k+1}_{x_i}} \circ_G \sigma_{f(x_1,t)} = \sigma_{f(x_1,t)^{k+1}_{x_i}}$. Consider

$$\begin{aligned}
(\sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,t)^{k+1}_{x_i}})(f) &= \hat{\sigma}_{f(x_1,t)}[f(x_1,t)^{k+1}_{x_i}] \\
&= \hat{\sigma}_{f(x_1,t)}[S^2(f(x_1,t), f(x_1,t)^k_{x_i}, f(x_1,t)^k_{x_i})] \\
&= S^2(\hat{\sigma}_{f(x_1,t)}[f(x_1,t)], \hat{\sigma}_{f(x_1,t)}[f(x_1,t)^k_{x_i}], \hat{\sigma}_{f(x_1,t)}[f(x_1,t)^k_{x_i}]) \\
&\quad \text{(by Proposition 2.2.10 (i))} \\
&= S^2(f(x_1,t), f(x_1,t)^k_{x_i}, f(x_1,t)^k_{x_i}) \quad \text{(by induction)} \\
&= f(x_1,t)^{k+1}_{x_i}.
\end{aligned}$$

So $\sigma_{f(x_1,t)} \circ_G \sigma_{f(x_1,t)^{k+1}_{x_i}} = \sigma_{f(x_1,t)^{k+1}_{x_i}}$. ■

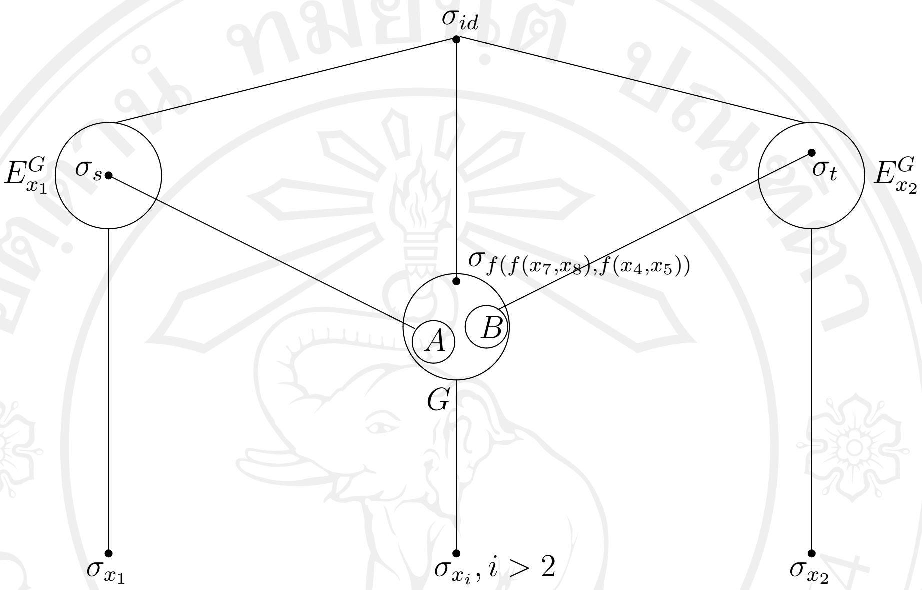
Proposition 4.5.10. *Let $t \in W_{(2)}(X)$ with $x_1 \notin var(t)$ and σ_s be an idempotent element with $f(t, x_2) \neq s \notin X$. Then $\sigma_s \leq \sigma_{f(t,x_2)}$ if and only if $s = f(t,x_2)^{length(Rp(s))}_{x_i}$ where $x_i = rightmost(s)$ with $i > 2$.*

Proof. The proof is similar to the proof of Proposition 4.5.9. ■

Proposition 4.5.11. *Let $s \in W_{(2)}(X) \setminus X$ and $\sigma_t \in G$. If $\sigma_s \leq \sigma_t$, then $s = t$.*

Proof. Let $\sigma_s \leq \sigma_t$. Then $\sigma_t \circ_G \sigma_s = \sigma_s$. By Lemma 4.1.4 (iii), we get $\sigma_t \circ_G \sigma_s = \sigma_t$. So $s = t$. ■

The following picture shows the natural partial ordering on the set of all idempotent elements of $Hyp_G(2)$.



$A = \{\sigma_u \mid u = s^n_{x_i}, n, i \in \mathbb{N}, i > 2\}$,

$B = \{\sigma_u \mid u = t^n_{x_i}, n, i \in \mathbb{N}, i > 2\}$.

Figure 2.