Chapter 5

Idempotent Pre-Generalized Hypersubstitutions of Type $\tau = (2, 2)$

In this chapter, we characterize all idempotent pre-generalized hypersubstitutions of type $\tau = (2, 2)$.

5.1 Pre-Generalized Hypersubstitutions of Type $\tau = (2, 2)$

In [14], K. Denecke and S. L. Wismath studied M-hyperidentities and M-solid varieties based on submonoids M of the monoid $\underline{Hyp}(\tau)$. They defined a number of natural such monoids based on various properties of hypersubstitutions. In the similar way, we can define these monoids for generalized hypersubstitutions of type $\tau = (2, 2)$.

Definition 5.1.1. Let $\tau = (2, 2)$ be a type with the binary operation symbols f and g. Any generalized hypersubstitution σ of type $\tau = (2, 2)$ is determined by the terms t_1, t_2 in $W_{(2,2)}(X)$ to which its maps the binary operation symbols f and g, denoted by σ_{t_1,t_2}.

(i) A generalized hypersubstitution σ of type $\tau = (2, 2)$ is called a *projection generalized hypersubstitution* if the terms $\sigma(f)$ and $\sigma(g)$ are variables, i.e. $\{\sigma(f), \sigma(g)\} \subseteq X$. We denote the set of all projection generalized hypersubstitutions of type $\tau = (2, 2)$ by $P_G(2, 2)$, i.e. $P_G(2, 2) := \{\sigma_{x_i,x_j} | x_i, x_j \in X\}$.

(ii) A generalized hypersubstitution σ of type $\tau = (2, 2)$ is called a *weak projection generalized hypersubstitution* if the terms $\sigma(f)$ or $\sigma(g)$ belongs to X. We denote the set of all weak projection generalized hypersubstitutions of type $\tau = (2, 2)$ by $WP_G(2, 2)$.

(iii) A generalized hypersubstitution σ of type $\tau = (2, 2)$ is called a *pre-generalized hypersubstitution* if the terms $\sigma(f)$ and $\sigma(g)$ are not belong to X. We denote the set of all pre-generalized hypersubstitutions of type $\tau = (2,2)$ by $Pre_G(2,2)$, i.e. $Pre_G(2,2) := Hyp_G(2,2) \setminus WP_G(2,2)$.

We introduce some notations. For $t \in W_{(2,2)}(X)$, we consider :

$ops(t)$:= the set of all operation symbols occurring in t,

$firstops(t)$:= the first operation symbol (from the left) which occurs in t.

Now, we assume that F is a variable over the two-element alphabet $\{f, g\}$. For an arbitrary term t of type $\tau = (2,2)$, we define two semigroup words $Lp(t)$ and $Rp(t)$ over the alphabet $\{f, g\}$ inductively as follows :

(i) If $t = F(x_i, t_2)$ where $t_2 \in W_{(2,2)}(X)$, $x_i \in X$, then $Lp(t) := F$.

(ii) if $t = F(t_1, x_i)$ where $t_1 \in W_{(2,2)}(X)$, $x_i \in X$, then $Rp(t) := F$.

(iii) If $t = F(t_1, t_2)$ where $t_1 \in W_{(2,2)}(X) \setminus X$, then $Lp(t) := F(Lp(t_1))$.

(iv) If $t = F(t_1, t_2)$ where $t_2 \in W_{(2,2)}(X) \setminus X$, then $Rp(t) := F(Rp(t_2))$.

We denote the number of symbols occurring in the semigroup word $Lp(t)$ ($Rp(t)$) by $length(Lp(t))$ ($length(Rp(t))$).

As an example, let $t, t_1, t_2 \in W_{(2,2)}(X)$ where $t_1 = f(x_1, g(x_3, x_4))$, $t_2 = g(f(x_1, x_2), f(x_1, x_5))$ and $t = f(t_1, t_2)$, then $Lp(t_1) = f$, $Rp(t_1) = fg$, $Lp(t_2) = gf$, $Rp(t_2) = gf$, $Lp(t) = ff$ and $Rp(t) = fgf$.

In [24], S. Leeratanavalee already proved that for any type τ, the set $P_G(\tau) \cup \{\sigma_{id}\}$ and $Pre_G(\tau)$ are submonoids of $\underline{Hyp_G(\tau)}$. It is easy to see that $WP_G(\tau) \cup \{\sigma_{id}\}$ is a submonoid of $\underline{Hyp_G(\tau)}$, and $P_G(\tau) \cup \{\sigma_{id}\}$ forms a submonoid of $WP_G(\tau) \cup \{\sigma_{id}\}$.

5.2 Idempotent Elements in $Pre_G(2,2)$

It is obvious that every projection generalized hypersubstitution is idempotent and σ_{id} is also idempotent. In Chapter 4 and [32], the authors characterized all idempotent elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ and S. Leeratanavalee characterized all idempotent elements of $WP_G(2,2) \cup \{\sigma_{id}\}$, see [22].

In this section, we consider the idempotent elements in $Pre_G(2,2)$. We have the following propositions.

Proposition 5.2.1. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. Then σ_{t_1,t_2} is idempotent if and only if $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.*

Proof. Assume that σ_{t_1,t_2} is idempotent, i.e. $\sigma^2_{t_1,t_2} = \sigma_{t_1,t_2}$. Then $\hat{\sigma}_{t_1,t_2}[t_1] = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(f)] = \sigma^2_{t_1,t_2}(f) = \sigma_{t_1,t_2}(f) = t_1$. Similarly, we get $\hat{\sigma}_{t_1,t_2}[t_2] = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(g)] = \sigma^2_{t_1,t_2}(g) = \sigma_{t_1,t_2}(g) = t_2$. Conversely, let $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Since $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, then $(\sigma_{t_1,t_2} \circ_G \sigma_{t_1,t_2})(f) = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(f)] = \hat{\sigma}_{t_1,t_2}[t_1] = t_1 = \sigma_{t_1,t_2}(f)$. Similarly, since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, then $(\sigma_{t_1,t_2} \circ_G \sigma_{t_1,t_2})(g) = \hat{\sigma}_{t_1,t_2}[\sigma_{t_1,t_2}(g)] = \hat{\sigma}_{t_1,t_2}[t_2] = t_2 = \sigma_{t_1,t_2}(g)$. Thus $\sigma^2_{t_1,t_2} = \sigma_{t_1,t_2}$. ■

Now, we assume that $t_1, t_2 \in W_{(2,2)}(X)$ where $op(t_1) = 1$, $op(t_2) = 1$, $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, $firstops(t_1) = g$ and $firstops(t_2) = f$. Then t_1 and t_2 have the forms $t_1 = g(x_i, x_j)$, $t_2 = f(x_k, x_l)$ where $i, j, k, l \in \mathbb{N}$. Since $t_1 = \hat{\sigma}_{t_1,t_2}[t_1] = S^2(\sigma_{t_1,t_2}(g), x_i, x_j) = S^2(t_2, x_i, x_j)$, it follows that $firstops(t_1) = f$. This is a contradiction and implies that if σ_{t_1,t_2} is idempotent, then the case $firstops(t_1) = g$ and $firstops(t_2) = f$ is impossible.

Then we will consider the following cases:

Case 1: $firstops(t_1) = f$ and $firstops(t_2) = f$.

Case 2: $firstops(t_1) = g$ and $firstops(t_2) = g$.

Case 3: $firstops(t_1) = f$ and $firstops(t_2) = g$.

For the three possible cases, we have the following results:

Proposition 5.2.2. *Let $t_1 = f(x_i, x_j)$ and $t_2 = f(x_k, x_l)$ with $i, j, k, l \in \mathbb{N}$. Then σ_{t_1,t_2} is idempotent if and only if the following conditions hold:*

(i) *If $x_1 \in var(t_1)$, then $x_i = x_1$ and if $x_2 \in var(t_1)$, then $x_j = x_2$.*

(ii) *If $x_i = x_j = x_1$ or $x_i = x_j = x_2$, then $x_k = x_l$.*

(iii) *If $x_i = x_1$ and $j > 2$, then $x_l = x_j$.*

(iv) *If $i > 2$ and $x_j = x_2$, then $x_k = x_i$.*

(v) *If $i, j > 2$, then $x_k = x_i$ and $x_l = x_j$.*

Proof. Assume that σ_{t_1,t_2} is idempotent, thus $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Then we obtain the equations $S^2(t_1, x_i, x_j) = t_1$ and $S^2(t_1, x_k, x_l) = t_2$.

(i) Assume that $x_1 \in var(t_1)$. Suppose that $x_i \neq x_1$. Then we have to replace x_1 in the term t_1 by x_i and then we conclude that $S^2(t_1, x_i, x_j) \neq t_1$. Hence $x_i = x_1$. By the same way we can prove that if $x_2 \in var(t_1)$, then $x_j = x_2$.

(ii) Assume that $x_i = x_j = x_1$. From $S^2(t_1, x_k, x_l) = t_2$, thus $S^2(f(x_1, x_1), x_k, x_l) = f(x_k, x_l)$ and then $f(x_k, x_k) = f(x_k, x_l)$. Hence $x_k = x_l$. By the same way we can prove that if $x_i = x_j = x_2$, then $x_k = x_l$.

(iii) Assume that $x_i = x_1$ and $j > 2$. From $S^2(t_1, x_k, x_l) = t_2$, thus $S^2(f(x_1, x_j), x_k, x_l) = f(x_k, x_l)$ and then $f(x_k, x_j) = f(x_k, x_l)$. Hence $x_l = x_j$.

(iv) Assume that $i > 2$ and $x_j = x_2$. From $S^2(t_1, x_k, x_l) = t_2$, thus $S^2(f(x_i, x_2), x_k, x_l) = f(x_k, x_l)$ and then $f(x_i, x_l) = f(x_k, x_l)$. Hence $x_k = x_i$.

(v) Assume that $i, j > 2$. From $S^2(t_1, x_k, x_l) = t_2$, thus $S^2(f(x_i, x_j), x_k, x_l) = f(x_k, x_l)$ and then $f(x_i, x_j) = f(x_k, x_l)$. Hence $x_k = x_i$ and $x_l = x_j$. Conversely, assume that (i), (ii), (iii), (iv) and (v) hold. Hence $\sigma_{t_1,t_2} \in \{\sigma_{f(x_1,x_1),f(x_k,x_k)}, \sigma_{f(x_1,x_2),f(x_k,x_l)}, \sigma_{f(x_1,x_j),f(x_k,x_j)}, \sigma_{f(x_2,x_2),f(x_k,x_k)}, \sigma_{f(x_i,x_2),f(x_i,x_l)}, \sigma_{f(x_i,x_j),f(x_i,x_j)} | i, j, k, l \in \mathbb{N}, i, j > 2\}$. It is easy to check that all these generalized hypersubstituitions are idempotent. ■

From Proposition 5.2.2, we obtain a similar result which solves the Case 2.

Proposition 5.2.3. *Let $t_1 = g(x_i, x_j)$ and $t_2 = g(x_k, x_l)$ with $i, j, k, l \in \mathbb{N}$. Then σ_{t_1,t_2} is idempotent if and only if the following conditions hold:*

(i) *If $x_1 \in var(t_2)$, then $x_k = x_1$ and if $x_2 \in var(t_2)$, then $x_l = x_2$.*

(ii) *If $x_k = x_l = x_1$ or $x_k = x_l = x_2$, then $x_i = x_j$.*

(iii) *If $x_k = x_1$ and $l > 2$, then $x_j = x_l$.*

(iv) *If $k > 2$ and $x_l = x_2$, then $x_i = x_k$.*

(v) *If $k, l > 2$, then $x_i = x_k$ and $x_j = x_l$.*

Proof. The proof is similar to the proof of Proposition 5.2.2. ■

For the Case 3 we have the following result:

Proposition 5.2.4. *Let $t_1 = f(x_i, x_j)$ and $t_2 = g(x_k, x_l)$ with $i, j, k, l \in \mathbb{N}$. Then σ_{t_1,t_2} is idempotent if and only if the following conditions hold:*

(i) *If $x_i = x_2$, then $x_j = x_2$.*

(ii) *If* $x_k = x_2$*, then* $x_l = x_2$.

(iii) *If* $i > 2$*, then* $x_j \neq x_1$.

(iv) *If* $k > 2$*, then* $x_l \neq x_1$.

Proof. Assume that σ_{t_1,t_2} is idempotent, thus $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Then we obtain the equations $S^2(t_1, x_i, x_j) = t_1$ and $S^2(t_2, x_k, x_l) = t_2$.

(i) Assume that $x_i = x_2$. From $S^2(t_1, x_i, x_j) = t_1$, thus $S^2(f(x_2, x_j), x_2, x_j) = f(x_2, x_j)$. Hence $x_j = x_2$.

(ii) The proof is similar to the proof of (i).

(iii) Assume that $i > 2$ and suppose that $x_j = x_1$. Thus $S^2(t_1, x_i, x_j) = S^2(f(x_i, x_1), x_i, x_1) = f(x_i, x_i) \neq f(x_i, x_1) = t_1$, which is a contradiction. Hence $x_j \neq x_1$.

(iv) The proof is similar to the proof of (iii).

Conversely, assume that (i), (ii), (iii) and (iv) hold. Hence $\sigma_{t_1,t_2} \in \{\sigma_{f(x_1,x_j),g(x_1,x_l)}, \sigma_{f(x_1,x_j),g(x_2,x_2)}, \sigma_{f(x_1,x_j),g(x_k,x_m)}, \sigma_{f(x_2,x_2),g(x_1,x_l)}, \sigma_{f(x_2,x_2),g(x_2,x_2)}, \sigma_{f(x_2,x_2),g(x_k,x_m)}, \sigma_{f(x_i,x_p),g(x_1,x_l)}, \sigma_{f(x_i,x_p),g(x_2,x_2)}, \sigma_{f(x_i,x_p),g(x_k,x_m)} | i, j, k, l, m, p \in \mathbb{N}, i, k > 2, m, p \neq 1\}$. It is easy to check that all these generalized hypersubstituitions are idempotent. ■

To consider the next cases, we first give the following definitions and some lemmas.

Definition 5.2.5. (i) $W^G_{(2,2)}(\{x_1\}) := \{t \in W_{(2,2)}(X) | x_1 \in var(t), x_2 \notin var(t)\}$.

(ii) $W^G_{(2,2)}(\{x_2\}) := \{t \in W_{(2,2)}(X) | x_2 \in var(t), x_1 \notin var(t)\}$.

Definition 5.2.6. Let $t \in W^G_{(2,2)}(\{x_1\})$ or $t \in W^G_{(2,2)}(\{x_2\})$. Then we define

(i) $t^1 := t$.

(ii) $t^n := S^2(t, t^{n-1}, t^{n-1})$ if $n > 1$.

(iii) $t^n_{x_i} := S^2(t^n, x_i, x_i)$ if $x_i \in X$, $n \in \mathbb{N}$.

Lemma 5.2.7. *Let* $t, t_1, t_2 \in W_{(2,2)}(X)$ *and* $x_i \in X$ *for all* $i \in \mathbb{N}$*. Then the following statements hold:*

(i) *If* $t_1 = f(x_1, t) \in W^G_{(2,2)}(\{x_1\})$*, then* $\hat{\sigma}_{t_1,t_2}[t_1^n] = t_1^n$ *and* $\hat{\sigma}_{t_1,t_2}[{t_1}^n_{x_i}] = {t_1}^n_{x_i}$ *for all* $n \in \mathbb{N}$.

(ii) *If* $t_1 = f(t, x_2) \in W^G_{(2,2)}(\{x_2\})$, *then* $\hat{\sigma}_{t_1,t_2}[t_1^n] = t_1^n$ *and* $\hat{\sigma}_{t_1,t_2}[{t_1}^n_{x_i}] = {t_1}^n_{x_i}$ *for all* $n \in \mathbb{N}$.

(iii) *If* $t_2 = g(x_1, t) \in W^G_{(2,2)}(\{x_1\})$, *then* $\hat{\sigma}_{t_1,t_2}[t_2^n] = t_2^n$ *and* $\hat{\sigma}_{t_1,t_2}[{t_2}^n_{x_i}] = {t_2}^n_{x_i}$ *for all* $n \in \mathbb{N}$.

(iv) *If* $t_2 = g(t, x_2) \in W^G_{(2,2)}(\{x_2\})$, *then* $\hat{\sigma}_{t_1,t_2}[t_2^n] = t_2^n$ *and* $\hat{\sigma}_{t_1,t_2}[{t_2}^n_{x_i}] = {t_2}^n_{x_i}$ *for all* $n \in \mathbb{N}$.

Proof. (i) Assume that $t_1 = f(x_1, t) \in W^G_{(2,2)}(\{x_1\})$. We first show that $\hat{\sigma}_{t_1,t_2}[t_1^n] = t_1^n$ by induction on $n \in \mathbb{N}$. For $n = 1$, since $t_1 \in W^G_{(2,2)}(\{x_1\})$, thus $\hat{\sigma}_{t_1,t_2}[t_1^1] = \hat{\sigma}_{t_1,t_2}[t_1] = \hat{\sigma}_{t_1,t_2}[f(x_1, t)] = S^2(t_1, x_1, \hat{\sigma}_{t_1,t_2}[t]) = t_1 = t_1^1$. Assume that $\hat{\sigma}_{t_1,t_2}[t_1^k] = t_1^k$. By Proposition 2.2.10 (i), we get $\hat{\sigma}_{t_1,t_2}[t_1^{k+1}] = \hat{\sigma}_{t_1,t_2}[S^2(t_1, t_1^k, t_1^k)] = S^2(\hat{\sigma}_{t_1,t_2}[t_1], \hat{\sigma}_{t_1,t_2}[t_1^k], \hat{\sigma}_{t_1,t_2}[t_1^k]) = S^2(t_1, t_1^k, t_1^k) = t_1^{k+1}$. Hence $\hat{\sigma}_{t_1,t_2}[t_1^n] = t_1^n$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$. From $\hat{\sigma}_{t_1,t_2}[t_1^n] = t_1^n$, thus $\hat{\sigma}_{t_1,t_2}[{t_1}^n_{x_i}] = \hat{\sigma}_{t_1,t_2}[S^2(t_1^n, x_i, x_i)] = S^2(\hat{\sigma}_{t_1,t_2}[t_1^n], x_i, x_i) = S^2(t_1^n, x_i, x_i) = {t_1}^n_{x_i}$.

The proof of (ii), (iii) and (iv) are similar to the proof of (i). ■

If ($op(t_1) = 1$ and $op(t_2) > 1$) or ($op(t_1) > 1$ and $op(t_2) = 1$), then we have

Lemma 5.2.8. *Let* σ_{t_1,t_2} *be a generalized hypersubstitution of type* $\tau = (2, 2)$. *Then the following statements hold:*

(i) *If* $op(t_1) = 1$ *and* $op(t_2) > 1$, *then* $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ *if and only if* $t_1 \in \{f(x_1, x_i), f(x_2, x_2), f(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$.

(ii) *If* $op(t_1) > 1$ *and* $op(t_2) = 1$, *then* $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$ *if and only if* $t_2 \in \{g(x_1, x_i), g(x_2, x_2), g(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$.

Proof. (i) Let $op(t_1) = 1$ and $op(t_2) > 1$ and assume that $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$. If $t_1 = g(x_i, x_j)$ where $i, j \in \mathbb{N}$, then $\hat{\sigma}_{t_1,t_2}[t_1] = S^2(t_2, x_i, x_j) \neq t_1$ because of $op(t_2) > 1$, which is a contradiction. If $t_1 = f(x_2, x_1)$, then $\hat{\sigma}_{t_1,t_2}[t_1] = S^2(t_1, x_2, x_1) = f(x_1, x_2) \neq t_1$, which is a contradiction. If $t_1 = f(x_i, x_1)$ where $i \in \mathbb{N}, i > 2$, then $\hat{\sigma}_{t_1,t_2}[t_1] = S^2(t_1, x_i, x_1) = f(x_i, x_i) \neq t_1$, which is a contradiction. If $t_1 = f(x_2, x_i)$ where $i \in \mathbb{N}, i > 2$, then $\hat{\sigma}_{t_1,t_2}[t_1] = S^2(t_1, x_2, x_i) = f(x_i, x_i) \neq t_1$, which is a contradiction. Thus $t_1 \in \{f(x_1, x_i), f(x_2, x_2), f(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$. Conversely, we can check easily that all of generalized hypersubstitutions σ_{t_1,t_2} where $t_1 \in \{f(x_1, x_i), f(x_2, x_2), f(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$ we have $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$.

(ii) The proof is similar to the proof of (i). ■

Lemma 5.2.8 shows that we have to consider the following cases if $op(t_1) = 1$ or $op(t_2) = 1$:

Case 1: $op(t_1) = 1$ and $op(t_2) > 1$.

Case 1.1: $firstops(t_2) = f$.

Case 1.2: $firstops(t_2) = g$.

Case 2: $op(t_1) > 1$ and $op(t_2) = 1$.

Case 2.1: $firstops(t_1) = f$.

Case 2.2: $firstops(t_1) = g$.

It is clear that Case 1.1 and Case 2.2 as well as Case 1.2 and Case 2.1 are similar. We consider at first the Case 1.2 and obtain:

Proposition 5.2.9. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2, 2)$. If $op(t_1) = 1$, $op(t_2) > 1$ and $t_2 = g(k_1, k_2)$, then σ_{t_1,t_2} is idempotent if and only if $t_1 \in \{f(x_1, x_i), f(x_2, x_2), f(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$ and the following conditions hold:*

(i) *$x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$.*

(ii) *If $x_1 \notin var(t_2)$ and $x_2 \in var(t_2)$, then $t_2 = g(k_1, x_2)$.*

(iii) *If $x_2 \notin var(t_2)$ and $x_1 \in var(t_2)$, then $t_2 = g(x_1, k_2)$.*

Proof. Assume that σ_{t_1,t_2} is idempotent. Since $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$, thus by Lemma 5.2.8 we have $t_1 \in \{f(x_1, x_i), f(x_2, x_2), f(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$. Suppose that $x_1, x_2 \in var(t_2)$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, thus we obtain the equation $t_2 = S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$. Since $op(t_2) > 1$, thus $k_1 \notin X$ or $k_2 \notin X$. This implies that $\hat{\sigma}_{t_1,t_2}[k_1] \notin X$ or $\hat{\sigma}_{t_1,t_2}[k_2] \notin X$. Since $x_1, x_2 \in var(t_2)$ and $\hat{\sigma}_{t_1,t_2}[k_1] \notin X$ or $\hat{\sigma}_{t_1,t_2}[k_2] \notin X$, thus $op(t_2) < op(S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2]))$ which contradicts to $t_2 = S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$. Hence $x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$. If $x_1 \notin var(t_2)$ and $x_2 \in var(t_2)$, then from $t_2 = S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$ there follows $t_2 = g(k_1, x_2)$. Similarly, for $x_2 \notin var(t_2)$ and $x_1 \in var(t_2)$ we have $t_2 = g(x_1, k_2)$. Conversely, we can check that all these generalized hypersubstitutions which satisfy the conditions of being idempotent by using Lemma 5.2.7. ■

From Proposition 5.2.9 we obtain a similar result which solves the Case 2.1.

Proposition 5.2.10. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. If $op(t_1) > 1$, $op(t_2) = 1$ and $t_1 = f(k_1, k_2)$, then σ_{t_1,t_2} is idempotent if and only if $t_2 \in \{g(x_1, x_i), g(x_2, x_2), g(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$ and the following conditions hold:*

(i) *$x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$.*

(ii) *If $x_1 \notin var(t_1)$ and $x_2 \in var(t_1)$, then $t_1 = f(k_1, x_2)$.*

(iii) *If $x_2 \notin var(t_1)$ and $x_1 \in var(t_1)$, then $t_2 = f(x_1, k_2)$.*

Proof. The proof is similar to the proof of Proposition 5.2.9. ■

For the Cases 1.1 and 2.2 we obtain the following necessary condition for the idempotency of σ_{t_1,t_2}:

Lemma 5.2.11. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type $\tau = (2,2)$ and $op(t_1) \geq 1$, $op(t_2) > 1$ and $t_2 = f(k_1, k_2)$. Then the following statements hold:*

(i) *If $x_1 \in var(t_1)$, then $firstops(k_1) = f$ or $k_1 \in X$.*

(ii) *If $x_2 \in var(t_1)$, then $firstops(k_2) = f$ or $k_2 \in X$.*

Proof. (i) Assume that $x_1 \in var(t_1)$. Since $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, thus we obtain the equation $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$. Let $k_1 = g(k_3, k_4)$. Thus $\hat{\sigma}_{t_1,t_2}[k_1] = S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4])$. From $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$, thus $t_2 = S^2(t_1, S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4]), \hat{\sigma}_{t_1,t_2}[k_2])$. Since $x_1 \in var(t_1)$, thus $op(t_2) < op(S^2(t_1, S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4]), \hat{\sigma}_{t_1,t_2}[k_2]))$, which contradicts to $t_2 = S^2(t_1, S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4]), \hat{\sigma}_{t_1,t_2}[k_2])$. Hence $firstops(k_1) = f$ or $k_1 \in X$.

(ii) The proof is similar to the proof of (i). ■

Lemma 5.2.12. *Let σ_{t_1,t_2} be an idempotent generalized hypersubstitution of type $\tau = (2,2)$ and $op(t_1) > 1$, $op(t_2) \geq 1$ and $t_1 = g(k_1, k_2)$. Then the following statements hold:*

(i) *If $x_1 \in var(t_2)$, then $firstops(k_1) = g$ or $k_1 \in X$.*

(ii) *If $x_2 \in var(t_2)$, then $firstops(k_2) = g$ or $k_2 \in X$.*

Proof. The proof is similar to the proof of Lemma 5.2.11. ∎

For the Cases 1.1 and 2.2 we have the following results:

Proposition 5.2.13. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. If $op(t_1) = 1$, $op(t_2) > 1$ and $t_2 = f(k_1,k_2)$, then σ_{t_1,t_2} is idempotent if and only if $t_1 \in \{f(x_1,x_i), f(x_2,x_2), f(x_j,x_2) | i,j \in \mathbb{N}, j > 2\}$ and the following conditions hold:*

(i) *If $t_1 = f(x_1,x_2)$, then $ops(t_2) = \{f\}$.*

(ii) *If $t_1 = f(x_1,x_i)$ with $i \neq 2$, then $t_2 = {t_1}_{x_k}{}^{length(Lp(t_2))}$ where $x_k = leftmost(t_2)$.*

(iii) *If $t_1 = f(x_j,x_2)$ with $j \neq 1$, then $t_2 = {t_1}_{x_k}{}^{length(Rp(t_2))}$ where $x_k = rightmost(t_2)$.*

Proof. Assume that σ_{t_1,t_2} is idempotent, thus $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. Then we obtain the equation $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$. Let $t_1 = f(x_i,x_j)$ where $i,j > 2$. Thus $t_2 = S^2(f(x_i,x_j), \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2]) = f(x_i,x_j)$, which contradicts to $op(t_2) > 1$. Hence $t_1 \neq f(x_i,x_j)$ where $i,j > 2$. Since $t_1 \neq f(x_i,x_j)$ where $i,j > 2$ and by Lemma 5.2.8, thus $t_1 \in \{f(x_1,x_i), f(x_2,x_2), f(x_j,x_2) | i,j \in \mathbb{N}, j > 2\}$.

(i) Let $t_1 = f(x_1,x_2)$. From $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$, we get $t_2 = f(\hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$. We consider the following three possible cases:

Case 1: $k_1 \notin X, k_2 \in X$.

Case 2: $k_1 \in X, k_2 \notin X$.

Case 3: $k_1, k_2 \notin X$.

Case 1: $k_1 \notin X, k_2 \in X$. From $t_2 = f(\hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$ there follows $t_2 = f(\hat{\sigma}_{t_1,t_2}[k_1], k_2)$. Then by Lemma 5.2.11 implies $firstops(k_1) = f$. We will show by induction on the complexity of k_1 occurring in $t_2 = f(k_1,k_2)$, that $ops(\hat{\sigma}_{t_1,t_2}[k_1]) = \{f\}$. If $k_1 = f(x_i,x_j)$ where $i,j \in \mathbb{N}$, then $\hat{\sigma}_{t_1,t_2}[k_1] = S^2(f(x_1x_2), x_i, x_j) = f(x_i,x_j)$. Hence $ops(\hat{\sigma}_{t_1,t_2}[k_1]) = \{f\}$. Let $k_1 = f(k_3,k_4)$ and assume that $ops(\hat{\sigma}_{t_1,t_2}[k_3]) = \{f\}$ and $ops(\hat{\sigma}_{t_1,t_2}[k_4]) = \{f\}$. Then $ops(\hat{\sigma}_{t_1,t_2}[k_1]) = ops(f(\hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4]))$. Hence $ops(t_2) = \{f\}$.

In the second case we obtain the result in a similar way.

Case 3: $k_1, k_2 \notin X$. By Lemma 5.2.11, we have $firstops(k_1) = f$ and $firstops(k_2) = f$. Then using $t_2 = f(\hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$ by induction on the complexities of k_1 and k_2, respectively, we can show that $ops(t_2) = \{f\}$.

(ii) Assume that $t_1 = f(x_1, x_1)$. From $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$ we have $t_2 = f(\hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_1])$. By Lemma 5.2.11, we get $firstops(k_1) = f$ or $k_1 \in X$. The last case is impossible since $op(t_2) > 1$. Then we can show that $ops(t_2) = \{f\}$. Let $k_1 = f(k_3, k_4)$. We get $t_2 = f(f(\hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_3]), f(\hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_3]))$. Now we set $k_3 = f(k_5, k_6)$. We obtain $t_2 = f(f(f(\hat{\sigma}_{t_1,t_2}[k_5], \hat{\sigma}_{t_1,t_2}[k_5]), f(\hat{\sigma}_{t_1,t_2}[k_5], \hat{\sigma}_{t_1,t_2}[k_5])), f(f(\hat{\sigma}_{t_1,t_2}[k_5], \hat{\sigma}_{t_1,t_2}[k_5]), f(\hat{\sigma}_{t_1,t_2}[k_5], \hat{\sigma}_{t_1,t_2}[k_5])))$. This procedure stops with a variable and then we have $t_2 = {t_1}_{x_k}{}^{length(Lp(t_2))}$ where $x_k = leftmost(t_2)$. Similarly, for $t_1 = f(x_1, x_i)$ where $i \in \mathbb{N}$ with $i > 2$ we have $t_2 = {t_1}_{x_k}{}^{length(Lp(t_2))}$ where $x_k = leftmost(t_2)$.

(iii) The proof is similar to the proof of (ii).

Conversely, we can check that all these generalized hypersubstitutions which satisfy the conditions are idempotent by using Lemma 5.2.7. ∎

From Proposition 5.2.13, we obtain a similar result which solves the Case 2.2.

Proposition 5.2.14. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2, 2)$. If $op(t_1) > 1$, $op(t_2) = 1$ and $t_1 = g(k_1, k_2)$, then σ_{t_1,t_2} is idempotent if and only if $t_2 \in \{g(x_1, x_i), g(x_2, x_2), g(x_j, x_2) | i, j \in \mathbb{N}, j > 2\}$ and the following conditions hold:*

(i) *If $t_2 = g(x_1, x_2)$, then $ops(t_1) = \{g\}$.*

(ii) *If $t_2 = g(x_1, x_i)$ with $i \neq 2$, then $t_1 = {t_2}_{x_k}{}^{length(Lp(t_1))}$ where $x_k = leftmost(t_1)$.*

(iii) *If $t_2 = g(x_j, x_2)$ with $j \neq 1$, then $t_1 = {t_2}_{x_k}{}^{length(Rp(t_1))}$ where $x_k = rightmost(t_1)$.*

Proof. The proof is similar to the proof of Proposition 5.2.13. ∎

Now, we assume that $op(t_1) > 1$ and $op(t_2) > 1$. We can prove that if σ_{t_1,t_2} is idempotent, then the case $firstops(t_1) = g$ and $firstops(t_2) = f$ is impossible.

Then we will consider the following cases:

Case 1: $firstops(t_1) = f$ and $firstops(t_2) = f$.

Case 2: $firstops(t_1) = g$ and $firstops(t_2) = g$.

Case 3: $firstops(t_1) = f$ and $firstops(t_2) = g$.

We obtain the following necessary condition for the idempotency of σ_{t_1,t_2}:

Lemma 5.2.15. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2, 2)$ and $op(t_1) > 1$, $op(t_2) > 1$. If σ_{t_1,t_2} is idempotent, then ($x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$) and ($x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$).*

Proof. Assume that σ_{t_1,t_2} is idempotent, thus $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. We consider into three cases :

Case 1. In this case we have $t_1 = f(k_1, k_2)$ and $t_2 = f(k_3, k_4)$. If $x_1, x_2 \in var(t_1)$ from $op(t_1) > 1$, then we obtain $op(t_1) = op(S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])) > op(t_1)$. This is a contradiction. Thus $x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$.

In the case $x_1, x_2 \notin var(t_1)$ we have $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4]) = t_1$. Thus $x_1 \notin var(t_2)$ and $x_2 \notin var(t_2)$.

In the case $t_1 \in W^G_{(2,2)}(\{x_1\})$ we have $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4])$. By Lemma 5.2.11 (i), we get $firstops(k_3) = f$ or $k_3 \in X$. If $k_3 \in X$, then from $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_3],$ $\hat{\sigma}_{t_1,t_2}[k_4])$ we get $t_2 = {t_1}^{length(Lp(t_2))}_{leftmost(t_2)}$ and so $x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$. Let $firstops(k_3)$ $= f$ and $k_3 = f(k_5, k_6)$. This gives $t_2 = S^2(t_1, S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_5], \hat{\sigma}_{t_1,t_2}[k_6]), \hat{\sigma}_{t_1,t_2}[k_4])$. Continuing in this way, we get $t_2 = {t_1}^{length(Lp(t_2))}_{leftmost(t_2)}$. Therefore $x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$. For $t_1 \in W^G_{(2,2)}(\{x_2\})$, we have $t_2 = {t_1}^{length(Rp(t_2))}_{rightmost(t_2)}$ and so $x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$.

Case 2 can be proved in a similar way.

Case 3. In this case t_1 and t_2 have the form $t_1 = f(k_1, k_2)$ and $t_2 = g(k_3, k_4)$ and if $x_1, x_2 \in var(t_1)$, then $op(t_1) < op(S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2]))$. Therefore $x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$. In the same way we can show that $x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$. ∎

For the three possible cases of the first operation symbol in t_1 and t_2 we have the following results:

Proposition 5.2.16. *Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2, 2)$, $op(t_1) > 1$, $op(t_2) > 1$ and $t_1 = f(k_1, k_2)$, $t_2 = f(k_3, k_4)$, then σ_{t_1,t_2} is idempotent if and only if ($x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$) and ($x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$) and the following conditions hold:*

(i) *If $t_1, t_2 \in W^G_{(2,2)}(\{x_1\})$, then $t_1 = f(x_1, k_2)$ and $t_2 = t_1^{length(Lp(t_2))}$.*

(ii) *If $t_1, t_2 \in W^G_{(2,2)}(\{x_2\})$, then $t_1 = f(k_1, x_2)$ and $t_2 = t_1^{length(Rp(t_2))}$.*

(iii) *If $t_1 \in W^G_{(2,2)}(\{x_1\})$, $t_2 \in W^G_{(2,2)}(\{x_2\})$, then $t_1 = f(x_1, k_2)$ and $t_2 = {t_1}_{x_2}{}^{length(Lp(t_2))}$.*

(iv) *If $t_1 \in W^G_{(2,2)}(\{x_2\})$, $t_2 \in W^G_{(2,2)}(\{x_1\})$, then $t_1 = f(k_1, x_2)$ and $t_2 = {t_1}_{x_1}{}^{length(Rp(t_2))}$.*

(v) *If $t_1 \in W^G_{(2,2)}(\{x_1\})$ and $x_1, x_2 \notin var(t_2)$, then $t_1 = f(x_1, k_2)$ and $t_2 = {t_1}_{x_k}{}^{length(Lp(t_2))}$ where $x_k = leftmost(t_2)$.*

(vi) *If* $t_1 \in W^G_{(2,2)}(\{x_2\})$ *and* $x_1, x_2 \notin var(t_2)$, *then* $t_1 = f(k_1, x_2)$ *and* $t_2 = {t_1}_{x_k}{}^{length(Rp(t_2))}$ *where* $x_k = rightmost(t_2)$.

(vii) *If* $x_1, x_2 \notin var(t_1)$, *then* $t_2 = t_1$.

Proof. Assume that σ_{t_1,t_2} is idempotent, thus $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$. By Lemma 5.2.15, we get ($x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$) and ($x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$).

(i) Assume that $t_1, t_2 \in W^G_{(2,2)}(\{x_1\})$. From $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, we obtain the equations $t_1 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_1])$ and $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_3])$. Since $x_1 \in var(t_1)$ and $t_1 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_1])$, thus $\hat{\sigma}_{t_1,t_2}[k_1] = x_1$. Since $op(t_1) > 1$, $op(t_2) > 1$, thus $k_1 = x_1$. So $t_1 = f(x_1, k_2)$. By Lemma 5.2.11 (i), we get $firstops(k_3) = f$ or $k_3 \in X$. If $k_3 \in X$, then from $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_3])$ we get $t_2 = {t_1}^{length(Lp(t_2))}_{leftmost(t_2)}$. Let $firstops(k_3) = f$ and $k_3 = f(k_5, k_6)$, we obtain $t_2 = S^2(t_1, S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_5], \hat{\sigma}_{t_1,t_2}[k_5]), S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_5], \hat{\sigma}_{t_1,t_2}[k_5]))$. This procedure stops after finitely many steps with the $leftmost(t_2)$. Hence $t_2 = {t_1}^{length(Lp(t_2))}_{leftmost(t_2)}$. But the $leftmost(t_2)$ must be x_1. Hence $t_2 = t_1^{length(Lp(t_2))}$.

The cases (ii), (iii), (iv), (v) and (vi) can be proved in the same manner.

(vii) Assume that $x_1, x_2 \notin var(t_1)$. From $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, thus $t_2 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4]) = t_1$.

Conversely, we can check that all these generalized hypersubstitutions which satisfy the conditions are idempotent by using Lemma 5.2.7. ∎

If $firstops(t_1) = firstops(t_2) = g$ we have a similar result:

Proposition 5.2.17. *Let* σ_{t_1,t_2} *be a generalized hypersubstitution of type* $\tau = (2,2)$, $op(t_1) > 1$, $op(t_2) > 1$ *and* $t_1 = g(k_1, k_2)$, $t_2 = g(k_3, k_4)$, *then* σ_{t_1,t_2} *is idempotent if and only if* ($x_1 \notin var(t_1)$ *or* $x_2 \notin var(t_1)$) *and* ($x_1 \notin var(t_2)$ *or* $x_2 \notin var(t_2)$) *and the following conditions hold:*

(i) *If* $t_1, t_2 \in W^G_{(2,2)}(\{x_1\})$, *then* $t_2 = g(x_1, k_4)$ *and* $t_1 = t_2^{length(Lp(t_1))}$.

(ii) *If* $t_1, t_2 \in W^G_{(2,2)}(\{x_2\})$, *then* $t_2 = g(k_3, x_2)$ *and* $t_1 = t_2^{length(Rp(t_1))}$.

(iii) *If* $t_1 \in W^G_{(2,2)}(\{x_1\})$, $t_2 \in W^G_{(2,2)}(\{x_2\})$, *then* $t_2 = g(k_3, x_2)$ *and* $t_1 = {t_2}_{x_1}{}^{length(Rp(t_1))}$.

(iv) *If* $t_1 \in W^G_{(2,2)}(\{x_2\})$, $t_2 \in W^G_{(2,2)}(\{x_1\})$, *then* $t_2 = g(x_1, k_4)$ *and* $t_1 = {t_2}_{x_2}{}^{length(Lp(t_1))}$.

(v) *If* $t_2 \in W^G_{(2,2)}(\{x_1\})$ *and* $x_1, x_2 \notin var(t_1)$*, then* $t_2 = g(x_1, k_4)$ *and* $t_1 = {t_2}_{x_k}^{length(Lp(t_1))}$ *where* $x_k = leftmost(t_1)$.

(vi) *If* $t_2 \in W^G_{(2,2)}(\{x_2\})$ *and* $x_1, x_2 \notin var(t_1)$*, then* $t_2 = g(k_3, x_2)$ *and* $t_1 = {t_2}_{x_k}^{length(Rp(t_1))}$ *where* $x_k = rightmost(t_1)$.

(vii) *If* $x_1, x_2 \notin var(t_2)$*, then* $t_1 = t_2$.

Proof. The proof is similar to the proof of Proposition 5.2.16. ■

In the last case we have:

Proposition 5.2.18. *Let* σ_{t_1,t_2} *be a generalized hypersubstitution of type* $\tau = (2, 2)$, $op(t_1) > 1$, $op(t_2) > 1$ *and* $t_1 = f(k_1, k_2)$, $t_2 = g(k_3, k_4)$*, then* σ_{t_1,t_2} *is idempotent if and only if* ($x_1 \notin var(t_1)$ *or* $x_2 \notin var(t_1)$) *and* ($x_1 \notin var(t_2)$ *or* $x_2 \notin var(t_2)$) *and the following conditions hold:*

(i) *If* $t_1 \in W^G_{(2,2)}(\{x_1\})$*, then* $t_1 = f(x_1, k_2)$.

(ii) *If* $t_1 \in W^G_{(2,2)}(\{x_2\})$*, then* $t_1 = f(k_1, x_2)$.

(iii) *If* $t_2 \in W^G_{(2,2)}(\{x_1\})$*, then* $t_2 = g(x_1, k_4)$.

(iv) *If* $t_2 \in W^G_{(2,2)}(\{x_2\})$*, then* $t_2 = g(k_3, x_2)$.

Proof. Assume that σ_{t_1,t_2} is idempotent. By Lemma 5.2.15, we get ($x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$) and ($x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$). Since $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$, thus we obtain the equations $t_1 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$ and $t_2 = S^2(t_2, \hat{\sigma}_{t_1,t_2}[k_3], \hat{\sigma}_{t_1,t_2}[k_4])$.

(i) Assume that $t_1 \in W^G_{(2,2)}(\{x_1\})$. From $t_1 = S^2(t_1, \hat{\sigma}_{t_1,t_2}[k_1], \hat{\sigma}_{t_1,t_2}[k_2])$, we get $\hat{\sigma}_{t_1,t_2}[k_1] = x_1$. Since $op(t_1) > 1$, $op(t_2) > 1$, thus $k_1 = x_1$. Hence $t_1 = f(x_1, k_2)$.

The cases (ii), (iii) and (iv) can be proved in the same manner.

Conversely, we can check that all these generalized hypersubstitutions which satisfy the conditions are idempotent by using Lemma 5.2.7. ■