Chapter 6

Monoids of Generalized Hypersubstitutions of Type $\tau = (n)$

In this chapter, we study the semigroup properties of $Hyp_G(n)$. In particular, we characterize the idempotent and regular elements of this monoid and describe some classes under Green's relations of this monoid.

6.1 Projection and Dual Generalized Hypersubstitutions of Type $\tau = (n)$

We assume that from now on the type $\tau = (n)$, for some $n \in \mathbb{N}$, i.e. we have only one n-ary operation symbol, say f. By σ_t we denote the generalized hypersubstitution which maps f to the term t in $W_{(n)}(X)$. A generalized hypersubstitution σ_t of type $\tau = (n)$ is called a *projection generalized hypersubstitution* if t is a variable. We denoted the set of all projection generalized hypersubstitutions of type $\tau = (n)$ by $P_G(n)$, i.e. $P_G(n) = \{\sigma_{x_i} | x_i \in X\}$.

Lemma 6.1.1. *For any $\sigma_t \in Hyp_G(n)$ and $\sigma_{x_i} \in P_G(n)$, we have*

(i) $\sigma_t \circ_G \sigma_{x_i} = \sigma_{x_i}$.

(ii) $\sigma_{x_i} \circ_G \sigma_t \in P_G(n)$ $(\hat{\sigma}_{x_i}[t] \in X)$.

Proof. The proof is similar to the proof of Lemma 4.1.4. ■

Corollary 6.1.2. *The following statements hold:*

(i) *$P_G(n)$ is the smallest two-sided ideal of $Hyp_G(n)$, called the kernel of $Hyp_G(n)$. Thus, $Hyp_G(n)$ is not simple.*

(ii) *$P_G(n)$ is the set of all right-zero elements of $Hyp_G(n)$, so that $P_G(n)$ itself is a right-zero band.*

(iii) *$Hyp_G(n)$ contains no left-zero elements.*

Proof. These follow immediately from Lemma 6.1.1. ■

Another special kind of generalized hypersubstitutions in $Hyp_G(n)$ are *dual generalized hypersubstitutions*, which are defined using permutations of the set $J := \{1, \ldots, n\}$. For any such permutation π, we let $\sigma_\pi = \sigma_{f(x_{\pi(1)},\ldots,x_{\pi(n)})}$. We let D_G be the set of all such dual generalized hypersubstitutions.

Lemma 6.1.3. *The following statements hold:*

(i) *For any two permutations π and $\rho, \sigma_\rho \circ_G \sigma_\pi = \sigma_{\pi\circ\rho}$.*

(ii) *For any permutation π with the inverse permutation π^{-1}, the generalized hypersubstitutions σ_π and $\sigma_{\pi^{-1}}$ are inverse of each other.*

Proof. (i) We have $(\sigma_\rho \circ_G \sigma_\pi)(f) = \hat{\sigma}_\rho[f(x_{\pi(1)}, \ldots, x_{\pi(n)})] = S^n(f(x_{\rho(1)}, \ldots, x_{\rho(n)}), x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_{\pi(\rho(1))}, \ldots, x_{\pi(\rho(n))}) = \sigma_{\pi\circ\rho}(f)$.

(ii) This follows from (i). ■

Lemma 6.1.4. *If $\sigma \circ_G \rho \in D_G$, then both σ and ρ are in D_G.*

Proof. Let $\sigma(f) = f(u_1, \ldots, u_n)$ and $\rho(f) = f(v_1, \ldots, v_n)$. Since $\sigma \circ_G \rho \in D_G$, thus there exists a permutation π such that $(\sigma \circ_G \rho)(f) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$. So $f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = (\sigma \circ_G \rho)(f) = S^n(f(u_1, \ldots, u_n), \hat{\sigma}[v_1], \ldots, \hat{\sigma}[v_n])$. Since π is a permutation, thus this forces all the u_i's to be distinct variables in X_n, and all the v_i's to be distinct variables in X_n. It follows that both σ and ρ are in D_G. ■

Corollary 6.1.5. *D_G is a submonoid of $Hyp_G(n)$ which forms a group, and no other elements of $Hyp_G(n)$ have inverses in $Hyp_G(n)$. Thus, D_G is a maximal subgroup of $Hyp_G(n)$.*

Proof. These follow immediately from Lemma 6.1.3 and 6.1.4. ■

Lemma 6.1.6. *Let F be the set of generalized hypersubstitutions of the form $\sigma_{f(x_i,\ldots,x_i)}$ for $i \in \mathbb{N}$. Let $M = P_G(n) \cup D_G \cup F$. Then M is a submonoid of $Hyp_G(n)$.*

Proof. It is straightforward to check that any product of two elements in M is also in M. ■

6.2 Idempotent and Regular Elements in $Hyp_G(n)$

All idempotent elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ were studied by W. Puninagool and S. Leeratanavalee in Chapter 4 and [32] and all regular elements of the monoid of all generalized hypersubstitutions of type $\tau = (2)$ were studied by W. Puninagool and S. Leeratanavalee in Chapter 4 and [27]. In this section, we characterize the idempotent and regular elements of $Hyp_G(n)$.

Proposition 6.2.1. *Let σ_t be a generalized hypersubstitution of type $\tau = (n)$. Then σ_t is idempotent if and only if $\hat{\sigma}_t[t] = t$.*

Proof. The proof is similar to the proof of Proposition 4.1.1 ■

Proposition 6.2.2. *For every $x_i \in X$, σ_{x_i} and σ_{id} are idempotent.*

Proof. The proof is similar to the proof of Proposition 4.1.2 ■

We let $G(n) := \{\sigma_t | t \notin X, var(t) \cap X_n = \emptyset\}$.

Proposition 6.2.3. *If $\sigma_t \in G(n)$ and $\sigma_s \in Hyp_G(n) \setminus P_G(n)$, then $\sigma_t \circ_G \sigma_s = \sigma_t$, i.e. $G(n)$ itself is a left zero band.*

Proof. The proof is similar to the proof of Lemma 4.1.4 (iii). ■

Then we consider only the case $\sigma_t \in Hyp_G(n) \setminus P_G(n)$ and $var(t) \cap X_n \neq \emptyset$.

Theorem 6.2.4. *Let $t = f(t_1, \ldots, t_n) \in W_{(n)}(X)$ and $\emptyset \neq var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}$. Then σ_t is idempotent if and only $t_{i_k} = x_{i_k}$ for all $k \in \{1, \ldots, m\}$.*

Proof. Assume that $\sigma_{f(t_1,\ldots,t_n)}$ is idempotent. Then $S^n(f(t_1, \ldots, t_n), \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_1], \ldots, \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_n]) = \sigma^2_{f(t_1,\ldots,t_n)}(f) = \sigma_{f(t_1,\ldots,t_n)}(f) = f(t_1, \ldots, t_n)$. Suppose that there exists $k \in \{1, \ldots, m\}$ such that $t_{i_k} \neq x_{i_k}$. If $t_{i_k} \in X$, then $\hat{\sigma}_{f(t_1,\ldots,t_n)}[t_{i_k}] = t_{i_k} \neq x_{i_k}$. So $S^n(f(t_1, \ldots, t_n), \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_1], \ldots, \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_n]) \neq f(t_1, \ldots, t_n)$ and we have a contradiction. If $t_{i_k} \notin X$, then $\hat{\sigma}_{f(t_1,\ldots,t_n)}[t_{i_k}] \notin X$. We obtain $op(t) = op(S^n(f(t_1, \ldots, t_n), \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_1], \ldots, \hat{\sigma}_{f(t_1,\ldots,t_n)}[t_n])) > op(t)$. This is a contradiction.

The proof of the converse direction is straightforward. ■

Now, we characterize the generalized hypersubstitutions of type $\tau = (n)$ which are regular.

Lemma 6.2.5. *Let $t \in W_{(n)}(X)$ and $\emptyset \neq var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}$ and let $a = f(a_1, \ldots, a_n) \in W_{(n)}(X)$. If $\hat{\sigma}_t[a] = t$, then $a_l = x_l$ for all $l = i_1, \ldots, i_m$.*

Proof. Assume that $\hat{\sigma}_t[a] = t$. Then $t = \hat{\sigma}_t[a] = S^n(t, \hat{\sigma}_t[a_1], \ldots, \hat{\sigma}_t[a_n])$. We will show that $a_l = x_l$ for all $l = i_1, \ldots, i_m$. Suppose that there exists $j \in \{i_1, \ldots, i_m\}$ such that $a_j \neq x_j$. If $a_j = x_k \in X$ where $x_k \neq x_j$, then $\hat{\sigma}_t[a_j] = x_k$. It follows that $t \neq S^n(t, \hat{\sigma}_t[a_1], \ldots, \hat{\sigma}_t[a_n])$. This is a contradiction. If $a_j \notin X$, then $\hat{\sigma}_t[a_j] \notin X$. It follows that $op(t) = op(\hat{\sigma}_t[a]) = S^n(t, \hat{\sigma}_t[a_1], \ldots, \hat{\sigma}_t[a_n])) > op(t)$ and we have a contradiction. ∎

Theorem 6.2.6. *Let $t = f(t_1, \ldots, t_n) \in W_{(n)}(X)$ and $\emptyset \neq var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}$. Then σ_t is regular if and only if there exist $j_1, \ldots, j_m \in \{1, \ldots, n\}$ such that $t_{j_1} = x_{i_1}, \ldots, t_{j_m} = x_{i_m}$.*

Proof. Assume that σ_t is regular. Then there exists $\sigma_s \in Hyp_G(n)$ such that $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. Since $t \notin X$, thus $s \notin X$. Then $s = f(s_1, \ldots, s_n)$ for some $s_1, \ldots, s_n \in W_{(n)}(X)$. From $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$, thus $\hat{\sigma}_t[\hat{\sigma}_s[t]] = t$. By Lemma 6.2.5, $\hat{\sigma}_s[t] = f(u_1, \ldots, u_n)$ for some $u_1, \ldots, u_n \in W_{(n)}(X)$ where $u_{i_1} = x_{i_1}, \ldots, u_{i_m} = x_{i_m}$. From $\hat{\sigma}_s[t] = f(u_1, \ldots, u_n)$, thus $S^n(f(s_1, \ldots, s_n), \hat{\sigma}_s[t_1], \ldots, \hat{\sigma}_s[t_n]) = f(u_1, \ldots, u_n)$. Since $u_{i_1} = x_{i_1}, \ldots, u_{i_m} = x_{i_m}$ thus $s_{i_1}, \ldots, s_{i_m} \in X_n$. Let $s_{i_1} = x_{j_1}, \ldots, s_{i_m} = x_{j_m}$. Hence $t_{j_1} = x_{i_1}, \ldots, t_{j_m} = x_{i_m}$. Conversely, assume the condition holds. Let $s = f(s_1, \ldots, s_n) \in W_{(n)}(X)$ where $s_1, \ldots, s_n \in W_{(n)}(X)$ such that $s_{i_1} = x_{j_1}, \ldots, s_{i_m} = x_{j_m}$. Then $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma}_t[\hat{\sigma}_s[t]] = \hat{\sigma}_t[S^n(f(s_1, \ldots, s_n), \hat{\sigma}_s[t_1], \ldots, \hat{\sigma}_s[t_n])] = \hat{\sigma}_t[f(k_1, \ldots, k_n)]$(where $k_{i_1} = x_{i_1}, \ldots, k_{i_m} = x_{i_m}$) $= S^n(t, \hat{\sigma}_t[k_1], \ldots, \hat{\sigma}_t[k_n]) = t$. Hence $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. ∎

6.3 Term Properties of the Composition Operation

We need to know more about the result of the composing two generalized hypersubstitutions in $Hyp_G(n)$. In particular, we want to know how long the term corresponding to $\sigma_s \circ_G \sigma_t$ is and what variables it uses, compared to the lengths of the terms s and t and the variables they use. We begin with the necessary definition.

Definition 6.3.1. Let $t \in W_{(n)}(X)$. We define some new terms, related to t, as follows. Recall that $J := \{1, \ldots, n\}$.

(i) Let α be any function from J to J. $C_\alpha[t]$ is the term formed from t by replacing each occurrence in t of a variable $x_i \in X_n$ by the variable $x_{\alpha(i)}$ i.e, $C_\alpha[t] =$

$S^n(t, x_{\alpha(1)}, \ldots, x_{\alpha(n)})$.

(ii) Let π be any permutation of J. $\pi[t]$ is the term defined inductively by $\pi[x_i] = x_i$ for any variable x_i, and $\pi[f(u_1, \ldots, u_n)] = f(\pi[u_{\pi(1)}], \ldots, \pi[u_{\pi(n)}])$.

The previous length results for the type $\tau = (2)$ were found by W. Puninagool and S. Leeratanavalee in Chapter 4 and [29] and S.L. Wismath in [34]. The next lemmas show how these results can be generalized to the type $\tau = (n)$.

Lemma 6.3.2. *Let $n \in \mathbb{N}$ with $n > 1$ and $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_w$. Then w is a longer term than $f(u_1, \ldots, u_n)$, unless the terms $f(u_1, \ldots, u_n)$ and $f(v_1, \ldots, v_n)$ satisfy the following condition* (Q):

(Q) *If a variable $x_i \in X_n$ is used anywhere in the term $f(u_1, \ldots, u_n)$, then the entry v_i in $f(v_1, \ldots, v_n)$ is a variable.*

Proof. If $var(f(u_1, \ldots, u_n)) \cap X_n = \emptyset$, then $f(u_1, \ldots, u_n)$ and $f(v_1, \ldots, v_n)$ satisfy the condition (Q). Let $var(f(u_1, \ldots, u_n)) \cap X_n = \{x_{i_1}, \ldots, x_{i_k}\}$. If $v_{i_j} \in X$ for all $j \in \{1, \ldots, k\}$, then $f(u_1, \ldots, u_n)$ and $f(v_1, \ldots, v_n)$ satisfy the condition (Q). If there exists $j \in \{1, \ldots, k\}$ where $v_{i_j} \notin X$, then $\hat{\sigma}_{f(u_1,\ldots,u_n)}[v_{i_j}] \notin X$. Since $n > 1$ and $\hat{\sigma}_{f(u_1,\ldots,u_n)}[v_{i_j}] \notin X$, thus $vb(\hat{\sigma}_{f(u_1,\ldots,u_n)}[v_{i_j}]) > 1$. So $vb(w) = vb(S^n(f(u_1, \ldots, u_n), \hat{\sigma}_{f(u_1,\ldots,u_n)}[v_1], \ldots, \hat{\sigma}_{f(u_1,\ldots,u_n)}[v_n])) > vb(f(u_1, \ldots, u_n))$. ■

Lemma 6.3.3. *Let $\sigma_t \in Hyp_G(n) \setminus P_G(n)$ and $x_1, \ldots, x_n \in var(t)$. Then for any $s \in W_{(n)}(X), vb(\hat{\sigma}_t[s]) \geq vb(s)$.*

Proof. We will prove by induction on the complexity of the term s. If $s \in X$, then $vb(\hat{\sigma}_t[s]) = vb(s)$. Assume that $s = f(u_1, \ldots, u_n)$ and $vb(\hat{\sigma}_t[u_i]) \geq vb(u_i)$ for all $1 \leq i \leq n$. Then $vb(\hat{\sigma}_t[s]) = vb(S^n(t, \hat{\sigma}_t[u_1], \ldots, \hat{\sigma}_t[u_n])) \geq vb(f(u_1, \ldots, u_n))$ since $x_1, \ldots, x_n \in var(t)$ and $vb(\hat{\sigma}_t[u_i]) \geq vb(u_i)$ for all $1 \leq i \leq n$. ■

Lemma 6.3.4. *Let $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_w$ where $vb(f(u_1, \ldots, u_n)) > n$. If $x_1, \ldots, x_n \in var(f(u_1, \ldots, u_n))$, then w is a longer term than $f(v_1, \ldots, v_n)$.*

Proof. We write $\sigma = \sigma_{f(u_1,\ldots,u_n)}$. From $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_w$, thus we get $w = S^n(f(u_1, \ldots, u_n), \hat{\sigma}[v_1], \ldots, \hat{\sigma}[v_n])$. Since $x_1, \ldots, x_n \in var(f(u_1, \ldots, u_n))$, thus $\hat{\sigma}[v_i]$ is used in w for all $1 \leq i \leq n$. We will prove by induction on the complexity of the term $f(v_1, \ldots, v_n)$. If $v_1, \ldots, v_n \in X$, then $vb(w) = vb(f(u_1, \ldots, u_n)) > n =$

$vb(f(v_1,\ldots,v_n))$. Assume that the claim holds for any term of length not less than n but less than k, and $f(v_1,\ldots,v_n)$ has length k. Since $vb(f(v_1,\ldots,v_n)) = k > n$, thus there exists $i \in \{1,\ldots,n\}$ such that $vb(v_i) \geq n$. By induction, we get $vb(\hat{\sigma}[v_i]) > vb(v_i)$. By Lemma 6.3.3, we get any other v_j has $vb(\hat{\sigma}[v_j]) \geq vb(v_j)$. Since all the $\hat{\sigma}[v_i]$ are used in w for all $1 \leq i \leq n$, thus w is longer than $f(v_1,\ldots,v_n)$. ■

Lemma 6.3.5. *Let $\sigma_s, \sigma_t \in Hyp_G(n)$.*

(i) $var((\sigma_s \circ_G \sigma_t)(f)) \cap X_n \subseteq var(t) \cap X_n$.

(ii) *If s uses only one variable, then the term for $\sigma_s \circ_G \sigma_t$ uses only one variable (not necessarily the same variable as s).*

Proof. We will prove by induction on the complexity of the term t.

(i) If $t \in X$, then $(\sigma_s \circ_G \sigma_t)(f) = t$. So $var((\sigma_s \circ_G \sigma_t)(f)) \cap X_n \subseteq var(t) \cap X_n$. Assume that $t = f(t_1,\ldots,t_n)$ and $var(\hat{\sigma}_s[t_i]) \cap X_n \subseteq var(t_i) \cap X_n$ for all $1 \leq i \leq n$. So $var((\sigma_s \circ_G \sigma_t)(f)) \cap X_n = var(S^n(s, \hat{\sigma}_s[t_1],\ldots,\hat{\sigma}_s[t_n])) \cap X_n \subseteq \bigcup_{i=1}^{n}(var(\hat{\sigma}_s[t_i])) \cap X_n = \bigcup_{i=1}^{n}(var(\hat{\sigma}_s[t_i]) \cap X_n) \subseteq \bigcup_{i=1}^{n}(var(t_i) \cap X_n) = \bigcup_{i=1}^{n} var(t_i) \cap X_n = var(t) \cap X_n$.

(ii) If $t \in X$, then $(\sigma_s \circ_G \sigma_t)(f) = t$. So the term for $\sigma_s \circ_G \sigma_t$ uses only one variable. Assume that $t = f(t_1,\ldots,t_n)$ and $\hat{\sigma}_s[t_i]$ uses only one variable for all $1 \leq i \leq n$. So $(\sigma_s \circ_G \sigma_t)(f) = S^n(s, \hat{\sigma}_s[t_1],\ldots,\hat{\sigma}_s[t_n])$. If $var(s) = \{x_i\}$ for some $x_i \in X_n$, then $var(\sigma_s \circ_G \sigma_t)(f) = var(\hat{\sigma}_s[t_i])$. If $var(s) = \{x_i\}$ where $i > n$, then $var((\sigma_s \circ_G \sigma_t)(f)) = var(s)$. ■

We conclude this section by extending the results from [34] to the case of $Hyp_G(n)$ on properties of the composition operation with a lemma describing the special role of the terms $\pi[t]$ and $C_{\alpha[t]}$ from Definition 6.3.1.

Lemma 6.3.6. *For $t \in W_{(n)}(X)$.*

(i) *Let π be any permutation on J. Then $\sigma_\pi \circ_G \sigma_t = \sigma_{\pi[t]}$.*

(ii) *Let α be any function on J. Define the generalized hypersubstitution σ_α by mapping the fundemental f to the term $f(x_{\alpha(1)},\ldots,x_{\alpha(n)})$. Then $\sigma_t \circ_G \sigma_\alpha = \sigma_{C_{\alpha[t]}}$.*

Proof. (i) We will prove by induction on the complexity of the term t. If $t \in X$, then by Lemma 6.1.1 (i), $\sigma_\pi \circ_G \sigma_t = \sigma_t = \sigma_{\pi[t]}$. Assume that $t = f(t_1,\ldots,t_n)$ and $\hat{\sigma}_\pi[t_i] =$

$\pi[t_i]$ for all $1 \leq i \leq n$. So $(\sigma_\pi \circ_G \sigma_t)(f) = S^n(f(x_{\pi(1)}, \ldots, x_{\pi(n)}), \hat{\sigma}_\pi[t_1], \ldots, \hat{\sigma}_\pi[t_n]) = f(\hat{\sigma}_\pi[t_{\pi(1)}], \ldots, \hat{\sigma}_\pi[t_{\pi(n)}]) = f(\pi[t_{\pi(1)}], \ldots, \pi[t_{\pi(n)}]) = \pi[f(t_1, \ldots, t_n)] = \pi[t]$.

(ii) We have $(\sigma_t \circ_G \sigma_\alpha)(f) = S^n(t, x_{\alpha(1)}, \ldots, x_{\alpha(n)}) = C_\alpha[t]$. So $\sigma_t \circ_G \sigma_\alpha = \sigma_{C_{\alpha[t]}}$. ■

6.4 Green's Ralations on $Hyp_G(n)$

Green's relations on $Hyp(n)$ have been studied by S.L. Wismath [34], and Green's relations on $Hyp_G(2)$ were study by W. Puninagool and S. Leeratanavalee in Chapter 4 and [29]. In this section, we describe some classes of the monoid of generalized hypersubstitutions of type $\tau = (n)$ with $n > 1$.

Proposition 6.4.1. *Any $\sigma_{x_i} \in P_G(n)$ is $\mathcal{L}$-related only to itself, but is $\mathcal{R}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $P_G(n)$, and not related to any other generalized hypersubstitutions. Moreover, the set $P_G(n)$ forms an $\mathcal{R}$-, $\mathcal{D}$- and $\mathcal{J}$-class.*

Proof. The proof is similar to the proof of Proposition 4.4.2. ■

Proposition 6.4.2. *Any $\sigma_t \in G(n)$ is $\mathcal{R}$-related only to itself, but is $\mathcal{L}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $G(n)$, and not related to any other generalized hypersubstitutions. Moreover, the set $G(n)$ forms an $\mathcal{L}$-, $\mathcal{D}$- and $\mathcal{J}$-class.*

Proof. The proof is similar to the proof of Proposition 4.4.7. ■

Theorem 6.4.3. *Let $\sigma_s, \sigma_t \in Hyp_G(n)$. Then $\sigma_s \mathcal{R} \sigma_t$ if and only if the following conditions hold:*

(i) *If $s \in X$, then $t \in X$.*

(ii) *If $s \notin X$, then $s = C_{\alpha[t]}$ for some bijection α on J.*

Proof. Assume that $\sigma_s \mathcal{R} \sigma_t$. If $s \in X$, then by Proposition 6.4.1 we get $t \in X$. Let $s \notin X$. Then by Proposition 6.4.1 we get $t \notin X$. Then there exist $p, q \in W_{(n)}(X) \setminus X$ such that $\sigma_s \circ_G \sigma_p = \sigma_t$ and $\sigma_t \circ_G \sigma_q = \sigma_s$. If $var(t) \cap X_n = \emptyset$, then from $\sigma_t \circ_G \sigma_q = \sigma_s$ we get $s = t$ and $s = C_{\alpha[t]}$ for all bijection α on J. Assume that $var(t) \cap X_n \neq \emptyset$. Let $p = f(p_1, \ldots, p_n)$ and $q = f(q_1, \ldots, q_n)$. So we have two equations

$$S^n(s, \hat{\sigma}_s[p_1], \ldots, \hat{\sigma}_s[p_n]) = t \quad (1)$$

$$S^n(t, \hat{\sigma}_t[q_1], \ldots, \hat{\sigma}_t[q_n]) = s \quad (2).$$

Now, if neither of these equations satisfies the condition (Q) of Lemma 6.3.2, we would have the length of the term t is longer than the length of the term s and also the length of s is longer than the length of t, which is clearly impossible. Thus, at least one of two equations must fit the condition (Q). But if one equation fits the condition (Q), Lemma 6.3.2 tells us that s and t have the same length, and therefore, the second equation also fits the condition (Q). By Lemma 6.3.2, if $x_i \in var(t) \cap X_n$, then $q_i \in X$. If such $q_i \notin X_n$, then from (2) we get $q_i \in var(s)$. So $S^n(s, \hat{\sigma}_s[p_1], \ldots, \hat{\sigma}_s[p_n]) \neq t$ which contradicts to (1). Thus such $q_i \in X_n$. Let $\alpha(i) = j$ if $x_i \in var(t) \cap X_n$ and $q_i = x_j$. This defines a partial function on J. It is clear that α is injective. Extending this map to a bijection on J, which we shall also call α. So $s = C_{\alpha[t]}$. Conversely, assume that the conditions hold. Then $s, t \in X$ or $s \notin X$ and $s = C_{\alpha[t]}$ for some bijection α on J. If $s, t \in X$, then by Proposition 6.4.1 we get $\sigma_s \mathcal{R} \sigma_t$. If $s \notin X$ and $s = C_{\alpha[t]}$ for some bijection α on J, then σ_α and $\sigma_{\alpha^{-1}}$ are inverse generalized hypersubstitutions. By Lemma 6.3.6 (ii), we get $\sigma_t \circ_G \sigma_\alpha = \sigma_{C_{\alpha[t]}} = \sigma_s$ and $\sigma_s \circ_G \sigma_{\alpha^{-1}} = \sigma_t$. Thus $\sigma_s \mathcal{R} \sigma_t$. ■

Lemma 6.4.4. *Let $t \in W_{(n)}(X)$ and π be a permutation on J. Then $\pi^{-1}[\pi[t]] = t$.*

Proof. We will prove by induction on the complexity of the term t. If $t \in X$ then $\pi^{-1}[\pi[t]] = \pi^{-1}[t] = t$. Assume that $t = f(t_1, \ldots, t_n)$ and $\pi^{-1}[\pi[t_i]] = t_i$ for all $1 \leq i \leq n$. So

$$\begin{aligned}
\pi^{-1}[\pi[t]] &= \pi^{-1}[\pi[f(t_1, \ldots, t_n)]] \\
&= \pi^{-1}[f(\pi[t_{\pi(1)}], \ldots, \pi[t_{\pi(n)}])] \\
&= f(\pi^{-1}[\pi[t_{\pi(\pi^{-1}(1))}]], \ldots, \pi^{-1}[\pi[t_{\pi(\pi^{-1}(n))}]]) \\
&= f(\pi^{-1}[\pi[t_1]], \ldots, \pi^{-1}[\pi[t_n]]) \\
&= f(t_1, \ldots, t_n) \\
&= t.
\end{aligned}$$

■

Lemma 6.4.5. *Let $\sigma_t \in Hyp_G(n) \setminus P_G(n)$. Then, for any permutation π on J, σ_t is $\mathcal{L}$-related to the generalized hypersubstitution $\sigma_{\pi[t]}$.*

Proof. We know from Lemma 6.3.6(i) that $\sigma_\pi \circ_G \sigma_t = \sigma_{\pi[t]}$. From Lemma 6.3.6(i) and Lemma 6.4.4, we get $\sigma_{\pi^{-1}} \circ_G \sigma_{\pi[t]} = \sigma_{\pi^{-1}[\pi[t]]} = \sigma_t$. So $\sigma_t \mathcal{L} \sigma_{\pi[t]}$. ■

Proposition 6.4.6. *Two idempotent elements σ_s and σ_t in $Hyp_G(n) \setminus P_G(n)$ are $\mathcal{L}$-related if and only if $var(s) \cap X_n = var(t) \cap X_n$.*

Proof. Assume that $\sigma_s \mathcal{L} \sigma_t$. Then there exist $u, v \in W_{(n)}(X)$ such that $\sigma_u \circ_G \sigma_t = \sigma_s$ and $\sigma_v \circ_G \sigma_s = \sigma_t$. By Lemma 6.3.5 (i), we get $var(s) \cap X_n \subseteq var(t) \cap X_n$ and $var(t) \cap X_n \subseteq var(s) \cap X_n$. So $var(s) \cap X_n = var(t) \cap X_n$. Conversely, we use the fact that for any two idempotent elements e and f in any semigroup, $e\mathcal{L}f$ if and only if $ef = e$ and $fe = f$. Since $var(s) \cap X_n = var(t) \cap X_n$, by Theorem 6.2.4 we can prove that $\sigma_t \circ_G \sigma_s = \sigma_t$ and $\sigma_s \circ_G \sigma_t = \sigma_s$. ■

Theorem 6.4.7. *Let σ_t be an idempotent element in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then $L_{\sigma_t} = \{\sigma_{\pi[w]} | \pi$ is a permutation of $J, w \notin X, var(w) \cap X_n = var(t) \cap X_n$ and σ_w is an idempotent element$\}$.*

Proof. Let $\sigma_{\pi[w]} \in Hyp_G(n)$ where π is a permutation of $J, w \notin X, var(w) \cap X_n = var(t) \cap X_n$ and σ_w is an idempotent element. By Proposition 6.4.6, we get $\sigma_w \mathcal{L} \sigma_t$. By Lemma 6.4.5, $\sigma_w \mathcal{L} \sigma_{\pi[w]}$. So $\sigma_{\pi[w]} \mathcal{L} \sigma_t$. Let $t = f(u_1, \ldots, u_n)$ and $s = f(v_1, \ldots, v_n)$ with $\sigma_s \mathcal{L} \sigma_t$. Then there exists $f(b_1, \ldots, b_n) \in W_{(n)}(X)$ such that $\sigma_{f(b_1,\ldots,b_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_{f(u_1,\ldots,u_n)}$. We write $\sigma = \sigma_{f(b_1,\ldots,b_n)}$. From $\sigma_{f(b_1,\ldots,b_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} = \sigma_{f(u_1,\ldots,u_n)}$, we get $S^n(f(b_1, \ldots, b_n), \hat{\sigma}[v_1], \ldots, \hat{\sigma}[v_n]) = f(u_1, \ldots, u_n)$. If $x_i \in var(t) \cap X_n$, then $u_i = x_i$ since σ_t is an idempotent element. So $b_i = x_j$ for some $x_j \in X_n$. This implies $\hat{\sigma}[v_j] = x_i$ and then $v_j = x_i$. Let β be a partial function on J defined by $\beta(i) = j$ if $x_i \in var(t) \cap X_n$ and $v_j = x_i$. If $\beta(i) = \beta(k) = j$, then $v_j = x_i = x_k$. So $i = k$ and β is injective. So β can be extended to a permutation α on J. Let $w = f(p_1, \ldots, p_n)$ where $p_i = x_i$ if $x_i \in var(t) \cap X_n$ and $p_i = \alpha[v_{\alpha(i)}]$ if $x_i \notin var(t) \cap X_n$. We will show that $var(w) \cap X_n = var(t) \cap X_n, \sigma_w$ is an idempotent element and $s = f(v_1, \ldots, v_n) = \pi[w]$ where $\pi = \alpha^{-1}$. We show first that $var(w) \cap X_n = var(t) \cap X_n$. Since $\sigma_s \mathcal{L} \sigma_t$, thus by Proposition 6.4.6, $var(s) \cap X_n = var(t) \cap X_n$. Let $x_j \in var(w) \cap X_n$. Then $x_j \in var(p_i)$ for some $i \in J$ and $x_j \in X_n$. If $p_i = x_i$ where $x_i \in var(t) \cap X_n$, then $x_j = x_i \in var(t)$. If $p_i = \alpha[v_{\alpha(i)}]$, then $x_j \in var(p_i) = var(\alpha[v_{\alpha(i)}]) = var(v_{\alpha(i)}) \subseteq var(s)$. But $var(s) \cap X_n = var(t) \cap X_n$, so $x_j \in var(t)$. Let $x_j \in var(t) \cap X_n$. Then $p_j = x_j$ and so $x_j \in var(s) \cap X_n$. Next, we show that σ_w is an idempotent element. Let $x_i \in var(w) \cap X_n$. Then $x_i \in var(t) \cap X_n$. So $p_i = x_i$. Thus σ_w is an idempotent element. Finally, we show that $s = f(v_1, \ldots, v_n) = \pi[w]$ where $\pi = \alpha^{-1}$. To do this we will show that for all $1 \leq k \leq n$, $v_k = \pi[p_{\pi(k)}]$. Let $1 \leq k \leq n$. If there exists $i \in J$ such that $\beta(i) = k$, then $\alpha(i) = k$ and $\pi(k) = \alpha^{-1}(k) = i$. So $p_i = x_i = v_k$. Thus $\pi[p_{\pi(k)}] = \pi[p_i] = \pi[x_i] = x_i = v_k$. If no such index i exists, then

$\pi[p_{\pi(k)}] = \pi[\alpha[v_{\alpha(\pi(k))}]] = \pi[\alpha[v_{\alpha(\alpha^{-1}(k))}]] = \pi[\alpha[v_k]] = \alpha^{-1}[\alpha[v_k]] = v_k.$ ∎

Corollary 6.4.8. *Let σ_t be an idempotent element in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then $D_{\sigma_t} = \{\sigma_w | w = C_{\alpha[\pi[s]]}$ for some α bijection on J, π a permutation on $J, s \notin X$, and σ_s an idempotent element with $var(s) \cap X_n = var(t) \cap X_n\}$.*

Proof. Put $K = \{\sigma_w | w = C_{\alpha[\pi[s]]}$ for some α bijection on J, π a permutation on $J, s \notin X$, and σ_s an idempotent element with $var(s) \cap X_n = var(t) \cap X_n\}$. Assume that $\sigma_s \in D_{\sigma_t}$. Then there exists $\sigma_u \in Hyp_G(n)$ such that $\sigma_s \mathcal{R} \sigma_u$ and $\sigma_u \mathcal{L} \sigma_t$. By Theorem 6.4.7, we get $u = \pi[v]$ for some a permutation π on J, an idempotent element σ_v with $v \notin X$ and $var(v) \cap X_n = var(t) \cap X_n$. By Theorem 6.4.3, we get $s = C_\alpha[u]$ for some a bijection α on J. So $\sigma_s \in K$. Assume that $\sigma_{C_{\alpha[\pi[s]]}} \in K$. By Theorem 6.4.3, we get $\sigma_{C_{\alpha[\pi[s]]}} \mathcal{R} \sigma_{\pi[s]}$. By Theorem 6.4.7, we get $\sigma_{\pi[s]} \mathcal{L} \sigma_t$. So $\sigma_{C_{\alpha[\pi[s]]}} \in D_{\sigma_t}$. ∎

Theorem 6.4.9. *Let σ_t be an idempotent element in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then its $\mathcal{J}$-class is equal to its $\mathcal{D}$-class.*

Proof. Let $t = f(u_1, \ldots, u_n)$ and let c be the number of distinct variables in X_n which occur in t. Let $s = f(v_1, \ldots, v_n)$ with $\sigma_s \mathcal{J} \sigma_t$. Then there exist $f(a_1, \ldots, a_n)$, $f(b_1, \ldots, b_n), f(p_1, \ldots, p_n), f(r_1, \ldots, r_n) \in W_{(n)}(X)$ such that

$$\sigma_{f(a_1,\ldots,a_n)} \circ_G \sigma_{f(v_1,\ldots,v_n)} \circ_G \sigma_{f(b_1,\ldots,b_n)} = \sigma_{f(u_1,\ldots,u_n)} \qquad (1)$$

$$\sigma_{f(p_1,\ldots,p_n)} \circ_G \sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(r_1,\ldots,r_n)} = \sigma_{f(v_1,\ldots,v_n)}. \qquad (2).$$

Let $f(q_1, \ldots, q_n)$ be the term for $\sigma_{f(v_1,\ldots,v_n)} \circ_G \sigma_{f(b_1,\ldots,b_n)}$. We write $\sigma = \sigma_{f(a_1,\ldots,a_n)}$. From (1), we get $S^n(f(a_1, \ldots, a_n), \hat{\sigma}[q_1], \ldots, \hat{\sigma}[q_n]) = f(u_1, \ldots, u_n)$. If $x_k \in var(t) \cap X_n$, then $u_k = x_k$ since σ_t is an idempotent element. So $a_k = x_j$ for some $x_j \in X_n$. This implies $\hat{\sigma}[q_j] = x_k$ and then $q_j = x_k$. Let α be a function from $J(t)$ to J defined by $\alpha(k) = j$ if $x_k \in var(t) \cap X_n$ and $a_k = x_j$ where $J(t) = \{k \in J | x_k \in var(t)\}$. So α can be extended to a permutation on J. We write $\sigma_1 = \sigma_{f(v_1,\ldots,v_n)}$. Since $f(q_1, \ldots, q_n)$ is the term for $\sigma_{f(v_1,\ldots,v_n)} \circ_G \sigma_{f(b_1,\ldots,b_n)}$, thus $S^n(f(v_1, \ldots, v_n), \hat{\sigma}_1[b_1], \ldots, \hat{\sigma}_1[b_n]) = f(q_1, \ldots, q_n)$. For each $k \in J(t), q_{\alpha(k)} = x_k$. So $v_{\alpha(k)} = x_l$ for some $x_l \in X_n$. So $\hat{\sigma}_1[b_l] = x_k$ and then $b_l = x_k$. Let $\beta : \alpha(J(t)) \to J$ defined by $\beta(\alpha(k)) = l$ where $k \in J(t)$ and $v_{\alpha(k)} = x_l$. So β can be extended to a permutation on J. Since α and β are injective, thus at least c distinct variables in X_n occur as v_i in entries of $s = f(v_1 \ldots, v_n)$. We claim that the only variables in X_n which occur in s are those c variables. Let $f(c_1, \ldots, c_n)$ be the term for $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(r_1,\ldots,r_n)}$. We write $\sigma_2 = \sigma_{f(p_1,\ldots,p_n)}$. From (2), we get

$S^n(f(p_1,\ldots,p_n),\hat{\sigma}_2[c_1],\ldots,\hat{\sigma}_2[c_n]) = f(v_1,\ldots,v_n)$. Since at least c distinct variables in X_n occur as v_i in entries $s = f(v_1\ldots,v_n)$, thus at least c distinct variables in X_n occur as p_i in entries $s = f(p_1\ldots,p_n)$ and then at least c distinct variables in X_n occur as c_i in entries $f(c_1\ldots,c_n)$. We write $\sigma_3 = \sigma_{f(u_1,\ldots,u_n)}$. Since $f(c_1,\ldots,c_n)$ is the term for $\sigma_{f(u_1,\ldots,u_n)} \circ_G \sigma_{f(r_1,\ldots,r_n)}$, thus $S^n(f(u_1,\ldots,u_n),\hat{\sigma}_3[r_1],\ldots,\hat{\sigma}_3[r_n]) = f(c_1,\ldots,c_n)$. But $f(u_1,\ldots,u_n)$ has only c distinct variables in X_n. Thus all the $r_j's$ used in the composition in (2) are variables in X_n. So the number of distinct variables in X_n which occur in $f(v_1,\ldots,v_n)$ is at most c. Thus the number of distinct variables in X_n which occur in $f(v_1,\ldots,v_n)$ is c and every variable in X_n which occurs in it occurs as a v_i. Let $w_1 = C_{(\beta\circ\alpha)^{-1}}[f(v_1,\ldots,v_n)]$. So $var(w_1)\cap X_n = var(t)\cap X_n$. From Theorem 6.4.3, we get $\sigma_{w_1}\mathcal{R}\sigma_s$. Let $w_2 = \alpha[w_1]$. From Lemma 6.4.5, $\sigma_{w_1}\mathcal{L}\sigma_{w_2}$. We will show that σ_{w_2} is an idempotent element. Let $w_1 = C_{(\beta\circ\alpha)^{-1}}[f(v_1,\ldots,v_n)] = f(d_1,\ldots,d_n)$. For each $x_k \in var(t)\cap X_n$, $v_{\alpha(k)} = x_{\beta(\alpha(k))}$. So $d_{\alpha(k)} = x_k$. From $w_2 = \alpha[w_1]$, we get $w_2 = \alpha[f(d_1,\ldots,d_n)] = f(\alpha[d_{\alpha(1)}],\ldots,\alpha[d_{\alpha(n)}])$ and $var(w_2)\cap X_n = var(t)\cap X_n$. Let $x_j \in var(w_2)\cap X_n$. Then $x_j \in var(t)\cap X_n$. So $\alpha[d_{\alpha(j)}] = \alpha[x_j] = x_j$. So σ_{w_2} is an idempotent element. By Proposition 6.4.6, we get $\sigma_{w_2}\mathcal{L}\sigma_t$. So $\sigma_{w_1}\mathcal{L}\sigma_t$. Thus $\sigma_s\mathcal{D}\sigma_t$. ■

Corollary 6.4.10. *Let σ_s,σ_t be idempotent elements in $Hyp_G(n)\setminus(P_G(n)\cup G(n))$. Then σ_s and σ_t are $\mathcal{J}$- or $\mathcal{D}$-related if and only if the number of distinct variables in X_n which occur in s and t are equal.*

Proof. One direction follows immediately from Corollary 6.4.8. Conversely, let $s = f(u_1,\ldots,u_n), t = f(v_1,\ldots,v_n), var(s)\cap X_n = \{x_{k_1},\ldots,x_{k_c}\}$ and $var(t)\cap X_n = \{x_{l_1},\ldots,x_{l_c}\}$. Since σ_s,σ_t are idempotent elements, thus $u_{k_j} = x_{k_j}$ and $v_{l_j} = x_{l_j}$ for all $1\le j\le c$. Let $s' = f(u'_1,\ldots,u'_n), t' = f(v'_1,\ldots,v'_n)$ where $u'_{k_j} = x_{k_j}$ and $v'_{l_j} = x_{l_j}$ for all $1\le j\le c$ and other $u'_j = x_{k_1}, v'_j = x_{l_1}$. By Proposition 6.4.6, we get $\sigma_s\mathcal{L}\sigma_{s'}$ and $\sigma_t\mathcal{L}\sigma_{t'}$. Let $\pi(l_j) = k_j$ for all $1\le j\le c$. Then π is injective. So π can be extended to a permutation on J, which we will also call π. So $\pi[s'] = f(p_1,\ldots,p_n)$ where $p_{l_j} = x_{k_j}$ for all $1\le j\le c$ and other $p_j = x_{k_1}$. Let $\alpha(k_j) = l_j$ for all $1\le j\le c$. So α can be extended to a bijection on J, which we will also call α. So $C_{\alpha[\pi[s']]} = t'$. Thus $\sigma_s\mathcal{J}\sigma_{s'}\mathcal{J}\sigma_{C_{\alpha[\pi[s']]}=t'}\mathcal{J}\sigma_t$. ■