Chapter 7

Conclusion

In this study, we found that

7.1 Complexity of Terms, Generalized Superpositions and Generalized Hypersubstitutions

(1) Let $s, t_1, \ldots, t_m \in W_\tau(X)$. Then,

(i) $mindepth(S^m(s, t_1, \ldots, t_m)) = min\{mindepth_j(s) + mindepth(t_j), mindepth_k(s) | 1 \leq j \leq m, k > m, x_j, x_k \in var(s)\}$.

(ii) $maxdepth(S^m(s, t_1, \ldots, t_m)) = max\{maxdepth_j(s) + maxdepth(t_j), maxdepth_k(s) | 1 \leq j \leq m, k > m, x_j, x_k \in var(s)\}$.

(iii) $vb(S^m(s, t_1, \ldots, t_m)) = \sum_{j=1}^{m} vb_j(s) vb(t_j) + \sum_{j>m} vb_j(s)$.

(iv) $op(S^m(s, t_1, \ldots, t_m)) = \sum_{j=1}^{m} vb_j(s) op(t_j) + op(s)$.

(2) Let $\tau = (n_i)_{i \in I}$ be a type and let t be a compound term of the form $t = f_i(t_1, \ldots, t_{n_i})$ where f_i is an n_i-ary operation symbol. Let σ be a generalized hypersubstitution of type τ. Then,

(i) $mindepth(\hat{\sigma}[t]) = min\{mindepth_j(\sigma(f_i)) + mindepth(\hat{\sigma}[t_j]), mindepth_k(\sigma(f_i)) | 1 \leq j \leq n_i, k > n_i, x_j, x_k \in var(\sigma(f_i))\}$.

(ii) $maxdepth(\hat{\sigma}[t]) = max\{maxdepth_j(\sigma(f_i)) + maxdepth(\hat{\sigma}[t_j]), maxdepth_k(\sigma(f_i)) | 1 \leq j \leq n_i, k > n_i, x_j, x_k \in var(\sigma(f_i))\}$.

(iii) $vb(\hat{\sigma}[t]) = \sum_{j=1}^{n_i} vb_j(\sigma(f_i))vb(\hat{\sigma}[t_j]) + \sum_{j>n_i} vb_j(\sigma(f_i))$.

(iv) $op(\hat{\sigma}[t]) = \sum_{j=1}^{n_i} vb_j(\sigma(f_i))op(\hat{\sigma}[t_j]) + op(\sigma(f_i))$.

(3) Let $\tau = (n_i)_{i\in I}$ be a type, $t \in W_\tau(X)$ be a term, and $\sigma \in Hyp_G(\tau)$ be a generalized hypersubstitution of type τ. Then the following statements hold:

(i) If σ is a regular generalized hypersubstitution and $n_i > 1$ for all $i \in I$, then $maxdepth(\hat{\sigma}[t]) \geq maxdepth(t)$.

(ii) If σ is a regular generalized hypersubstitution, then $vb(\hat{\sigma}[t]) \geq vb(t)$.

(iii) If σ is a regular generalized hypersubstitution and $n_i > 1$ for all $i \in I$, then $op(\hat{\sigma}[t]) \geq op(t)$.

(4) Let $\tau = (n_i)_{i\in I}$ be a type and V be a non-trivial M-strongly solid variety of type τ. Let $k \geq 1$. Then the following statements hold:

(i) For the maximum depth c, if $n_i > 1$ for all $i \in I$, then each $N_k^c(V)$ is $(M \cap Reg)$-strongly solid.

(ii) For the variable count c, each $N_k^c(V)$ is $(M \cap Reg)$-strongly solid.

(iii) For the operation count c, if $n_i > 1$ for all $i \in I$, then each $N_k^c(V)$ is $(M \cap Reg)$-strongly solid.

7.2 Monoids of Generalized Hypersubstitutions of Type $\tau = (2)$

(1) $P_G(2) \cup E_{x_1}^G \cup E_{x_2}^G \cup G \cup \{\sigma_{id}\}$ is the set of all idempotent elements in $Hyp_G(2)$.

(2) The order of any generalized hypersubstitution of type $\tau = (2)$ is 1,2 or infinite.

(3) $P_G(2) \cup E_{x_1}^G \cup E_{x_2}^G \cup \overline{E_{x_1}^G} \cup \overline{E_{x_2}^G} \cup G \cup \{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$ is the set of all regular elements in $Hyp_G(2)$.

(4) For any $\sigma_t \in Hyp_G(2) \setminus P_G(2)$, we have $\sigma_t \mathcal{R} \sigma_{\bar{t}}$, $\sigma_t \mathcal{L} \sigma_{t'}$ and $\sigma_t \mathcal{D} \sigma_{\bar{t}} \mathcal{D} \sigma_{t'} \mathcal{D} \sigma_{\overline{t'}}$.

(5) Any $\sigma_{x_i} \in P_G(2)$ is $\mathcal{L}$-related only to itself, but is $\mathcal{R}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $P_G(2)$, and not related to any other generalized hypersubstitutions. Moreover, the set $P_G(2)$ forms an $\mathcal{R}$-, $\mathcal{D}$- and $\mathcal{J}$- class.

(6) All of $\mathcal{R}$-, $\mathcal{L}$- and $\mathcal{D}$-class of σ_{id} are equal to $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

(7) $(\sigma_{id})_i = Hyp_G(2) = (\sigma_{f(x_2,x_1)})_i$, and if $\sigma \in Hyp_G(2)$ and $(\sigma)_i = Hyp_G(2)$, then σ is one of σ_{id} or $\sigma_{f(x_2,x_1)}$. Moreover, the $\mathcal{J}$-class of σ_{id} is equal to its $\mathcal{D}$-class, $\{\sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

(8) Any $\sigma_t \in G$ is $\mathcal{R}$-related only to itself, but is $\mathcal{L}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of G, and not related to any other generalized hypersubstitutions. Moreover, the set G forms an $\mathcal{L}$-, $\mathcal{D}$- and $\mathcal{J}$- class.

(9) Let $\tau = (n_i)_{i\in I}$ be a type and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then $\sigma_1 \mathcal{R} \sigma_2$ if and only if $Im\hat{\sigma}_1 = Im\hat{\sigma}_2$.

(10) For any $\sigma_s, \sigma_t \in Hyp_G(2)$, $\sigma_s \mathcal{R} \sigma_t$ if and only if the following conditions hold:

(i) If $s \in X$, then $t \in X$.

(ii) If $s \notin X$, then $s = t$ or $s = \overline{t}$.

(11) The $\mathcal{L}$-class of the element $\sigma_{f(x_1,x_1)}$ is precisely the set $E^G_{x_1} \cup \overline{E^G_{x_2}}$.

(12) The $\mathcal{D}$-class of the element $\sigma_{f(x_1,x_1)}$ is precisely the set $E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$.

(13) The following statements hold:

(i) $(\sigma_{f(x_1,x_1)})_i = I := \{\sigma_t \in Hyp_G(2) | t \in W^G_{(2)}(\{x_1\}) \cup W^G_{(2)}(\{x_2\})$ or $x_1, x_2 \notin var(t)\}$.

(ii) If $\sigma \in I$ where $\sigma \notin E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$, then $(\sigma)_i \subsetneq I$.

(iii) The $\mathcal{J}$-class of $\sigma_{f(x_1,x_1)}$ is equal to its $\mathcal{D}$-class, $E^G_{x_1} \cup E^G_{x_2} \cup \overline{E^G_{x_1}} \cup \overline{E^G_{x_2}}$.

(14) For any $\sigma_t \in E^G(\{x_1, x_2\})$, the elements which are $\mathcal{L}$-related to σ_t are only σ_t itself and $\sigma_{t'}$.

(15) For $\sigma_t \in E^G(\{x_1, x_2\})$, $D_{\sigma_t} = \{\sigma_t, \sigma_{t'}, \sigma_{\overline{t}}, \sigma_{\overline{t'}}\}$.

(16) For $\sigma_t \in E^G(\{x_1, x_2\})$, the $\mathcal{J}$-class of σ_t is equal to its $\mathcal{D}$-class, $\{\sigma_t, \sigma_{t'}, \sigma_{\bar{t}}, \sigma_{\overline{t'}}\}$.

(17) Let $t \in W_{(2)}(X)\backslash X$ and $x_1 \in var(t)$ or $x_2 \in var(t)$. Then the following statements are equivalent:

(i) σ_t has an $\mathcal{H}$-class of size 2,

(ii) $t' = \bar{t}$,

(iii) $t = f(u, v)$ for some $u, v \in W_{(2)}(X)$ with $v = \overline{u'}$.

(18) For all $x_i \in X$, σ_{x_i} is primitive.

(19) Let σ_t be an idempotent element with $t \notin X$. Then σ_t is not primitive.

(20) Let σ_t be an idempotent element. Then $\sigma_{x_1} \leq \sigma_t$ if and only if $leftmost(t) = x_1$.

(21) Let σ_t be an idempotent element. Then $\sigma_{x_2} \leq \sigma_t$ if and only if $rightmost(t) = x_2$.

(22) Let $x_i \in X$ where $i > 2$ and σ_t be an idempotent element. Then $\sigma_{x_i} \leq \sigma_t$ if and only if $t = x_i$ or $t \notin X$.

(23) Let $t \in W_{(2)}(X)$ with $x_2 \notin var(t)$ and σ_s be an idempotent element. Then $\sigma_{f(x_1,t)} \leq \sigma_s$ if and only if $s = f(x_1, x_2)$ or $s = f(x_1, t)$.

(24) Let $t \in W_{(2)}(X)$ with $x_1 \notin var(t)$ and σ_s be an idempotent element. Then $\sigma_{f(t,x_2)} \leq \sigma_s$ if and only if $s = f(x_1, x_2)$ or $s = f(t, x_2)$.

(25) Let $t \in W_{(2)}(X)$ with $x_2 \notin var(t)$ and σ_s be an idempotent element with $f(x_1, t) \neq s \notin X$. Then $\sigma_s \leq \sigma_{f(x_1,t)}$ if and only if $s = f(x_1, t)_{x_i}^{length(Lp(s))}$ where $x_i = leftmost(s)$ with $i > 2$.

(26) Let $t \in W_{(2)}(X)$ with $x_1 \notin var(t)$ and σ_s be an idempotent element with $f(t, x_2) \neq s \notin X$. Then $\sigma_s \leq \sigma_{f(t,x_2)}$ if and only if $s = f(t, x_2)_{x_i}^{length(Rp(s))}$ where $x_i = rightmost(s)$ with $i > 2$.

(27) Let $s \in W_{(2)}(X) \setminus X$ and $\sigma_t \in G$. If $\sigma_s \leq \sigma_t$, then $s = t$.

7.3 Idempotent Elements in $Pre_G(2,2)$

(1) Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. Then σ_{t_1,t_2} is idempotent if and only if $\hat{\sigma}_{t_1,t_2}[t_1] = t_1$ and $\hat{\sigma}_{t_1,t_2}[t_2] = t_2$.

(2) Let $t_1 = f(x_i, x_j)$ and $t_2 = f(x_k, x_l)$ with $i, j, k, l \in \mathbb{N}$. Then σ_{t_1,t_2} is idempotent if and only if the following conditions hold:

(i) If $x_1 \in var(t_1)$, then $x_i = x_1$ and if $x_2 \in var(t_1)$, then $x_j = x_2$.

(ii) If $x_i = x_j = x_1$ or $x_i = x_j = x_2$, then $x_k = x_l$.

(iii) If $x_i = x_1$ and $j > 2$, then $x_l = x_j$.

(iv) If $i > 2$ and $x_j = x_2$, then $x_k = x_i$.

(v) If $i, j > 2$, then $x_k = x_i$ and $x_l = x_j$.

(3) Let $t_1 = g(x_i, x_j)$ and $t_2 = g(x_k, x_l)$ with $i, j, k, l \in \mathbb{N}$. Then σ_{t_1,t_2} is idempotent if and only if the following conditions hold:

(i) If $x_1 \in var(t_2)$, then $x_k = x_1$ and if $x_2 \in var(t_2)$, then $x_l = x_2$.

(ii) If $x_k = x_l = x_1$ or $x_k = x_l = x_2$, then $x_i = x_j$.

(iii) If $x_k = x_1$ and $l > 2$, then $x_j = x_l$.

(iv) If $k > 2$ and $x_l = x_2$, then $x_i = x_k$.

(v) If $k, l > 2$, then $x_i = x_k$ and $x_j = x_l$.

(4) Let $t_1 = f(x_i, x_j)$ and $t_2 = g(x_k, x_l)$ with $i, j, k, l \in \mathbb{N}$. Then σ_{t_1,t_2} is idempotent if and only if the following conditions hold:

(i) If $x_i = x_2$, then $x_j = x_2$.

(ii) If $x_k = x_2$, then $x_l = x_2$.

(iii) If $i > 2$, then $x_j \neq x_1$.

(iv) If $k > 2$, then $x_l \neq x_1$.

(5) Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. If $op(t_1) = 1$, $op(t_2) > 1$ and $t_2 = g(k_1, k_2)$, then σ_{t_1,t_2} is idempotent if and only if $t_1 \in$

$\{f(x_1, x_i), f(x_2, x_2), f(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$ and the following conditions hold:

(i) $x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$.

(ii) If $x_1 \notin var(t_2)$ and $x_2 \in var(t_2)$, then $t_2 = g(k_1, x_2)$.

(iii) If $x_2 \notin var(t_2)$ and $x_1 \in var(t_2)$, then $t_2 = g(x_1, k_2)$.

(6) Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. If $op(t_1) > 1$, $op(t_2) = 1$ and $t_1 = f(k_1, k_2)$, then σ_{t_1,t_2} is idempotent if and only if $t_2 \in \{g(x_1, x_i), g(x_2, x_2), g(x_j, x_k) | i, j, k \in \mathbb{N}, j > 2, k \neq 1\}$ and the following conditions hold:

(i) $x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$.

(ii) If $x_1 \notin var(t_1)$ and $x_2 \in var(t_1)$, then $t_1 = f(k_1, x_2)$.

(iii) If $x_2 \notin var(t_1)$ and $x_1 \in var(t_1)$, then $t_2 = f(x_1, k_2)$.

(7) Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. If $op(t_1) = 1$, $op(t_2) > 1$ and $t_2 = f(k_1, k_2)$, then σ_{t_1,t_2} is idempotent if and only if $t_1 \in \{f(x_1, x_i), f(x_2, x_2), f(x_j, x_2) | i, j \in \mathbb{N}, j > 2\}$ and the following conditions hold:

(i) If $t_1 = f(x_1, x_2)$, then $ops(t_2) = \{f\}$.

(ii) If $t_1 = f(x_1, x_i)$ with $i \neq 2$, then $t_2 = {t_{1x_k}}^{length(Lp(t_2))}$ where $x_k = leftmost(t_2)$.

(iii) If $t_1 = f(x_j, x_2)$ with $j \neq 1$, then $t_2 = {t_{1x_k}}^{length(Rp(t_2))}$ where $x_k = rightmost(t_2)$.

(8) Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$. If $op(t_1) > 1$, $op(t_2) = 1$ and $t_1 = g(k_1, k_2)$, then σ_{t_1,t_2} is idempotent if and only if $t_2 \in \{g(x_1, x_i), g(x_2, x_2), g(x_j, x_2) | i, j \in \mathbb{N}, j > 2\}$ and the following conditions hold:

(i) If $t_2 = g(x_1, x_2)$, then $ops(t_1) = \{g\}$.

(ii) If $t_2 = g(x_1, x_i)$ with $i \neq 2$, then $t_1 = {t_{2x_k}}^{length(Lp(t_1))}$ where $x_k = leftmost(t_1)$.

(iii) If $t_2 = g(x_j, x_2)$ with $j \neq 1$, then $t_1 = {t_2}_{x_k}{}^{length(Rp(t_1))}$ where $x_k = rightmost(t_1)$.

(9) Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$, $op(t_1) > 1$, $op(t_2) > 1$ and $t_1 = f(k_1, k_2)$, $t_2 = f(k_3, k_4)$, then σ_{t_1,t_2} is idempotent if and only if ($x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$) and ($x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$) and the following conditions hold:

(i) If $t_1, t_2 \in W^G_{(2,2)}(\{x_1\})$, then $t_1 = f(x_1, k_2)$ and $t_2 = t_1^{length(Lp(t_2))}$.

(ii) If $t_1, t_2 \in W^G_{(2,2)}(\{x_2\})$, then $t_1 = f(k_1, x_2)$ and $t_2 = t_1^{length(Rp(t_2))}$.

(iii) If $t_1 \in W^G_{(2,2)}(\{x_1\})$, $t_2 \in W^G_{(2,2)}(\{x_2\})$, then $t_1 = f(x_1, k_2)$ and $t_2 = {t_1}_{x_2}{}^{length(Lp(t_2))}$.

(iv) If $t_1 \in W^G_{(2,2)}(\{x_2\})$, $t_2 \in W^G_{(2,2)}(\{x_1\})$, then $t_1 = f(k_1, x_2)$ and $t_2 = {t_1}_{x_1}{}^{length(Rp(t_2))}$.

(v) If $t_1 \in W^G_{(2,2)}(\{x_1\})$ and $x_1, x_2 \notin var(t_2)$, then $t_1 = f(x_1, k_2)$ and $t_2 = {t_1}_{x_k}{}^{length(Lp(t_2))}$ where $x_k = leftmost(t_2)$.

(vi) If $t_1 \in W^G_{(2,2)}(\{x_2\})$ and $x_1, x_2 \notin var(t_2)$, then $t_1 = f(k_1, x_2)$ and $t_2 = {t_1}_{x_k}{}^{length(Rp(t_2))}$ where $x_k = rightmost(t_2)$.

(vii) If $x_1, x_2 \notin var(t_1)$, then $t_2 = t_1$.

(10) Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$, $op(t_1) > 1$, $op(t_2) > 1$ and $t_1 = g(k_1, k_2)$, $t_2 = g(k_3, k_4)$, then σ_{t_1,t_2} is idempotent if and only if ($x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$) and ($x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$) and the following conditions hold:

(i) If $t_1, t_2 \in W^G_{(2,2)}(\{x_1\})$, then $t_2 = g(x_1, k_4)$ and $t_1 = t_2^{length(Lp(t_1))}$.

(ii) If $t_1, t_2 \in W^G_{(2,2)}(\{x_2\})$, then $t_2 = g(k_3, x_2)$ and $t_1 = t_2^{length(Rp(t_1))}$.

(iii) If $t_1 \in W^G_{(2,2)}(\{x_1\})$, $t_2 \in W^G_{(2,2)}(\{x_2\})$, then $t_2 = g(k_3, x_2)$ and $t_1 = {t_2}_{x_1}{}^{length(Rp(t_1))}$.

(iv) If $t_1 \in W^G_{(2,2)}(\{x_2\})$, $t_2 \in W^G_{(2,2)}(\{x_1\})$, then $t_2 = g(x_1, k_4)$ and $t_1 = {t_2}_{x_2}{}^{length(Lp(t_1))}$.

(v) If $t_2 \in W^G_{(2,2)}(\{x_1\})$ and $x_1, x_2 \notin var(t_1)$, then $t_2 = g(x_1, k_4)$ and $t_1 = {t_2}_{x_k}^{length(Lp(t_1))}$ where $x_k = leftmost(t_1)$.

(vi) If $t_2 \in W^G_{(2,2)}(\{x_2\})$ and $x_1, x_2 \notin var(t_1)$, then $t_2 = g(k_3, x_2)$ and $t_1 = {t_2}_{x_k}^{length(Rp(t_1))}$ where $x_k = rightmost(t_1)$.

(vii) If $x_1, x_2 \notin var(t_2)$, then $t_1 = t_2$.

(11) Let σ_{t_1,t_2} be a generalized hypersubstitution of type $\tau = (2,2)$, $op(t_1) > 1$, $op(t_2) > 1$ and $t_1 = f(k_1, k_2)$, $t_2 = g(k_3, k_4)$, then σ_{t_1,t_2} is idempotent if and only if ($x_1 \notin var(t_1)$ or $x_2 \notin var(t_1)$) and ($x_1 \notin var(t_2)$ or $x_2 \notin var(t_2)$) and the following conditions hold:

(i) If $t_1 \in W^G_{(2,2)}(\{x_1\})$, then $t_1 = f(x_1, k_2)$.

(ii) If $t_1 \in W^G_{(2,2)}(\{x_2\})$, then $t_1 = f(k_1, x_2)$.

(iii) If $t_2 \in W^G_{(2,2)}(\{x_1\})$, then $t_2 = g(x_1, k_4)$.

(iv) If $t_2 \in W^G_{(2,2)}(\{x_2\})$, then $t_2 = g(k_3, x_2)$.

7.4 Monoids of Generalized Hypersubstitutions of Type $\tau = (n)$

(1) Let σ_t be a generalized hypersubstitution of type $\tau = (n)$. Then σ_t is idempotent if and only if $\hat{\sigma}_t[t] = t$.

(2) For every $x_i \in X$, σ_{x_i} and σ_{id} are idempotent.

(3) If $\sigma_t \in G(n)$ and $\sigma_s \in Hyp_G(n) \setminus P_G(n)$, then $\sigma_t \circ_G \sigma_s = \sigma_t$, i.e. $G(n)$ itself is a left zero band.

(4) Let $t = f(t_1, \ldots, t_n) \in W_{(n)}(X)$ and $\emptyset \neq var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}$. Then σ_t is idempotent if and only $t_{i_k} = x_{i_k}$ for all $k \in \{1, \ldots, m\}$.

(5) Let $t = f(t_1, \ldots, t_n) \in W_{(n)}(X)$ and $\emptyset \neq var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}$. Then σ_t is regular if and only if there exist $j_1, \ldots, j_m \in \{1, \ldots, n\}$ such that $t_{j_1} = x_{i_1}, \ldots, t_{j_m} = x_{i_m}$.

(6) Any $\sigma_{x_i} \in P_G(n)$ is $\mathcal{L}$-related only to itself, but is $\mathcal{R}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $P_G(n)$, and not related to any other generalized hypersubstitutions. Moreover, the set $P_G(n)$ forms an $\mathcal{R}$-, $\mathcal{D}$- and $\mathcal{J}$-class.

(7) Any $\sigma_t \in G(n)$ is $\mathcal{R}$-related only to itself, but is $\mathcal{L}$-related, $\mathcal{D}$-related and $\mathcal{J}$-related to all elements of $G(n)$, and not related to any other generalized hypersubstitutions. Moreover, the set $G(n)$ forms an $\mathcal{L}$-, $\mathcal{D}$- and $\mathcal{J}$-class.

(8) Let $\sigma_s, \sigma_t \in Hyp_G(n)$. Then $\sigma_s \mathcal{R} \sigma_t$ if and only if the following conditions hold:

(i) If $s \in X$, then $t \in X$.

(ii) If $s \notin X$, then $s = C_{\alpha[t]}$ for some bijection α on J.

(9) Let $\sigma_t \in Hyp_G(n) \setminus P_G(n)$. Then, for any permutation π on J, σ_t is $\mathcal{L}$-related to the generalized hypersubstitution $\sigma_{\pi[t]}$.

(10) Two idempotent elements σ_s and σ_t in $Hyp_G(n) \setminus P_G(n)$ are $\mathcal{L}$-related if and only if $var(s) \cap X_n = var(t) \cap X_n$.

(11) Let σ_t be an idempotent element in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then $L_{\sigma_t} = \{\sigma_{\pi[w]} | \pi$ is a permutation of $J, w \notin X, var(w) \cap X_n = var(t) \cap X_n$ and σ_w is an idempotent element$\}$.

(12) Let σ_t be an idempotent element in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then $D_{\sigma_t} = \{\sigma_w | w = C_{\alpha[\pi[s]]}$ for some α bijection on J, π a permutation on $J, s \notin X$, and σ_s an idempotent element with $var(s) \cap X_n = var(t) \cap X_n\}$.

(13) Let σ_t be an idempotent element in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then its $\mathcal{J}$-class is equal to its $\mathcal{D}$-class.

(14) Let σ_s, σ_t be idempotent elements in $Hyp_G(n) \setminus (P_G(n) \cup G(n))$. Then σ_s and σ_t are $\mathcal{J}$- or $\mathcal{D}$-related if and only if the number of distinct variables in X_n which occur in s and t are equal.