Chapter 2

Preliminaries

In this chapter, we present the basic concepts about some important notations and mathematical definition of LPD system, uncertain LPD system, uncertain linear system with nonlinear perturbation, uncertain linear non-autonomous systems, uncertain linear impulsive switched systems with nonlinear perturbations and uncertain impulsive switched LPD system with nonlinear perturbations (with delay-independent or delay-dependent). We investigates the case of continuous, discrete and discontinuous time for the linear delay systems. Finally, we show some general concepts of stability, important definitions and lemmas because it will be recalled.

2.1 Notations

We give some important notations for using in the thesis.
R^+ denotes the set of all non-negative real numbers;
Z^+ denotes the set of all non-negative integer numbers;
R^n denotes the n-dimensional Euclidean space;
Matrix M is positive definite ($M > 0$) if $x^T M x > 0$ for all $x \in R^n$, $x \neq 0$;
Matrix M is semi-positive definite ($M \geq 0$) if $x^T M x \geq 0$ for all $x \in R^n$;
Matrix M is negative definite ($M < 0$) if $x^T M x < 0$ for all $x \in R^n$, $x \neq 0$;
Matrix M is semi-negative definite ($M \leq 0$) if $x^T M x \leq 0$ for all $x \in R^n$;
$M > 0$ ($M \geq 0$) denotes the square symmetric positive (semi-) definite matrix;
$M < 0$ ($M \leq 0$) denotes the square symmetric negative (semi-) definite matrix;
$M > N$ ($M \geq N$) denotes the $M - N$ matrix is square symmetric positive (semi-) definite matrix;
$M < N$ ($M \leq N$) denotes the $M - N$ matrix is square symmetric negative (semi-)

definite matrix;

$\langle x, y\rangle$ or $x^T y$ denotes the scalar product of two vector $x, y \in R^n$;

$\|x\|$ denotes the Euclidean vector norm of $x \in R^n$;

$M^{n\times m}$ denotes the space of all $(n \times m)$ matrices;

M^T denotes the transpose of the matrix M;

M^{-1} denotes the inverse of a non-singular matrix M;

M is symmetric if $M = M^T$;

$C([-h, 0], R^n)$ denotes the space of all piecewise continuous vector functions mapping $[-h, 0]$ into R^n;

$\lambda(M)$ denotes the set of all eigenvalues of M;

$\lambda_{max}(M)$ denotes max $\{Re\lambda : \lambda \in \lambda(M)\}$;

$\lambda_{min}(M)$ denotes min $\{Re\lambda : \lambda \in \lambda(M)\}$;

I denotes the identity matrix;

$s(\rho) = \{x \in R^n : \|x\| < \rho, \rho > 0\}$;

$x_t = \{x(t + s) : s \in [-h, 0]\}, \|x_t\| = \sup_{s\in[-h,0]} \|x(t + s)\|$.

2.2 Several Linear Systems with Time-delay

2.2.1 LPD Systems with Time-delay

• Continuous Case

Consider the following linear parameter dependent (LPD) system with time-delay of the form

$$\begin{cases} \dot{x}(t) = A(\alpha)x(t) + B(\alpha)x(t-h), & \forall t \geq 0; \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases}$$

where $x(t) \in R^n$ is the state variable, $h \in R^+$ is the delay and $\phi(t)$ is a continuous vector-valued initial condition on $[-h, 0]$. $A(\alpha)$ and $B(\alpha)$ are matrices belonging to the polytope Ω_1 or Ω_2 where

$$\Omega_1 := \{A(\alpha), B(\alpha)\} = \Big\{\sum_{i=1}^{N} \alpha_i A_i, \sum_{i=1}^{N} \alpha_i B_i\Big\},$$

$$\sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0, A_i, B_i \in M^{n \times n}, i = 1, ..., N,$$

and

$$\Omega_2 := \{A(\alpha), B(\alpha)\} = \Big\{ \sum_{i=1}^{N} \alpha_i(t) A_i, \sum_{i=1}^{N} \alpha_i(t) B_i \Big\},$$

$$\sum_{i=1}^{N} \alpha_i(t) = 1, \alpha_i(t) \geq 0, A_i, B_i \in M^{n \times n}, i = 1, ..., N.$$

•Discrete Case

We consider the discrete-time LPD system with time-delay of the form

$$\begin{cases} x(k+1) = A(\alpha)x(k) + B(\alpha)x(k-h), & \forall k \geq 0; \\ x(k) = \phi(k), & \forall k \in [-h, 0], \end{cases}$$

where $x(k) \in R^n$, $k \in Z^+$ is the state variable, $h \in Z^+$ is the delay and $\phi(k)$ is a vector-valued initial condition on $[-h, 0]$. $A(\alpha)$ and $B(\alpha)$ are uncertain matrices belonging to the polytope Ω_1.

2.2.2 Uncertain LPD Systems with Time-varying Delays

• Continuous Case

Consider the uncertain linear parameter dependent (LPD) system with time-varying delays of the form

$$\begin{cases} \dot{x}(t) = \Big[A(\alpha) + \Delta A(t)\Big] x(t) + \Big[B(\alpha) + \Delta B(t)\Big] x(t - h(t)), & \forall t \geq 0; \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases}$$

where $x(t) \in R^n$, $h(t)$ is the time-varying delays for all $t \geq 0$, and satisfy $0 \leq h(t) \leq h$, $h \in R^+$. $A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i$, $B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i$ where A_i and B_i are the given matrices in $M^{n \times n}$ for all $i = 1, 2, ..., N$ and $\sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0$ for all $i = 1, 2, ..., N$. $\phi(t)$ is a continuous vector-valued initial condition on $[-h, 0]$. $\Delta A(t)$ and $\Delta B(t)$ are unknown matrices representing time-varying parameter uncertainties, we are assumed to be of the form

$$\Delta A(t) = K(\alpha)\Delta(t)A_1(\alpha), \quad \Delta B(t) = K(\alpha)\Delta(t)B_1(\alpha),$$

where $A_1(\alpha)=\sum_{i=1}^{N}\alpha_iA_i^1$, $B_1(\alpha)=\sum_{i=1}^{N}\alpha_iB_i^1$, $K(\alpha)=\sum_{i=1}^{N}\alpha_iK_i$; A_i^1, B_i^1 and K_i are the given matrices in $M^{n\times n}$ for all $i=1,2,...,N$ and $\Delta(t)$ satisfies

$$\Delta(t)=F(t)[I-JF(t)]^{-1}.$$

The uncertain matrix $F(t)$ satisfies

$$F(t)^TF(t)\leq I,\quad I-JJ^T>0. \tag{2.1}$$

Remark 1. The condition (2.1) guarantees that $I-JF(t)$ to be invertible. It is easy to know that when $J=0$, the parametric uncertainty of linear fractional form reduces to $\Delta(t)=F(t)$.

• **Discrete Case**

Consider the uncertain discrete-time linear parameter dependent (LPD) system with time-varying delays of the form

$$\begin{cases} x(k+1)=\Big[A(\alpha)+\Delta A(k)\Big]x(k)+\Big[B(\alpha)+\Delta B(k)\Big]x(k-h(k)), & \forall k\geq 0;\\ x(k)=\phi(k), & \forall k\in[-h,0], \end{cases}$$

where $x(k)\in R^n$, $h(k)$ is a positive integer for each $k\in Z^+$, and satisfy $0\leq h(k)\leq h$, $h\in Z^+$. $\phi(k)$ is a vector-valued initial condition on $[-h,0]$, $A(\alpha)=\sum_{i=1}^{N}\alpha_iA_i$, $B(\alpha)=\sum_{i=1}^{N}\alpha_iB_i$ where A_i and B_i are the given matrices in $M^{n\times n}$ for all $i=1,2,...,N$ and $\sum_{i=1}^{N}\alpha_i=1, \alpha_i\geq 0$ for all $i=1,2,...,N$. $\Delta A(k)$ and $\Delta B(k)$ are unknown matrices representing time-varying parameter uncertainties, we are assumed to be of the form

$$\Delta A(k)=K(\alpha)\Delta(k)A_1(\alpha), \Delta B(k)=K(\alpha)\Delta(k)B_1(\alpha),$$

where $A_1(\alpha)=\sum_{i=1}^{N}\alpha_iA_i^1$, $B_1(\alpha)=\sum_{i=1}^{N}\alpha_iB_i^1$, $K(\alpha)=\sum_{i=1}^{N}\alpha_iK_i$; A_i^1, B_i^1 and K_i are the given matrices in $M^{n\times n}$ for all $i=1,2,...,N$ and $\Delta(k)$ satisfies

$$\Delta(k)=F(k)[I-JF(k)]^{-1}.$$

The uncertain matrix $F(k)$ satisfies

$$F(k)^TF(k)\leq I,\quad I-JJ^T>0. \tag{2.2}$$

Remark 2. The condition (2.2) guarantees that $I - JF(k)$ to be invertible. It is easy to know that when $J = 0$, the parametric uncertainty of linear fractional form reduces to $\Delta(k) = F(k)$.

2.2.3 Uncertain Linear System with Interval Time-varying Delays and Non-linear Perturbations.

• Continuous Case

Consider systems described by the following state equation of the form

$$\begin{cases} \dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t - h(t)) \\ \qquad\quad + f(t, x(t)) + g(t, x(t - h(t))), & \forall t \geq 0; \\ x(t) = \phi(t), \quad \dot{x}(t) = \psi(t), & \forall t \in [-h_2, 0], \end{cases} \tag{2.3}$$

where $x(t) \in R^n$ is the state, A and B are given matrices of appropriate dimensions, $0 \leq h_1 \leq h(t) \leq h_2, h_2 > 0$. Consider the initial function $\phi(t), \psi(t) \in C([-h_2, 0], R^n)$ with the norm $\|\phi\| = \sup_{t\in[-h_2,0]} \|\phi(t)\|$ and $\|\psi\| = \sup_{t\in[-h_2,0]} \|\psi(t)\|$. $\Delta A(t)$ and $\Delta B(t)$ are unknown matrices representing time-varying parameter uncertainties, we are assumed to be of the form

$$\Delta A(t) = KF(t)A_1, \quad \Delta B(t) = KF(t)B_1,$$

where K, A_1, B_1 are known constant matrices and $F(t)$ is an unknown time-varying matrix satisfying $\|F(t)\| \leq 1$. The uncertainties $f(.), g(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t - h(t))$, respectively, and are bounded in magnitude of the form

$$f^T(t, x(t))f(t, x(t)) \leq \eta x^T(t)x(t),$$

$$g^T(t, x(t - h(t)))g(t, x(t - h(t))) \leq \rho x^T(t - h(t))x(t - h(t)),$$

where η, ρ are given positive real constants. The delay $h(t)$ is a continuous differentiable function satisfying

$$0 \leq h_1 \leq h(t) \leq h_2, \quad \dot{h}(t) \leq \delta < 1.$$

• Discrete Case

Consider the uncertain linear system with time-varying delays and nonlinear perturbations described by the following state equation of the form

$$\begin{cases} x(k+1) = \big[A+\Delta A\big]x(k) + \big[B+\Delta B\big]x(k-h(k)) \\ \qquad\qquad +f(k,x(k)) + g(t,x(k-h(k))), \qquad\qquad \forall k \in Z^+; \\ x(k) = \phi(k), \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \forall k \in [-h_2, 0], \end{cases} \tag{2.4}$$

where $x(k) \in R^n$, $h(k)$ is a positive integer representing the time delay. A and B are given matrix of appropriate dimensions. $\phi(k)$ is a real valued initial function on $[-h_2, 0]$ with the norm $\|\phi\| = \sup_{t\in[-h_2,0]} \|\phi(k)\|$. The uncertainties $f(.), g(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(k)$ and the delayed state $x(k-h(k))$, respectively, and are bounded in magnitude:

$$f^T(k,x(k))f(k,x(k)) \le \eta x^T(k)x(k),$$

$$g^T(k,x(k-h(k)))g(k,x(k-h(k))) \le \rho x^T(k-h(k))x(k-h(k)),$$

where η, ρ are given nonnegative integers. The uncertainties $\Delta A(k)$ and $\Delta B(k)$ are unknown matrices representing time-varying parameter uncertainties, we are assumed to be of the form

$$\Delta A(k) = K\Delta(k)A_1, \quad \Delta B(k) = K\Delta(k)B_1,$$

where $K, A_1, B_1 \in M^{n\times n}$ and $\Delta(k)$ satisfies

$$\Delta(k) = F(k)[I - JF(k)]^{-1}, \quad I - JJ^T > 0.$$

The uncertain matrix $F(k)$ satisfies

$$F(k)^T F(k) \le I.$$

In addition, we assume that the time-varying delays $h(k)$ are upper and lower bounded. It satisfies the following assumption of the form

$$h_1 \le h(k) \le h_2,$$

where h_1 and h_2 are known positive integer.

2.2.4 Uncertain Impulsive Switched Systems with Time-varying Delays and Nonlinear Perturbations

Consider the uncertain impulsive switched system with time-varying delays and nonlinear perturbations of the form

$$\begin{cases} \dot{x}(t) = \Big[A_{i_k} + \Delta A_{i_k}(t)\Big]x(t) + \Big[B_{i_k} + \Delta B_{i_k}(t)\Big]x(t - h_{i_k}(t)) & \\ \qquad +f_{i_k}(t, x(t)) + g_{i_k}(t, x(t - h_{i_k}(t))), & t \neq t_k; \\ \Delta x(t) = I_k(x(t)) = D_k x(t), & t = t_k; \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases}$$

where $x(t) \in R^n$ is the state and $\phi(t)$ is a piecewise continuous vector-valued initial function ($\phi(t) \in C([-h, 0], R^n)$). A_{i_k}, B_{i_k} and D_k are given real matrices of appropriate dimensions. $\Delta x(t) = x(t^+) - x(t^-)$, $x(t^+) = \lim_{\nu \to 0^+} x(t + \nu)$, $x(t^-) = \lim_{\nu \to 0^+} x(t - \nu)$. We assume the solution of the impulsive switched system is left continuous, i.e., $\lim_{\nu \to 0^+} x(t_k - \nu) = x(t_k^-) = x(t_k)$. $i_k \in \{1, 2, ..., m\}$, $k \in N$, $m \in N$, i_k is an impulsive switching time and $t_0 < t_1 < t_2 < \cdots < t_k < \cdots, t_k \to +\infty$ as $k \to +\infty$. Under the switching law of system, at the time t_k, the system switches to the i_k subsystem from the i_{k-1} subsystem. The delay $h_{i_k}(t)$ is a time varying bounded continuous function satisfying,

$$0 \leq h_{i_k}(t) \leq h, \quad \dot{h}_{i_k}(t) \leq \delta < +\infty,$$

for all i_k and $t > 0$. The uncertainty $\Delta A_{i_k}(t)$ and $B_{i_k}(t)$ are time varying matrices and satisfy the condition $\Delta A_{i_k}(t) = E_{i_k}\Delta_{i_k}(t)H_{i_k}$, $\Delta B_{i_k}(t) = E_{i_k}\Delta_{i_k}(t)M_{i_k}$, $E_{i_k}, H_{i_k}, M_{i_k} \in M^{n \times n}$ and $\Delta_{i_k}(t)$ satisfies

$$\Delta_{i_k}(t) = F_{i_k}(t)[I - JF_{i_k}(t)]^{-1}, \quad I - JJ^T > 0.$$

The uncertain matrix $F_{i_k}(t)$ satisfies

$$F_{i_k}(t)^T F_{i_k}(t) \leq I.$$

The uncertainties $f_{i_k}(.), g_{i_k}(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t - h_{i_k}(t))$, respectively, and are bounded in magnitude:

$$f_{i_k}^T(t, x(t)) f_{i_k}(t, x(t)) \leq \eta x^T(t) x(t),$$

$$g_{i_k}^T(t, x(t-h(t)))g_{i_k}(t, x_{i_k}(t-h(t))) \leq \rho x^T(t-h_{i_k}(t))x(t-h_{i_k}(t)),$$

where η, ρ are given nonnegative constants.

Next, Consider the uncertain impulsive switched LPD system with time-varying delays and nonlinear perturbations of the form

$$\begin{cases} \dot{x}(t) = \hat{A}_{i_k}(\alpha)x(t) + \hat{B}_{i_k}(\alpha)x(t-h_{i_k}(t)) + f_{i_k}(t, x(t)) & \\ \qquad +g_{i_k}(t, x(t-h_{i_k}(t))), & t \neq t_k; \\ \Delta x(t) = x(t) - x(t^-) = G_k(\alpha)x(t^- - h_{i_k}(t^-)), & t = t_k; \\ x(t) = \phi(t), & \forall t \in [-h, 0], \\ \hat{A}_{i_k}(\alpha) = \Big[A_{i_k}(\alpha) + \Delta A_{i_k}(t)\Big], \hat{B}_{i_k}(\alpha) = \Big[B_{i_k}(\alpha) + \Delta B_{i_k}(t)\Big], & \end{cases}$$

where $x(t) \in R^n$ is the state, $n \in Z^+$ and $h_{i_k}(t)$ is a positive function representing the time-varying delays. $\phi(t)$ is a piecewise continuous vector-valued initial function. $A_{i_k}(\alpha)$, $B_{i_k}(\alpha)$ and $G_k(\alpha)$ are uncertain $M^{n\times n}$ polytope matrices of the form

$$[A_{i_k}(\alpha), B_{i_k}(\alpha)] = \Big[\sum_{j=1}^{N} \alpha_j A_{i_k,j}, \sum_{j=1}^{N} \alpha_j B_{i_k,j}\Big],$$

$$\sum_{j=1}^{N} \alpha_j = 1, \alpha_j \geq 0, \quad A_{i_k,j}, B_{i_k,j} \in M^{n\times n}, j = 1, ..., N,$$

and

$$G_k(\alpha) = \sum_{j=1}^{N} \alpha_j G_{k,j}, \quad G_{k,j} \in M^{n\times n}, j = 1, ..., N.$$

The uncertainties $f_{i_k}(.), g_{i_k}(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h_{i_k}(t))$, respectively, and are bounded in magnitude:

$$f_{i_k}^T(t, x(t))f_{i_k}(t, x(t)) \leq \eta x^T(t)x(t),$$

$$g_{i_k}^T(t, x(t-h(t)))g_{i_k}(t, x_{i_k}(t-h(t))) \leq \rho x^T(t-h_{i_k}(t))x(t-h_{i_k}(t)),$$

where η, ρ are given nonnegative constants. $\Delta x(t) = x(t_k^+) - x(t_k^-)$, $\lim_{\nu\to 0^+} x(t_k+\nu) = x(t_k^+)$, $x(t_k^-) = \lim_{\nu\to 0^+} x(t-\nu)$. We assume that the solution of the impulsive switched system is right continuous i.e., $x(t_k^+) = x(t_k)$. $i_k \in \{1, 2, ..., m\}$, $k \in N$, $m \in N$, t_k is an impulsive switching time point and $t_0 < t_1 < t_2 < \cdots < t_k < \cdots, t_k \to +\infty$

as $k \to +\infty$. Under the switching law of system, at the time t_k, the system switches to the i_k subsystem from the i_{k-1} subsystem. The delay $h_{i_k}(t)$ is a time varying bounded continuous function satisfying,

$$0 \le h_{i_k}(t) \le h,$$

for all i_k and $t > 0$. The uncertainties $\Delta A_{i_k}(t)$, $\Delta B_{i_k}(t)$ and $\Delta C_{i_k}(t)$ are time varying matrices and satisfy the condition,

$$\Delta A_{i_k}(t) = E_{i_k}(\alpha)\Delta_{i_k}(t)M_{i_k}(\alpha), \quad \Delta B_{i_k}(t) = E_{i_k}(\alpha)\Delta_{i_k}(t)N_{i_k}(\alpha),$$

$$\Delta C_{i_k}(t) = E_{i_k}(\alpha)\Delta_{i_k}(t)F_{i_k}(\alpha),$$

where $\Delta_{i_k}(t)$ satisfies $\Delta_{i_k}(t) = F_{i_k}(t)[I - JF_{i_k}(t)]^{-1}, \quad I - JJ^T > 0$. The uncertain matrix $F_{i_k}(t)$ satisfies

$$F_{i_k}(t)^T F_{i_k}(t) \le I.$$

2.2.5 Uncertain Linear Non-autonomous System with Time-varying Delays

Consider the uncertain linear non-autonomous system with time-varying delays

$$\begin{cases} \dot{x}(t) = [A_0(t) + \Delta A_0(t)]x(t) + [A_1(t) + \Delta A_1(t)]x(t - h(t)), & t \ge 0; \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $A_i(t)$, $i = 0, 1$ are given matrix functions continuous on $[0, \infty)$, $0 \le h(t) \le h, h > 0$. Consider the initial function $\phi(t) \in C([-h, 0], \mathbb{R}^n)$ with the norm $\|\phi\| = \sup_{t \in [-h,0]} \|\phi(t)\|$. The delay $h(t)$ is a continuously differentiable function satisfying

$$0 \le h(t) \le h, \quad \dot{h}(t) \le \delta < 1.$$

The uncertainties $\Delta A_0, \Delta A_1$ are time-varying and satisfy the condition:

$$\Delta A_i(t) = G_i(t)F(t)H_i(t), \quad i = 0, 1,$$

$$\|F(t)\| \le 1, \quad \forall t \in R^+,$$

where $G_i(t), H_i(t)$, $i = 0, 1$ are given matrix functions of appropriate dimensions.

2.3 Definitions and Lemmas

Consider a dynamical system which satisfies

$$\dot{x}(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x \in R^n. \tag{2.5}$$

The system defined by (2.5) is said to be autonomous, or time-invariant, if f does not depend on time (does not depend on the independent variable), and non-autonomous, or time-varying, otherwise. It is said to be linear if $f(t, x(t)) = A(t)x(t)$ for some $A(.) : R^+ \to R^{n\times n}$ and nonlinear, otherwise.

We will assume that $f(t, x(t)) : R \times R^n \to R^n$ satisfies the standard conditions for the existence and uniqueness of solutions. Such conditions are, for instance, that $f(t, x(t))$ is Lipschitz continuous with respect to x, uniformly in t, and piecewise continuous in t. A point $x^* \in R^n$ is an equilibrium point of (2.5) if $f(t, x^*) = 0$. Intuitively and somewhat crudely speaking, we say an equilibrium point is locally stable if all solutions which start near x^* (meaning that the initial conditions are in a neighborhood of x^*) remain near x^* for all time. The equilibrium point x^* is said to be locally asymptotically stable if x^* is locally stable and, furthermore, all solutions starting near x^* tend towards x^* as $t \to +\infty$. We say somewhat crude because the time-varying nature of equation (2.5) introduces all kinds of additional subtleties. Nonetheless, it is intuitive that a pendulum has a locally stable equilibrium point when the pendulum is hanging straight down and an unstable equilibrium point when it is pointing straight up. If the pendulum is damped, the stable equilibrium point is locally asymptotically stable.

By shifting the origin of the system, we may assume that the equilibrium point of interest occurs at $x^* = 0$. If multiple equilibrium points exist, we will need to study the stability of each by appropriately shifting the origin. Therefore, we assume $f(t, 0) = 0$ so that system (2.5) admits the trivial solution [30].

Remark 3. In this thesis, the robust stability problem considers the stability problem of systems which contain some uncertainties. Uncertainties of systems are explained by including polytopic uncertainties and norm-bounded uncertainties.

Definition 2.3.1 [27] A function $w(\cdot) : R^n \to R$ is called *positive (negative) definite* if $w(0) = 0$ and $w(x) > 0$ $(w(x) < 0)$ whenever $x \neq 0$.

Definition 2.3.2 [27] A function $w(\cdot) : R^n \to R$ is called *positive (negative) semi-definite* if $w(0) = 0$ and $w(x) \geq 0$ $(w(x) \leq 0)$.

Definition 2.3.3 [33] The equilibrium point $x^* = 0$ of the system (2.5) is

(i) *stable* (in the sense of Lyapunov) at $t = t_0$ if $\forall \epsilon > 0$, $\exists \delta(t_0, \epsilon) > 0$ such that

$$\|x_0\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0;$$

(ii) *asymptotically stable* (in the sense of Lyapunov) if $x^* = 0$ is stable and $\exists \delta(t_0)$ such that

$$\|x_0\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0;$$

(iii) *exponentially stable* if there exist three positive real constants ϵ, K and λ such that

$$\|x(t)\| \leq K\|x_0\|e^{-\lambda(t-t_0)}, \quad \forall \|x_0\| < \epsilon, \quad t \geq t_0;$$

The largest constant λ which may be utilized in above inequality is called the rate of convergence.

(iv) *globally asymptotically stable* if $x^* = 0$ is stable and $\forall x_0 \in R^n$

$$\lim_{t \to \infty} x(t) = 0.$$

Lyapunovs direct method (also called the second method of Lyapunov) allows us to determine the stability of a system without explicitly integrating the differential equation (2.5). The method is a generalization of the idea that if there is some measure of energy in a system, then we can study the rate of change of the energy of the system to ascertain stability. To make this precise, we need to define exactly what one means by a measure of energy. Let B_ϵ be a ball of size ϵ around the origin,

$$B_\epsilon = \{x \in R^n : \|x\| < \epsilon\}.$$

Definition 2.3.4 [27] [Class K function]

A function $\varphi(\cdot) : R^+ \to R^+$ belongs to class K (denoted $\varphi(\cdot) \in K$), if it is continuous, strictly monotone increasing, and $\varphi(0) = 0$.

Definition 2.3.5 [33] [Locally positive definite functions]

A continuous function $V(t, x(t)) : R^+ \times R^n \to R$ is a *locally positive definite function* if for some $\epsilon > 0$ and some continuous, strictly increasing function $\alpha(\cdot) : R^+ \to R$,

$$V(t, 0) = 0, \quad V(t, x(t)) \geq \alpha(\|x\|), \quad \forall x \in B_\epsilon, \forall t \geq 0.$$

A locally positive definite function is locally like an energy function. Functions which are globally like energy functions are called positive definite functions:

Definition 2.3.6 [33] [Positive definite functions]

A continuous function $V(t, x(t)) : R^+ \times R^n \to R$ is a *positive definite function* if it satisfies the conditions of Definition 2.3.5 and, additionally, $\alpha(p) \to +\infty$ as $p \to +\infty$.

To bound the energy function from above, we define decrescence as follows:

Definition 2.3.7 [33] [Decrescent functions]

A continuous function $V(t, x(t)) : R^+ \times R^n \to R$ is a *decrescent* if for some $\epsilon > 0$ and some continuous, strictly increasing function $\beta(\cdot) : R^+ \to R$,

$$V(t, 0) = 0, \quad V(t, x(t)) \leq \beta(\|x\|), \quad \forall x \in B_\epsilon, \forall t \geq 0.$$

Lemma 2.3.8 [27] Given a positive definite function $W(x)$, there exist two functions, $\varphi_1, \varphi_2 \in K$ such that

$$\varphi_1\|x\| \leq W(x) \leq \varphi_2\|x\|, \quad x \in R^n.$$

Definition 2.3.9 [29] A function $V(\cdot) : R^+ \times R^n \to R$ is said to be *Lyapunov function* if it satisfies the following:

(i) $V(t, x)$ and all its partial derivatives $\dfrac{\partial V}{\partial t}, \dfrac{\partial V}{\partial x_i}$ are continuous for all $i = 1, 2, 3, ..., n$.

(ii) $V(t, x)$ is locally positive definite function, i.e., $V(0) = 0$ and $V(t, x) > 0$, $x \neq 0$, $\forall x \in B_\epsilon$.

(iii) The derivative of $V(t,x)$ with respect to system (2.5), namely

$$\begin{aligned}\dot{V}(t,x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + \dots + \frac{\partial V}{\partial x_n}\dot{x_n} \\ &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1}f_1 + \frac{\partial V}{\partial x_2}f_2 + \dots + \frac{\partial V}{\partial x_n}f_n. \end{aligned} \qquad (2.6)$$

$\dot{V}(t,x)$ is negative semi-definite i.e., $\dot{V}(t,0) = 0$, and $\forall x \in B_\epsilon$, $\dot{V}(t,x) \leq 0$.

Notice that in (2.6) the f_i are the components of f in system (2.5), so $\dot{V}$ can be determined directly from the system equations.

Using these definitions, the following theorem allows us to determine stability for a system by studying an appropriate energy function. Roughly, this theorem states that when $V(t,x(t))$ is a locally positive definite function and $\dot{V}(t,x(t)) \leq 0$ then we can conclude stability of the equilibrium point. The time derivative of V is taken along the trajectories of the system,

$$\dot{V} = \dot{V}|_{\dot{x}(t)=f(t,x(t))} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f.$$

Lemma 2.3.10 [33] [Basic theorem of Lyapunov]

Let $V(t,x(t)) : R^+ \times R^n \to R$ be a non-negative function with derivative $\dot{V}$ along the trajectories of the system (2.5).

(i) If $V(t,x(t))$ is locally positive definite function and $\dot{V}(t,x(t)) \leq 0$, locally in x and for all t, then the origin of the system (2.5) is locally stable (in the sense of Lyapunov).

(ii) If $V(t,x(t))$ is locally positive definite function and decrescent, and $\dot{V}(t,x(t)) \leq 0$ locally in x and for all t, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).

(iii) If $V(t,x(t))$ is locally positive definite function and decrescent, and $-\dot{V}(t,x(t))$ is locally positive definite function, then the origin of the system is uniformly locally asymptotically stable.

(iv) If $V(t,x(t))$ is positive definite function and decrescent, and $-\dot{V}(t,x(t))$ is positive definite function, then the origin of the system is globally uniformly asymptotically stable.

Lemma 2.3.11 [13] Consider the non autonomous system with time-delay of the form

$$\begin{cases} \dot{x}(t) = f(t, x(t-h)), & \forall t \geq 0; \\ x(t) = \phi(t), & \forall t \in [-h, 0] \end{cases} \tag{2.7}$$

where $x(t) \in R^n$ is the state variable, $h \in R^+$ is the delay and $f : R^+ \times C(C([-h, 0], R^n)) \to R^n$. $\phi(t)$ is a continuous vector-valued initial condition. We assume $f(t, 0) = 0$ so that system (2.7) admits the trivial solution. We also assume that system (2.7) has an existence and uniqueness solution. Suppose that $u, v, w : R^+ \to R^+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there exist a continuous differentiable functional $V : R^+ \times C \to R$ such that

$$u(\|\phi(0)\|) \leq V(x) \leq v(\|\phi\|),$$

the equilibrium point $x^* = 0$ of system (2.7) is

(i) uniformly stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|).$$

(ii) uniformly asymptotically stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

where $w(s) > 0$ for $s > 0$.

(iii) globally uniformly asymptotically stable if

$$\dot{V}(t, x(t)) \leq -w(\|\phi(0)\|),$$

and $\lim\limits_{s \to +\infty} u(s) = +\infty$.

Definition 2.3.12 [18] A functional $V : R \times C \to R^+$ is called a Lyapunov-Krasovskii functional for the system (2.7) if it has the following properties. There exist $\lambda_1, \lambda_2, \lambda_3 > 0$ such that

(i) $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(t, x_t) \leq -\lambda_3 \|x(t)\|^2$.

Lemma 2.3.13 [13] Consider the non autonomous time-delay system (2.7). If there exist a Lyapunov function $V(t, x_t)$ and $\lambda_1, \lambda_2 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold,

(i) $\lambda_1 \|x(t)\|^2 \leq V(t, x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(t, x_t) \leq 0$,

then the solution of the system is bounded, i.e., there exists $N > 0$ such that $\|x(t, \phi)\| \leq N\|\phi\|, \forall t \geq 0$.

Lemma 2.3.14 [18] Consider the autonomous time-delay system (2.7). If there exist a Lyapunov-Krasovskii function $V(x_t)$ and $\lambda_1, \lambda_2, \lambda_3 > 0$ such that for every solution $x(t)$ of the system, the following conditions hold,

(i) $\lambda_1 \|x(t)\|^2 \leq V(x_t) \leq \lambda_2 \|x_t\|^2$,

(ii) $\dot{V}(x_t) \leq -\lambda_3 \|x(t)\|^2$,

then the solution of the system (2.7) is exponentially stable.

Definition 2.3.15 [41] For given $\beta > 0$, the system (2.7) is said to be $\beta-$ stable, if there exists a function $\xi(\cdot) : R^+ \to R^+$ such that for each $\phi(t) \in C([-h, 0], R^n)$, the solution $x(t, \phi)$ of the system satisfies,

$$\|x(t, \phi)\| \leq \xi(\|\phi\|)e^{-\beta t}, \quad \forall t \in R^+.$$

Definition 2.3.16 [22] The system (2.7) is exponentially stable, if there exist constants $\alpha > 0$, $\beta > 0$ such that for each $\phi(t) \in C([-h, 0], R^n)$, the solution $x(t, \phi)$ of the system satisfies

$$\|x(t, \phi)\| \leq \alpha\|\phi\|e^{-\beta t}, \quad \forall t \in R^+.$$

Consider the discrete delay system of the form

$$\begin{cases} x(n+1) = f(n, x_n), & \forall n \geq n_0; \\ x_{n_0} = \phi, \end{cases} \tag{2.8}$$

where $x(n) \in R^n$ is the state variable, $n, n_0 \in Z^+$, $f \in C(Z^+ \times C([-m, 0], R^n), R^n)$, $\phi \in C([-m, 0], R^n)$ where $m \in Z^+$ represents the delay in system (2.8). $x_n \in C([-m, 0], R^n)$ is defined by $x_n(s) = x(n+s)$ for any $s \in C[-m, 0]$. We assume $f(t, 0) = 0$, so that system (2.8) admits the trivial solution. We also assume that system (2.8) has an unique solution, denoted by $x(n) = x(n, n_0, \phi)$ for any given initial data $n_0 \in Z^+$.

Definition 2.3.17 [26] The trivial solution of system (2.8) is said to be uniformly stable (US) if, for any given initial data, $n_0 \in Z^+$, $x_{n_0} = \phi$ and for any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ independent of n_0 such that when $\|\phi\|_m < \delta$, the following inequality holds:

$$\|x(n, n_0, \phi)\| \leq \epsilon,$$

for any $n \geq n_0$, $n, n_0 \in Z^+$.

Definition 2.3.18 [26] The trivial solution of system (2.8) is said to be uniformly attractive if, for each given initial data, $n_0 \in Z^+$, $x_{n_0} = \phi$ and for any $\eta > 0$, there exists there exist a positive real number $\sigma = \sigma(\eta) > 0$ and a positive integer $K = K(\eta) > 0$, where both σ and K are independent of n_0, such that when $\|\phi\|_m < \sigma$ and $n \geq n_0 + K$, the following inequality holds,

$$\|x(n, n_0, \phi)\| < \eta,$$

i.e.,

$$\lim_{n \to \infty} x(n, n_0, \phi) = 0.$$

Definition 2.3.19 [26] The trivial solution of system (2.8) is said to be uniformly asymptotically stable (UAS) if for any initial data $n_0 \in Z^+$, $x_{n_0} = \phi$, the trivial solution of system (2.8) is US and uniformly attractive.

Definition 2.3.20 [26] The trivial solution of system (2.8) is said to be uniformly exponentially stable (UES) if, for any initial data $n_0 \in Z^+$, $x_{n_0} = \phi$, there exist two positive numbers $\alpha > 0, M > 0$, where both α and M are independent of n_0, such that for all $n \geq n_0$, $n, n_0 \in Z^+$,

$$\|x(n, n_0, \phi)\| \leq M \|\phi\|_m e^{-\alpha(n-n_0)}.$$

Definition 2.3.21 [49] The discrete delay system (2.8) is said to be *asymptotically stable* if there exists a positive definite function $V(x) : R^n \to R^+$ such that

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0,$$

along the solution of the system (2.8).

Lemma 2.3.22 [2] Consider the impulsive system of the form

$$\begin{cases} \dot{x}(t) = f(t,x), & t \neq t_k; \\ \Delta x(t) = I_k(x), & t = t_k, \end{cases} \tag{2.9}$$

where $x \in R^n$ denotes the state and $f \in C[R \times R^n, R^n]$ satisfies a Lipschitz condition with respect to x which guarantees the existence and uniqueness of solutions of system (2.9) for given initial conditions. ($C[U;W]$ denotes the set of all continuous functions from set U to set W; and $C^k[U;W]$ denotes the set of all functions from U to W which have continuous derivatives up to order k.) The set $E = \{t_1, t_2, t_3, ... : t_1 < t_2 < t_3 < \cdots\} \subset R^+$ is an unbounded, closed, discrete subset of R^+ which denotes the set of times when jumps occur, and $I_k : R^n \to R^n$ denotes the incremental change of the state at the time t_k. It should be pointed out that in general E depends on a specific motion and that for different motions the corresponding sets E are in general different. Assume that for system (2.9) satisfying $f(t,0) = 0$ and $I_k(0) = 0$ for all $t \in R^+$ and $k \in Z^+$, there exists a $V : R^+ \times R^n \to R^+$ and $u, v \in K$ such that

$$u(\|x\|) \leq V(t,x) \leq v(\|x\|),$$

for all $(t,x) \in R^+ \times R^n$.

(i) If for any solution $x(t)$ of (2.9) which is defined on $[t_0, +\infty)$, it is true that $V(t,x)$ is left continuous on $[t_0, +\infty)$ and is differentiable everywhere on $(t_0, +\infty)$ except on the set E, and if it is also true that

$$\dot{V}(t, x(t)) \leq 0, \quad t \neq t_k,$$

$$V(t^+, x(t^+)) \leq V(t, x(t)), \quad t = t_k,$$

then the equilibrium $x = 0$ of system (2.9) is uniformly stable.

(ii) If in addition, we assume that there exists a $w \in K$ such that

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|), \quad t \neq t_k,$$

then the equilibrium $x = 0$ of system (2.9) is uniformly asymptotically stable.

Lemma 2.3.23 [3] [Completing the square]
Assume that $S \in M^{n\times n}$ is a symmetric positive definite matrix. Then for every $Q \in M^{n\times n}$:

$$2\langle Qy, x\rangle - \langle Sy, y\rangle \leq \langle QS^{-1}Q^T x, x\rangle, \quad \forall x, y \in R^n.$$

Lemma 2.3.24 [13] [Jensen's inequality]
For any constant symmetric matrix $R > 0$, scalar $h > 0$, and vector function $\dot{x}(t) : [-h, 0] \to \mathbb{R}^n$ such that the following integral is well defined, then

$$-h\int_{t-h}^{t} \dot{x}^T(s)R\dot{x}(s)ds \leq \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \begin{bmatrix} -R & R \\ R & -R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}.$$

Lemma 2.3.25 [50] For given matrices $Q = Q^T, H, E$ and $R = R^T > 0$ of appropriate dimension, then

$$Q + HFE + E^T F^T H^T < 0,$$

for all F satisfies $F^T F \leq R$, if and only if there exist a positive number $\epsilon > 0$, such that

$$Q + \epsilon^{-1}HH^T + \epsilon E^T RE < 0.$$

Lemma 2.3.26 [4] Give constant matrices M_1, M_2 and M_3 of appropriate dimensions with $M_1 = M_1^T$. Then,

$$M_1 + M_2\Delta(k)M_3 + M_3^T\Delta(k)^T M_2^T < 0,$$

where $\Delta(k) = F(k)[I - JF(k)]^{-1}$, $F(k)^T F(k) \leq I$, $\forall k \in Z^+$, if and only if for some scalar $\epsilon > 0$,

$$M_1 + \begin{bmatrix} \epsilon^{-1}M_3^T & \epsilon M_2 \end{bmatrix} \begin{bmatrix} I & -J \\ -J^T & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1}M_3^T & \epsilon M_2 \end{bmatrix}^T < 0.$$

Lemma 2.3.27 [13] Let G, H, F be real matrices of appropriate dimensions with $\|F\| < 1$. Then

(i) For any $\epsilon > 0 : GFH + H^T F^T G^T \leq \frac{1}{\epsilon} GG^T + \epsilon H^T H$.

(ii) For any $\epsilon > 0$ such that $\epsilon I - HH^T > 0$,

$$(A + GFH)(A + GFH)^T \leq AA^T + AH^T(\epsilon I - HH^T)^{-1}HA^T + \epsilon GG^T.$$

Lemma 2.3.28 [3] [Schur complement lemma]

Given constant symmetric matrices X, Y, Z where $Y > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

Lemma 2.3.29 [25] [Halanay lemma]

Let $m(t)$ be a scale positive function and assume that the following condition holds:

$$\dot{m}(t) \leq -am(t) + b\bar{m}(t), \quad t \geq t_0,$$

where constants $a > b > 0$. Then, there exist two condition $\gamma > 0$, $\alpha > 0$, such that for all $t \geq t_0$,

$$m(t) \leq \gamma \bar{m}(t_0) e^{-\alpha(t-t_0)}.$$

Here, $\bar{m}(t) = \sup_{t-h \leq s \leq t}\{m(s)\}$ and $\alpha > 0$ satisfies $\alpha - a + be^{\alpha h} = 0$.