Chapter 3

Continuous Time Delay Systems

In this chapter, we study the problem of robust stability for continuous-time delay systems such as linear parameter dependent (LPD) system with time-delay, uncertain LPD system with time-varying delays, uncertain linear system with interval time-varying delay and nonlinear perturbation. We use appropriate Lyapunov functions and derive stability conditions in terms of linear matrix inequalities (LMIs). Based on combination of the Riccati equation approach and the use of suitable Lyapunov functional, sufficient conditions for robust stability of linear non-autonomous delay systems with time-varying and norm-bounded uncertainties have been established. The conditions are formulated in terms of the solution of curtain Riccati differential equations. Numerical examples are presented to illustrate the effectiveness of the theoretical results.

3.1 Stability Criteria of LPD Systems with Time-delay

Consider the linear parameter dependent (LPD) system with time-delay of the form

$$\begin{cases} \dot{x}(t) = A(\alpha)x(t) + B(\alpha)x(t-h), & \forall t \geq 0; \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \tag{3.1}$$

where $x(t) \in R^n$ is the state, $h \in R^+$ is the delay and $\phi(t)$ is a continuous vector-valued initial condition on $[-h, 0]$. $A(\alpha)$ and $B(\alpha)$ are matrices belonging to the polytope Ω_1 or Ω_2 where

$$\Omega_1 := \{A(\alpha), B(\alpha)\} = \Big\{ \sum_{i=1}^{N} \alpha_i A_i, \sum_{i=1}^{N} \alpha_i B_i \Big\},$$

$$\sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0, A_i, B_i \in M^{n\times n}, i = 1, ..., N,$$

and

$$\Omega_2 := \{A(\alpha), B(\alpha)\} = \Big\{\sum_{i=1}^{N}\alpha_i(t)A_i, \sum_{i=1}^{N}\alpha_i(t)B_i\Big\},$$

$$\sum_{i=1}^{N}\alpha_i(t) = 1, \alpha_i(t) \geq 0, A_i, B_i \in M^{n\times n}, i = 1, ..., N.$$

We assume the following bounds of the parameter values:

$$\exists \beta_i > 0 : \quad \|\dot{\alpha}_i(t)\| \leq \beta_i, \quad \forall t > 0.$$

From system (3.1), we let

$$y(t) = e^{\beta t}x(t), \quad t \in R^+,$$

then the system (3.1) is transformed to the following delay system of the form

$$\dot{y}(t) = A_\beta(\alpha)y(t) + B_\beta(\alpha)y(t-h), \quad t \in R^+, \tag{3.2}$$

where

$$A_\beta(\alpha) = A(\alpha) + \beta I, \quad B_\beta(\alpha) = e^{\beta h}B(\alpha).$$

Theorem 3.1.1 *For given $\beta > 0$, the system (3.1) with $A(\alpha), B(\alpha) \in \Omega_1$ is $\beta-$stable if there exist symmetric positive definite matrices P, Q and positive real numbers h, ϵ such that the following conditions hold.*

$$\begin{aligned}
(i) \quad & A_i^T P_\epsilon + 2\beta P_\epsilon + P_\epsilon A_i + Q + e^{2\beta h}P_\epsilon B_i Q^{-1}B_i^T P_\epsilon \\
& \leq -I, \quad i = 1, ..., N. \\
(ii) \quad & A_i^T P_\epsilon + 4\beta P_\epsilon + P_\epsilon A_i + 2Q + A_j^T P_\epsilon + P_\epsilon A_j \\
& + e^{2\beta h}P_\epsilon B_i Q^{-1}B_j^T P_\epsilon + e^{2\beta h}P_\epsilon B_j Q^{-1}B_i^T P_\epsilon \\
& \leq \frac{2I}{N-1}, \quad i = 1, ..., N-1, j = i+1, ..., N.
\end{aligned}$$

Proof. We define the following Lyapunov function for system (3.2) of the form

$$V(t, y(t)) = y^T(t)Py(t) + \epsilon\|y(t)\|^2 + \int_{t-h}^{t} y^T(s)Qy(s)ds.$$

The derivative of $V(t, y(t))$ along the trajectories of system (3.2) is given by

$$\begin{aligned}\dot{V}(t,y(t)) &= 2\dot{y}^T(t)Py(t) + 2\epsilon\dot{y}^T(t)y(t) + y^T(t)Qy(t) - y^T(t-h)Qy(t-h)\\ &= 2y^T(t)A_\beta^T(\alpha)P_\epsilon y(t) + 2y^T(t-h)B_\beta^T(\alpha)P_\epsilon y(t)\\ &\quad + y^T(t)Qy(t) - y^T(t-h)Qy(t-h).\end{aligned}$$

Since $P_\epsilon = P + \epsilon I$ and Lemma 2.3.11, we have

$$\begin{aligned}&2y^T(t-h)B_\beta^T(\alpha)P_\epsilon y(t) - y^T(t-h)Qy(t-h)\\ &\quad \le y^T(t)P_\epsilon B_\beta(\alpha)Q^{-1}B_\beta^T(\alpha)P_\epsilon y(t).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\dot{V}(t,y(t)) &\le y^T(t)\Big[A_\beta^T(\alpha)P_\epsilon + P_\epsilon A_\beta(\alpha)\\ &\quad + P_\epsilon B_\beta(\alpha)Q^{-1}B_\beta^T(\alpha)P_\epsilon + Q\Big]y(t)\\ &= y^T(t)\Big[\{A^T(\alpha)+\beta I\}P_\epsilon + P_\epsilon\{A(\alpha)+\beta I\}\\ &\quad + P_\epsilon\{e^{\beta h}B(\alpha)\}Q^{-1}\{e^{\beta h}B^T(\alpha)\}P_\epsilon + Q\Big]y(t)\\ &= y^T(t)\Big[\{\sum_{i=1}^N \alpha_i A_i^T + \beta I\}P_\epsilon + P_\epsilon\{\sum_{i=1}^N \alpha_i A_i + \beta I\}\\ &\quad + P_\epsilon\{e^{\beta h}\sum_{i=1}^N \alpha_i B_i\}Q^{-1}\{e^{\beta h}\sum_{i=1}^N \alpha_i B_i^T\}P_\epsilon + Q\Big]y(t).\end{aligned}$$

Then, we get that

$$\begin{aligned}\dot{V}(t,y(t)) &\le y^T(t)\Big[\sum_{i=1}^N \alpha_i[\sum_{i=1}^N \alpha_i[A_i^T P_\epsilon + 2\beta P_\epsilon + P_\epsilon A_i + Q]]\\ &\quad + \{\sum_{i=1}^N \alpha_i e^{\beta h}P_\epsilon B_i Q^{-1}\}\{\sum_{i=1}^N \alpha_i e^{\beta h}B_i^T P_\epsilon\}\Big]y(t)\\ &= y^T(t)\Big[\sum_{i=1}^N \alpha_i^2[A_i^T P_\epsilon + 2\beta P_\epsilon + P_\epsilon A_i + Q\\ &\quad + e^{2\beta h}P_\epsilon B_i Q^{-1}B_i^T P_\epsilon] + \sum_{i=1}^{N-1}\sum_{j=i+1}^N \alpha_i\alpha_j[A_i^T P_\epsilon + 4\beta P_\epsilon\\ &\quad + P_\epsilon A_i + 2Q + A_j^T P_\epsilon + P_\epsilon A_j] + \sum_{i=1}^{N-1}\sum_{j=i+1}^N \alpha_i\alpha_j[e^{2\beta h}P_\epsilon\\ &\quad \times B_i Q^{-1}B_j^T P_\epsilon + e^{2\beta h}P_\epsilon B_j Q^{-1}B_i^T P_\epsilon]\Big]y(t).\end{aligned}$$

By assumption (i) and (ii), and from the facts that $\sum_{i=1}^{N} \alpha_i = 1$,

$$\sum_{i=1}^{N} \alpha_i A_i \sum_{i=1}^{N} \alpha_i B_i = \sum_{i=1}^{N} \alpha^2 A_i B_i + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j [A_i B_j + A_j B_i],$$

and

$$(N-1) \sum_{i=1}^{N} \alpha_i^2 - 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} [\alpha_i - \alpha_j]^2 \geq 0.$$

Hence, we conclude that

$$\dot{V}(t, y(t)) \leq 0, \quad \forall t \in R^+.$$

Integrating both sides of this inequality from 0 to t, we find

$$V(t, y(t)) - V(0, y(0)) \leq 0, \quad \forall t \in R^+,$$

and hence

$$\begin{aligned} y^T(t)Py(t) + \epsilon \|y(t)\|^2 + \int_{t-h}^{t} y^T(s)Qy(s)ds \\ \leq y^T(0)Py(0) + \epsilon \|y(0)\|^2 + \int_{0-h}^{0} y^T(s)Qy(s)ds. \end{aligned}$$

Since, we know that

$$y^T Py \geq 0, \quad \int_{t-h}^{t} y^T(s)Qy(t)ds \geq 0,$$

and

$$\int_{-h}^{0} y^T(s)Qy(s)ds \leq \lambda_{max}(Q)\|\phi\| \int_{-h}^{0} e^{\beta s} ds = \frac{\lambda_{max}(Q)}{\beta}(1 - e^{-\beta h})\|\phi\|^2,$$

we have

$$\epsilon \|y(t)\|^2 \;\leq\; \lambda_{max}(P)\|y(0)\|^2 + \epsilon \|y(0)\|^2 + \frac{\lambda_{max}(Q)}{\beta}(1 - e^{-\beta h})\|\phi\|.$$

Therefore, the solution $y(t, \phi)$ of the system (3.2) is bounded. Returning to the solution $x(t, \phi)$ of system (3.1) and noting that

$$\|y(0)\| = \|x(0)\| = \|\phi(0)\| \leq \|\phi\|,$$

we summarize that

$$\|x(t,\phi)\| \le \xi(\|\phi\|)e^{-\beta t}, \quad \forall t \in R^+,$$

where

$$\xi(\|\phi\|) \;\; := \;\; \{\epsilon^{-1}\lambda_{max}(P)\|\phi\|^2 + \|\phi\|^2 + \frac{\lambda_{max}(Q)}{\beta\epsilon}(1-e^{-\beta h})\|\phi\|\}^{\frac{1}{2}}.$$

This means that the system (3.1) is $\beta-$stable. The proof of the theorem is complete. □

Example 3.1.1 Consider the following LPD system with time-delay of the form

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)x(t-\frac{1}{2}), \quad t \in R^+, \tag{3.3}$$

with initial function $\phi(t) = \begin{bmatrix} 5 \\ -6 \end{bmatrix} \in C([-\frac{1}{2}, 0], R^+)$ where

$$A(\alpha) = \alpha_1 \begin{bmatrix} -5 & 1 \\ 1 & -3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 & 1 \\ -2 & -3 \end{bmatrix},$$

$$B(\alpha) = \alpha_1 \begin{bmatrix} -0.5 & 0 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.05 & 0 \\ 1 & 0 \end{bmatrix}.$$

We have $h = \frac{1}{2}, N = 2$. By taking $\epsilon = \beta = 1$ and $\alpha_1 = \alpha_1 = \frac{1}{2}$, we can verify that $P = \begin{bmatrix} 4 & -1 \\ -1 & 3 \end{bmatrix}$ and $Q = \begin{bmatrix} 16 & -4 \\ -4 & 9 \end{bmatrix}$ satisfy all conditions of Theorem 3.1.1. Therefore, the system (3.3) is 1- stable. □

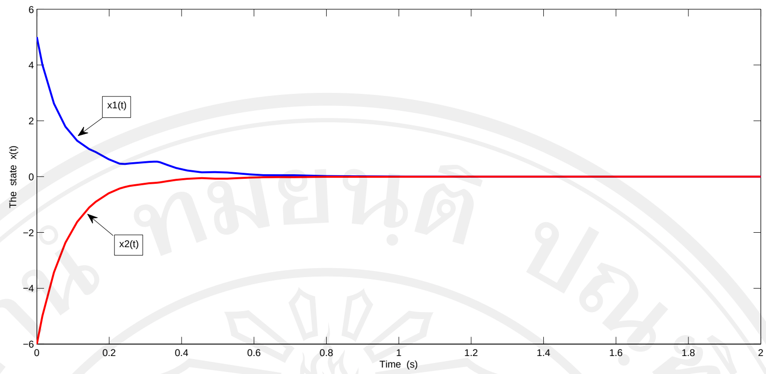

Figure 3.1: The simulation of solutions for the states $x_1(t)$ and $x_2(t)$ in the LPD delay system (3.3) with initial conditions $x_1(t) = 5$ and $x_2(t) = -6$, $-\frac{1}{2} \leq t \leq 0$ by using dde45 in Matlab.

We introduce the following notations for later use,

$$M_i(P_j, Q_j) = \begin{bmatrix} \sum_{k=1}^{N} \beta_k P_k + A_i^T P_j + P_j A_i + Q_j & P_j B_i \\ B_i^T P_j & -Q_j \end{bmatrix},$$

$$N_{i,j}(R, h) = \begin{bmatrix} hA_i^T RA_j - \frac{R}{h} & hA_i^T RB_j + \frac{R}{h} \\ hB_i^T RA_j + \frac{R}{h} & hB_i^T RB_j - \frac{R}{h} \end{bmatrix}, S \in R^{2n\times 2n}.$$

Theorem 3.1.2 *The system* (3.1) *with* $A(\alpha), B(\alpha) \in \Omega_2$ *is asymptotically stable if there exist symmetric positive definite matrices* $P_j, Q_j, j = 1, 2, ..., N$, R, *symmetric positive semi-definite matrix* S *and positive real numbers* h, ϵ *such that the following LMIs hold.*

$$(i) \quad M_i(P_i, Q_i) + N_{i,i}(R, h) < -S - \epsilon I, \quad i = 1, ..., N.$$

$$(ii) \quad M_j(P_i, Q_i) + M_i(P_i, Q_i) + N_{j,i}(R, h) + N_{i,j}(R, h) < \frac{2S}{N-1} - 2\epsilon I, i = 1, ..., N-1, j = i+1, ..., N.$$

Proof. We define the Lyapunov function for system (3.1) of the form

$$V(t, x(t)) = V_1(t, x(t)) + V_2(t, x(t)) + V_3(t, x(t))$$

where

$$V_1(t, x(t)) = x^T(t)P(\alpha)x(t), V_2(t, x(t)) = \int_{t-h}^{t} x^T(\theta)Q(\alpha)x(\theta)d\theta$$

and

$$V_3(t,x(t)) = \int_{t-h}^{t}\int_{s}^{t} \dot{x}^T(\theta)R\dot{x}(\theta)d\theta ds$$

with $P(\alpha) = \sum_{i=1}^{N}\alpha_i(t)P_i, Q(\alpha) = \sum_{i=1}^{N}\alpha_i(t)Q_i$. The derivative of $V(t,x(t))$ along the trajectories of system (3.1) is given by $\dot{V}(t,x(t)) = \dot{V}_1(t,x(t)) + \dot{V}_2(t,x(t)) + \dot{V}_3(t,x(t))$. Therefore, we obtain

$$\begin{aligned}
\dot{V}_1(t,x(t)) &= x^T(t)\dot{P}(\alpha)x(t) + 2\dot{x}^T(t)P(\alpha)x(t) \\
&= x^T(t)\dot{P}(\alpha)x(t) + 2x^T(t)A^T(\alpha)P(\alpha)x(t) \\
&\quad +2x^T(t-h)B^T(\alpha)P(\alpha)x(t) \\
\dot{V}_2(t,x(t)) &= x^T(t)Q(\alpha)x(t) - x^T(t-h)Q(\alpha)x(t-h)
\end{aligned}$$

and

$$\dot{V}_3(t,x(t)) = h\dot{x}^T(t)R\dot{x}(t) - \int_{t-h}^{t}\dot{x}^T(\theta)R\dot{x}(\theta)d\theta.$$

By using the Jensen's inequality, the last term can be bounded as follows:

$$-\int_{t-h}^{t}\dot{x}^T(\theta)R\dot{x}(\theta)d\theta < -[x(t)-x(t-h)]^T\frac{R}{h}[x(t)-x(t-h)].$$

We obtain that

$$\begin{aligned}
\dot{V}(t,x(t)) &< x^T(t)\dot{P}(\alpha)x(t) + 2x^T(t)A^T(\alpha)P(\alpha)x(t) \\
&\quad +2x^T(t-h)B^T(\alpha)P(\alpha)x(t) + x^T(t)Q(\alpha)x(t) \\
&\quad -x^T(t-h)Q(\alpha)x(t-h) + x^T(t)A^T(\alpha)hRA(\alpha)x(t) \\
&\quad +2x^T(t-h)B^T(\alpha)hRA(\alpha)x(t) + x^T(t-h)B^T(\alpha)hR \\
&\quad \times B(\alpha)x(t-h) - x^T(t)\frac{R}{h}x(t) + 2x^T(t)\frac{R}{h}x^T(t-h) \\
&\quad -x^T(t-h)\frac{R}{h}x(t-h).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\dot{V}(t,x(t)) &< \sum_{i=1}^{N}\alpha_i(t)\Big[x^T(t)\sum_{k=1}^{N}\beta_k P_k x(t) + 2x^T(t)\sum_{j=1}^{N}\alpha_j(t)A_j^T P_i x(t) \\
&\quad +2x^T(t-h)\sum_{j=1}^{N}\alpha_j(t)B_j^T P_i x(t) + x^T(t)Q_i x(t)
\end{aligned}$$

$$
\begin{aligned}
&-x^T(t-h)Q_ix(t-h)+x^T(t)\sum_{j=1}^{N}\alpha_j(t)A_j^ThRA_ix(t)\\
&+2x^T(t-h)\sum_{j=1}^{N}\alpha_j(t)B_j^ThRA_ix(t)\\
&+x^T(t-h)\sum_{j=1}^{N}\alpha_j(t)B_j^ThRB_ix(t-h)-x^T(t)\frac{R}{h}x(t)\\
&+2x^T(t-h)\frac{R}{h}x(t)-x^T(t-h)\frac{R}{h}x(t-h)\Big]
\end{aligned}
$$

since $\sum_{i=1}^{N}\dot{\alpha}_i(t)P_i \le \sum_{i=1}^{N}\beta_iP_i$. We can rewrite this inequality as

$$
\begin{aligned}
\dot{V} \quad < \quad & \sum_{i=1}^{N}\alpha_i(t)\Big[\sum_{j=1}^{N}\alpha_j(t)\{x^T(t)\sum_{k=1}^{N}\beta_kP_kx(t)+2x^T(t)A_j^TP_ix(t)\\
&+x^T(t)A_j^ThRA_ix(t)+x^T(t)Q_ix(t)\\
&+2x^T(t-h)B_j^ThRA_ix(t)+2x^T(t-h)B_j^TP_ix(t)\\
&-x^T(t-h)Q_ix(t-h)+x^T(t-h)B_j^ThRB_ix(t-h)\\
&-x^T(t)\frac{R}{h}x(t)+2x^T(t-h)\frac{R}{h}x(t)-x^T(t-h)\frac{R}{h}x(t-h)\}\Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\dot{V} \quad < \quad & \sum_{i=1}^{N}\alpha_i^2(t)\Big[x^T(t)\sum_{k=1}^{N}\beta_kP_kx(t)+2x^T(t)A_i^\top P_ix(t)\\
&+x^T(t)Q_ix(t)-x^T(t-h)Q_ix(t-h)\\
&+2x^T(t-h)B_i^TP_ix(t)+x^T(t)A_i^ThRA_ix(t)\\
&+2x^T(t-h)B_i^ThRA_ix(t)-x^T(t)\frac{R}{h}x(t)\\
&+2x^T(t-h)\frac{R}{h}x(t)+x^T(t-h)B_i^ThRB_ix(t-h)\\
&-x^T(t-h)\frac{R}{h}x(t-h)\Big]\\
&+\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i(t)\alpha_j(t)\Big[x^T(t)\sum_{k=1}^{N}\beta_kP_kx(t)\\
&+2x^T(t)A_j^TP_ix(t)+2x^T(t-h)B_j^TP_ix(t)\\
&+x^T(t)Q_ix(t)-x^T(t-h)Q_ix(t-h)\\
&+x^T(t)A_j^ThRA_ix(t)+2x^T(t-h)B_j^ThRA_ix(t)
\end{aligned}
$$

$$
\begin{aligned}
&+x^T(t-h)B_j^ThRB_ix(t-h) - x^T(t)\frac{R}{h}x(t)\\
&+2x^T(t-h)\frac{R}{h}x(t) - x^T(t-h)\frac{R}{h}x(t-h)\\
&+x^T(t)\sum_{k=1}^{N}\beta_kP_kx(t) + 2x^T(t)A_i^TP_jx(t)\\
&+2x^T(t-h)B_i^TP_jx(t) + x^T(t)Q_jx(t)\\
&+x^T(t)A_i^ThRA_jx(t) + 2x^T(t-h)B_i^ThRA_jx(t)\\
&+x^T(t-h)B_i^ThRB_jx(t-h) - x^T(t)\frac{R}{h}x(t)\\
&+2x^T(t-h)\frac{R}{h}x(t) - x^T(t-h)Q_jx(t-h)\\
&-x^T(t-h)\frac{R}{h}x(t-h)\Big].
\end{aligned}
$$

We can rewrite this inequality as

$$
\begin{aligned}
\dot{V} \;<\; & \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \Big\{ \sum_{i=1}^{N} \alpha_i^2(t)[M_i(P_i,Q_i) + N_{i,i}(R)] \\
& + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \alpha_i(t)\alpha_j(t)[M_j(P_i,Q_i) \\
& + M_i(P_i,Q_i) + N_{j,i}(R,h) + N_{i,j}(R,h)] \Big\} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}.
\end{aligned}
$$

From the identities

$$
(N-1)\sum_{i=1}^{N}\alpha_i^2(t) - 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i(t)\alpha_j(t) = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}[\alpha_i(t)-\alpha_j(t)]^2 \geq 0
$$

and

$$
-\epsilon\sum_{i=1}^{N}\alpha_i^2(t)I - 2\epsilon\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i(t)\alpha_j(t)I = -\epsilon I,
$$

we obtain

$$\begin{aligned}\dot{V} &< \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}^T \Big[(-\sum_{i=1}^{N}\alpha_i^2(t)S - \epsilon\sum_{i=1}^{N}\alpha_i^2(t)I) \\ &\quad + \frac{2}{N-1}\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i(t)\alpha_j(t)S \\ &\quad -2\epsilon\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i(t)\alpha_j(t)I\Big] \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ &\leq -\epsilon\|x\|^2.\end{aligned}$$

Therefore, this means that the system (3.1) is asymptotically stable. The proof of the theorem is complete. □

Example 3.1.2 Consider the following LPD delay system of the form

$$\dot{x}(t) = A(\alpha)x(t) + B(\alpha)x(t-1), \quad t \in R^+, \tag{3.4}$$

where

$$A(\alpha) = \alpha_1(t)\begin{bmatrix} -9 & 1 \\ 1 & -6 \end{bmatrix} + \alpha_2(t)\begin{bmatrix} -9 & 1 \\ 1 & -5 \end{bmatrix},$$

$$B(\alpha) = \alpha_1(t)\begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} + \alpha_2(t)\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

We have $h = 1, N = 2$. By taking $\epsilon = \frac{1}{8}, \beta = \frac{1}{2}$ (for instance, we choose $\alpha_1 = e^{-\frac{t}{2}}$ and $\alpha_2 = 1 - e^{-\frac{t}{2}}$), we can verify that

$$Q_1 = \begin{bmatrix} 30 & -1 \\ -1 & 25 \end{bmatrix}, Q_2 = \begin{bmatrix} 30 & -1 \\ -1 & 26 \end{bmatrix}, R = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, P_1 = P_2 = \begin{bmatrix} 100 & -1 \\ -1 & 80 \end{bmatrix},$$

and $S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ satisfy all conditions of Theorem 3.1.2. Therefore, the system (3.4) is asymptotically stable. □

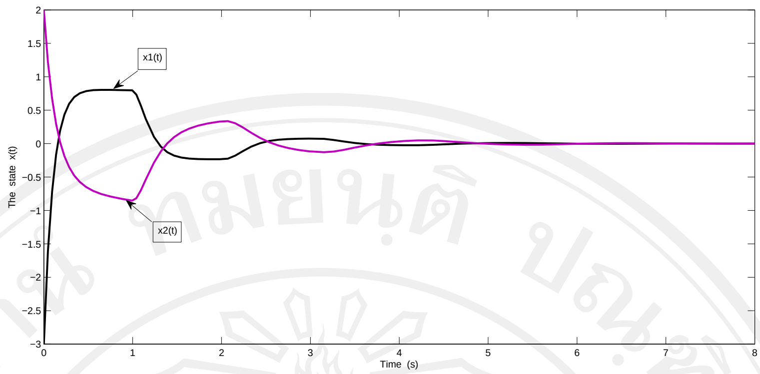

Figure 3.2: The simulation of solutions for the states $x_1(t)$ and $x_2(t)$ in the LPD delay system (3.4) with initial conditions $x_1(t) = -3$ and $x_2(t) = 2$, $-1 \leq t \leq 0$ by using dde45 in Matlab.

3.2 Stability Criteria of Uncertain LPD Systems with Time-varying Delays

Consider the uncertain linear parameter dependent (LPD) system with time-varying delays of the form

$$\begin{cases} \dot{x}(t) = \Big[A(\alpha) + \Delta A(t)\Big]x(t) + \Big[B(\alpha) + \Delta B(t)\Big]x(t - h(t)), & \forall t \geq 0; \\ x(t) = \phi(t), & \forall t \in [-h, 0], \end{cases} \tag{3.5}$$

where $x(t) \in R^n$, $A(\alpha) = \sum_{i=1}^{N} \alpha_i A_i$, $B(\alpha) = \sum_{i=1}^{N} \alpha_i B_i$ and $\phi(t)$ is a continuous vector-valued initial condition on $[-h, 0]$. The delay $h(t)$ is a time varying bounded continuous function satisfying

$$0 \leq h(t) \leq h, \quad \dot{h}(t) \leq \delta < 1,$$

for all $t > 0$. $\Delta A(t)$ and $\Delta B(t)$ are unknown matrices representing time-varying parameter uncertainties such as we assume to be of the form

$$\Delta A(t) = E(\alpha)\Delta(t)H(\alpha), \Delta B(t) = E(\alpha)\Delta(t)M(\alpha),$$

where $H(\alpha) = \sum_{i=1}^{N} \alpha_i H_i$, $M(\alpha) = \sum_{i=1}^{N} \alpha_i M_i$, $E(\alpha) = \sum_{i=1}^{N} \alpha_i E_i$ and $\Delta(k)$ satisfies

$$\Delta(t) = F(t)[I - JF(t)]^{-1}, \quad I - JJ^T > 0.$$

The uncertain matrix $F(t)$ satisfies

$$F(t)^T F(t) \leq I. \tag{3.6}$$

Remark 1. The condition (3.6) guarantees that $I - JF(t)$ to be invertible. It is easy to know that when $J = 0$, the parametric uncertainty of linear fractional form reduces to $\Delta(t) = F(t)$.

We change the form of the state variable

$$y(t) = e^{\beta t} x(t), \quad t \in R^+,$$

then the system (3.5) is transformed to the following delay system of the form

$$\dot{y}(t) = \overline{A}(\alpha, \beta) y(t) + \overline{B}(\alpha, \beta) y(t - h(t)), \quad t \in R^+, \tag{3.7}$$

where

$$\overline{A}(\alpha, \beta) = A(\alpha) + \Delta A(t) + \beta I, \quad \overline{B}(\alpha, \beta) = e^{\beta h(t)}[B(\alpha) + \Delta B(t)].$$

We introduce the following notations for using this Section,

$$\Phi(\alpha) = \begin{bmatrix} \overline{A}^T(\alpha, \beta) P(\alpha) + P(\alpha) \overline{A}(\alpha, \beta) + Q(\alpha) & P(\alpha) \widetilde{B}(\alpha) \\ \widetilde{B}^T(\alpha) P(\alpha) & -(1 - \delta) e^{-2\beta h} Q(\alpha) \end{bmatrix},$$

where

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i, \quad Q(\alpha) = \sum_{i=1}^{N} \alpha_i Q_i, \quad \widetilde{B}(\alpha) = B(\alpha) + \Delta B(t).$$

Lemma 3.2.1 *Let $\overline{A}(\alpha, \beta), \overline{B}(\alpha, \beta) \in R^{n \times n}$ be given matrices as in (3.7). Let $P_i, Q_i, i = 1, 2, ..., N$ be symmetric positive definite matrices and positive real numbers $\delta, \beta, h, \epsilon, \zeta$. Then $\Phi(\alpha) \leq 0$ if and only if*

$$(i) \quad \begin{bmatrix} \Theta_{i,i} & P_i B_i & \epsilon^{-1} H_i^T & \epsilon P_i E_i \\ B_i^T P_i & -(1 - \delta) e^{-2\beta h} Q_i & \epsilon^{-1} M_i^T & 0 \\ \epsilon^{-1} H_i & \epsilon^{-1} M_i & -I & J \\ \epsilon E_i^T P_i & 0 & J^T & -I \end{bmatrix} \leq -\zeta I,$$

$i = i, 2, \ldots, N.$

$$(ii) \quad \begin{bmatrix} \Theta_{i,j} & P_iB_j & \epsilon^{-1}H_i^T & \epsilon P_iE_j \\ B_i^TP_j & -(1-\delta)e^{-2\beta h}Q_i & \epsilon^{-1}M_i^T & 0 \\ \epsilon^{-1}H_i & \epsilon^{-1}M_i & -I & J \\ \epsilon E_i^TP_j & 0 & J^T & -I \end{bmatrix} + \begin{bmatrix} \Theta_{j,i} & P_jB_i & \epsilon^{-1}H_j^T & \epsilon P_jE_i \\ B_j^TP_i & -(1-\delta)e^{-2\beta h}Q_j & \epsilon^{-1}M_j^T & 0 \\ \epsilon^{-1}H_j & \epsilon^{-1}M_j & -I & J \\ \epsilon E_j^TP_i & 0 & J^T & -I \end{bmatrix} \leq \frac{2\zeta I}{N-1},$$

$i = 1, ..., N-1, j = i+1, ..., N,$

where $\Theta_{i,j} = (A_i^T + \beta I)P_j + P_i(A_j + \beta I) + Q_i$.

Proof. We consider $\Phi(\alpha)$ define as

$$\Phi(\alpha) = \begin{bmatrix} \overline{A}^T(\alpha,\beta)P(\alpha) + P(\alpha)\overline{A}(\alpha,\beta) + Q(\alpha) & P(\alpha)\widetilde{B}(\alpha) \\ \widetilde{B}^T(\alpha)P(\alpha) & -(1-\delta)e^{-2h}Q(\alpha) \end{bmatrix}.$$

Then, we have

$$\begin{aligned} &\Phi(\alpha) \\ = &\begin{bmatrix} A^T(\alpha,\beta)P(\alpha) + P(\alpha)A(\alpha,\beta) + Q(\alpha) & P(\alpha)B(\alpha) \\ B^T(\alpha)P(\alpha) & -(1-\delta)e^{-2\beta h}Q(\alpha) \end{bmatrix} \\ &+ \begin{bmatrix} P(\alpha)E(\alpha)\Delta(t)H(\alpha) + H^T(\alpha)\Delta^T(t)E^T(\alpha)P(\alpha) & P(\alpha)E(\alpha)\Delta(t)M(\alpha) \\ M^T(\alpha)\Delta^T(t)E^T(\alpha)P(\alpha) & 0 \end{bmatrix} \\ = &\begin{bmatrix} A^T(\alpha,\beta)P(\alpha) + P(\alpha)A(\alpha,\beta) + Q(\alpha) & P(\alpha)B(\alpha) \\ B^T(\alpha)P(\alpha) & -(1-\delta)e^{-2\beta h}Q(\alpha) \end{bmatrix} \\ &+ \begin{bmatrix} P(\alpha)E(\alpha) \\ 0 \end{bmatrix} \Delta(t) \begin{bmatrix} H(\alpha) & M(\alpha) \end{bmatrix} + \begin{bmatrix} H^T(\alpha) \\ M^T(\alpha) \end{bmatrix} \Delta^T(t) \begin{bmatrix} E^T(\alpha)P(\alpha) & 0 \end{bmatrix}. \end{aligned}$$

We assume $\Phi(\alpha) \leq 0$. By lemma 2.3.14, $\Phi(\alpha) \leq 0$ is equivalent to below inequality as

$$\begin{bmatrix} A^T(\alpha,\beta)P(\alpha)+P(\alpha)A(\alpha,\beta)+Q(\alpha) & P(\alpha)B(\alpha) \\ B^T(\alpha)P(\alpha) & -(1-\delta)e^{-2\beta h}Q(\alpha) \end{bmatrix} + \begin{bmatrix} \epsilon^{-1}H^T(\alpha) & \epsilon P(\alpha)E(\alpha) \\ \epsilon^{-1}M^T(\alpha) & 0 \end{bmatrix} \begin{bmatrix} I & -J \\ -J^T & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1}H^T(\alpha) & \epsilon P(\alpha)E(\alpha) \\ \epsilon^{-1}M^T(\alpha) & 0 \end{bmatrix}^T \leq 0.$$

By using Schur complement Lemma 2.3.8 in above inequality, it becomes that

$$\begin{bmatrix} \Theta_{11}(\alpha) & P(\alpha)B(\alpha) & \epsilon^{-1}H^T(\alpha) & \epsilon P(\alpha)E(\alpha) \\ B^T(\alpha)P(\alpha) & -(1-\delta)e^{-2\beta h}Q(\alpha) & \epsilon^{-1}M^T(\alpha) & 0 \\ \epsilon^{-1}H(\alpha) & \epsilon^{-1}M(\alpha) & -I & J \\ \epsilon E^T(\alpha)P(\alpha) & 0 & J^T & -I \end{bmatrix} \leq 0,$$

where $\Theta_{11}(\alpha) = A^T(\alpha,\beta)P(\alpha)+P(\alpha)A(\alpha,\beta)+Q(\alpha)$. We know from the facts that $\sum_{i=1}^{N}\alpha_i = 1$,

$$\sum_{i=1}^{N}\alpha_i A_i \sum_{i=1}^{N}\alpha_i B_i = \sum_{i=1}^{N}\alpha^2 A_i B_i + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j[A_iB_j + A_jB_i],$$

$$(N-1)\sum_{i=1}^{N}\alpha_i^2\zeta - 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\zeta = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}[\alpha_i-\alpha_j]^2\zeta \geq 0.$$

Therefore, we obtain

$$\begin{bmatrix} \Theta_{i,i} & P_iB_i & \epsilon^{-1}H_i^T & \epsilon P_iE_i \\ B_i^TP_i & -(1-\delta)e^{-2\beta h}Q_i & \epsilon^{-1}M_i^T & 0 \\ \epsilon^{-1}H_i & \epsilon^{-1}M_i & -I & J \\ \epsilon E_i^TP_i & 0 & J^T & -I \end{bmatrix} \leq -\zeta I,$$

$i = i, 2, \ldots, N.$

$$\begin{bmatrix} \Theta_{i,j} & P_iB_j & \epsilon^{-1}H_i^T & \epsilon P_iE_j \\ B_i^TP_j & -(1-\delta)e^{-2\beta h}Q_i & \epsilon^{-1}M_i^T & 0 \\ \epsilon^{-1}H_i & \epsilon^{-1}M_i & -I & J \\ \epsilon E_i^TP_j & 0 & J^T & -I \end{bmatrix} + \begin{bmatrix} \Theta_{j,i} & P_jB_i & \epsilon^{-1}H_j^T & \epsilon P_jE_i \\ B_j^TP_i & -(1-\delta)e^{-2\beta h}Q_j & \epsilon^{-1}M_j^T & 0 \\ \epsilon^{-1}H_j & \epsilon^{-1}M_j & -I & J \\ \epsilon E_j^TP_i & 0 & J^T & -I \end{bmatrix} \le \frac{2\zeta I}{N-1},$$

$$i = 1, ..., N-1, j = i+1, ..., N,$$

where $\Theta_{i,j} = (A_i^T + \beta I)P_j + P_i(A_j + \beta I) + Q_i$. The proof of the lemma is complete.□

Theorem 3.2.2 *For given $\beta > 0$, the system (3.5) is robustly $\beta-$ stable, if there exist symmetric positive definite matrices P_i, Q_i, $i = 1, 2, ..., N$ and positive real numbers δ, ϵ, ζ such that the following conditions hold.*

$$(i) \quad \begin{bmatrix} \Theta_{i,i} & P_iB_i & \epsilon^{-1}H_i^T & \epsilon P_iE_i \\ B_i^TP_i & -(1-\delta)e^{-2\beta h}Q_i & \epsilon^{-1}M_i^T & 0 \\ \epsilon^{-1}H_i & \epsilon^{-1}M_i & -I & J \\ \epsilon E_i^TP_i & 0 & J^T & -I \end{bmatrix} \le -\zeta I,$$

$$i = i, 2, \ldots, N.$$

$$(ii) \quad \begin{bmatrix} \Theta_{i,j} & P_iB_j & \epsilon^{-1}H_i^T & \epsilon P_iE_j \\ B_i^TP_j & -(1-\delta)e^{-2\beta h}Q_i & \epsilon^{-1}M_i^T & 0 \\ \epsilon^{-1}H_i & \epsilon^{-1}M_i & -I & J \\ \epsilon E_i^TP_j & 0 & J^T & -I \end{bmatrix} + \begin{bmatrix} \Theta_{j,i} & P_jB_i & \epsilon^{-1}H_j^T & \epsilon P_jE_i \\ B_j^TP_i & -(1-\delta)e^{-2\beta h}Q_j & \epsilon^{-1}M_j^T & 0 \\ \epsilon^{-1}H_j & \epsilon^{-1}M_j & -I & J \\ \epsilon E_j^TP_i & 0 & J^T & -I \end{bmatrix} \le \frac{2\zeta I}{N-1},$$

$$i = 1, ..., N-1, j = i+1, ..., N,$$

where $\Theta_{i,j} = (A_i^T + \beta I)P_j + P_i(A_j + \beta I) + Q_i$.

Proof. We define the following Lyapunov function for system (3.7) of the form

$$V(t, y(t)) = y^T(t)P(\alpha)y(t) + \int_{t-h(t)}^{t} y^T(s)Q(\alpha)y(s)ds.$$

The derivative of $V(t, y(t))$ along the trajectories of system (3.7) is given by

$$\begin{aligned}
\dot{V}(t, y(t)) &= \dot{y}^T(t)P(\alpha)y(t) + y^T(t)P(\alpha)\dot{y}(t) \\
&\quad + y^T(t)Q(\alpha)y(t) - (1 - \dot{h}(t))y^T(t - h(t))Q(\alpha)y(t - h(t)) \\
&\leq y^T(t)\overline{A}^T(\alpha, \beta)P(\alpha)y(t) + y^T(t - h(t))\overline{B}^T(\alpha, \beta)P(\alpha)y(t) \\
&\quad + y^T(t)P(\alpha)\overline{A}(\alpha, \beta)y(t) + y^T(t)P(\alpha)\overline{B}(\alpha, \beta)y(t - h(t)) \\
&\quad + y^T(t)Q(\alpha)y(t) - (1 - \delta)y^T(t - h(t))Q(\alpha)y(t - h(t)) \\
&\leq \begin{bmatrix} y(t) \\ e^{\beta h(t)}y(t - h(t)) \end{bmatrix}^T \Phi(\alpha) \begin{bmatrix} y(t) \\ e^{\beta h(t)}y(t - h(t)) \end{bmatrix}
\end{aligned}$$

where

$$\Phi(\alpha) = \begin{bmatrix} \overline{A}^T(\alpha, \beta)P(\alpha) + P(\alpha)\overline{A}(\alpha, \beta) + Q(\alpha) & P(\alpha)\widetilde{B}(\alpha) \\ \widetilde{B}^T(\alpha)P(\alpha) & -(1 - \delta)e^{-2\beta h}Q(\alpha) \end{bmatrix},$$

and

$$\overline{A}(\alpha, \beta) = A(\alpha) + \Delta A(t) + \beta I, \quad \overline{B}(\alpha, \beta) = e^{\beta h(t)}\widetilde{B}(\alpha) = e^{\beta h(t)}\Big[B(\alpha) + \Delta B(t)\Big].$$

By assumption $(i), (ii)$ and Lemma 3.2.1, we conclude that

$$\dot{V}(t, y(t)) \leq 0.$$

Integrating both sides of above inequality from 0 to t, we find

$$V(t, y(t)) - V(0, y(0)) \leq 0,$$

and hence

$$\begin{aligned}
&y^T(t)P(\alpha)y(t) + \int_{t-h(t)}^{t} y^T(s)Q(\alpha)y(s)ds \\
&\leq y^T(0)P(\alpha)y(0) + \int_{-h(0)}^{0} y^T(s)Q(\alpha)y(s)ds,
\end{aligned}$$

$$\int_{-h}^{0} y^T(s)Q(\alpha)y(s)ds \le \max(\lambda_{max}(Q_i))\|\phi\|^2 \int_{-h}^{0} e^{2\beta s}ds$$
$$= \frac{\max(\lambda_{max}(Q_i))}{2\beta}(1 - e^{-2\beta h})\|\phi\|^2.$$

We conclude that

$$\min(\lambda_{min}(P_i))\|y(t)\|^2 \le \max(\lambda_{max}(P_i))\|y(0)\|^2 + \frac{\max(\lambda_{max}(Q_i))}{2\beta}(1 - e^{-2\beta h})\|\phi\|^2,$$

for $i = 1, 2, 3, ..., N$. Therefore, the solution $y(t, \phi)$ of the system (3.7) is bounded. Returning to the solution $x(t, \phi)$ of system (3.5), it is easy to see that

$$\|y(0)\| = \|x(0)\| = \phi(0) \le \|\phi\|,$$

we conclude

$$\|x(t, \phi)\| \le \xi(\|\phi\|)e^{-\beta t},$$

where

$$\xi(\|\phi\|) := \left\{\frac{\max(\lambda_{max}(Q_i))\|\phi\|^2 + \frac{\max(\lambda_{max}(Q_i))}{2\beta}(1 - e^{-2\beta h})\|\phi\|}{\min(\lambda_{min}(P_i))}\right\}^{\frac{1}{2}}.$$

This means that the system (3.5) is robustly $\beta-$ stable. The proof of the theorem is complete. □

Example 3.2.2 Consider the following uncertain linear parameter dependent (LPD) system with time-varying delays of the form

$$\dot{x}(t) = \Big[A(\alpha) + \Delta A(t)\Big]x(t) + \Big[B(\alpha) + \Delta B(t)\Big]x(t - h(t)), \quad t \in R^+, \qquad (3.8)$$

where

$$A(\alpha) = \alpha_1 \begin{bmatrix} -4 & 1 \\ 2 & -3.9 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 & 0.3 \\ -0.6 & -0.2 \end{bmatrix},$$

$$B(\alpha) = \alpha_1 \begin{bmatrix} -0.4 & 0.1 \\ -0.6 & -0.2 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1.1 & -0.3 \\ 2.1 & -1.3 \end{bmatrix},$$

$$E(\alpha) = \alpha_1 \begin{bmatrix} -2.1 & 0.1 \\ 0 & -0.1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix},$$

$$H(\alpha) = \alpha_1 \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$M(\alpha) = \alpha_1 \begin{bmatrix} -0.1 & 0 \\ -0.1 & -0.1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.11 & 0 \\ 0.21 & -0.1 \end{bmatrix},$$

$$J = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad F(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By taking $\epsilon = 1, \beta = 0.9, \delta = 0.1$, we choose $\alpha_1 = \alpha_2 = \frac{1}{2}$. By using LMI Toolbox in MATLAB, we use the condition (i) and (ii) in Theorem 3.2.2 for this example. The solutions of LMI verify as follows of the form

$$P(\alpha) = \alpha_1 \begin{bmatrix} 0.6876 & -0.0722 \\ -0.0722 & 0.1706 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.9396 & 0.1174 \\ 0.1174 & 0.2178 \end{bmatrix},$$

$$Q(\alpha) = \alpha_1 \begin{bmatrix} 2.0002 & 0.1645 \\ 0.1645 & 0.6791 \end{bmatrix} + \alpha_2 10^{-3} \begin{bmatrix} 0.5298 & 0.0594 \\ 0.0594 & 0.1686 \end{bmatrix},$$

and $\zeta = 4.7267 \times 10^{-6}$. We obtain the maximum upper bound of time-varying delays ($h_{max} = 0.6363$). Therefore, the system (3.8) is asymptotically stable. $\square$

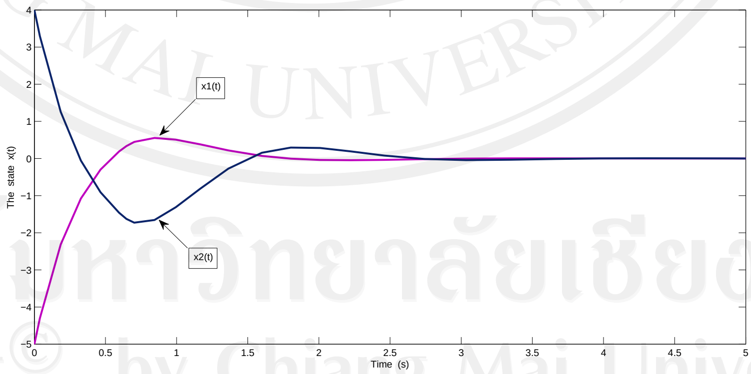

Figure 3.3: The simulation of solutions for the states $x_1(t)$ and $x_2(t)$ in the LPD delay system (3.8) with initial conditions $x_1(t) = -5$ and $x_2(t) = 4$, $-0.6363 \leq t \leq 0$ by using dde45 in Matlab.

3.3 Stability Criteria of Uncertain Linear System with Interval Time-varying Delays and Nonlinear Perturbations.

Consider systems described by the following state equation of the form

$$\begin{cases} \dot{x}(t) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t - h(t)) \\ \qquad + f(t, x(t)) + g(t, x(t - h(t))), & \forall t \geq 0; \\ x(t) = \phi(t), \quad \dot{x}(t) = \psi(t), & \forall t \in [-h_2, 0], \end{cases} \tag{3.9}$$

where $x(t) \in R^n$ is the state, A and B are given matrices of appropriate dimensions, $0 \leq h_1 \leq h(t) \leq h_2, h_2 > 0$. Consider the initial function $\phi(t), \psi(t) \in C([-h_2, 0], R^n)$ with the norm $\|\phi\| = \sup_{t\in[-h_2,0]} \|\phi(t)\|$ and $\|\psi\| = \sup_{t\in[-h_2,0]} \|\psi(t)\|$. $\Delta A(t)$ and $\Delta B(t)$ are unknown matrices representing time-varying parameter uncertainties, we are assumed to be of the form

$$\Delta A(t) = KF(t)A_1, \Delta B(t) = KF(t)B_1,$$

where K, A_1, B_1 are known constant matrices and $F(t)$ is an unknown time-varying matrix satisfying $\|F(t)\| \leq 1$. The uncertainties $f(.), g(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t - h(t))$, respectively, and are bounded in magnitude of the form

$$f^T(t, x(t))f(t, x(t)) \leq \eta x^T(t)x(t),$$

$$g^T(t, x(t - h(t)))g(t, x(t - h(t))) \leq \rho x^T(t - h(t))x(t - h(t)),$$

where η, ρ are given positive real constants. The delay $h(t)$ is a continuous differentiable function satisfying

$$0 \leq h_1 \leq h(t) \leq h_2, \quad \dot{h}(t) \leq \delta < 1.$$

In this section, we present the exponentially stability conditions dependent on interval time-varying delays of uncertain linear system (3.9) with nonlinear perturbations vie linear matrix inequality (LMIs) approach.

Theorem 3.3.1 *The system (3.9) is robust exponentially stable with a decay rate* α*, if there exist symmetric positive definite matrices* P_i*,* $i = 1, 2, ..., 8$*, matrices* Q_1, Q_2 *of appropriate dimension and positive real numbers* $h_1, h_2, \alpha, \epsilon_1, \epsilon_2, \delta, \gamma, \eta, \varepsilon$ *such that the following condition holds.*

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & 0 & 0 & A_{15} & 0 & 0 & Q_1^T & Q_1^T & Q_1^T E \\ * & \bar{A}_{22} & 0 & 0 & A_{25} & 0 & 0 & 0 & 0 & 0 \\ * & * & A_{33} & A_{34} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & A_{44} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & A_{55} & 0 & 0 & Q_2^T & Q_2^T & Q_2^T E \\ * & * & * & * & * & -e^{-2\alpha h_2} P_7 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -e^{-2\alpha h_2} P_8 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\epsilon_1 I & 0 & 0 \\ * & * & * & * & * & * & * & * & -\epsilon_2 I & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon I \end{bmatrix} \leq 0, \tag{3.10}$$

where

$$\begin{aligned}
\bar{A}_{11} &= 2\alpha P_1 + Q_1^T A + A^T Q_1 + \epsilon_1 \eta I + P_2 + P_3 + P_4 - e^{-2\alpha h_2} P_5 + \varepsilon A_1^T A_1, \\
\bar{A}_{22} &= -(1-\delta) e^{-2\alpha h_2} P_2 - e^{-2\alpha h_2} P_5 + \epsilon_2 \gamma I + \varepsilon B_1^T B_1, \\
A_{33} &= -e^{-2\alpha h_1} P_3 - e^{-2\alpha h_2} P_6, \\
A_{44} &= -e^{-2\alpha h_2} P_4 - e^{-2\alpha h_2} P_6, \\
A_{55} &= -Q_2^T - Q_2 + h_2^2 [P_5 + P_7] + (h_2 - h_1)^2 [P_6 + P_8], \\
\bar{A}_{12} &= Q_1^T B + \varepsilon A_1^T B_1, \\
A_{15} &= P_1 - Q_1^T + A^T Q_2, \\
A_{25} &= B^T Q_2, \\
A_{34} &= e^{-2\alpha h_2} P_6.
\end{aligned}$$

Proof. Rewrite system (3.9) in the following descriptor system:

$$
\begin{aligned}
\dot{x}(t) &= y(t),\\
y(t) &= A(t)x(t) + B(t)x(t-h(t)) + f(t, x(t)) + g(t, x(t-h(t))),\\
&\quad A(t) = [A + \Delta A(t)], \quad B(t) = [B + \Delta B(t)]. \qquad (3.11)
\end{aligned}
$$

It is easy to see that (3.9) is equivalent to (3.11). We consider a Lyapunov functional candidate for system (3.11) of the form

$$V(t, x(t)) = \sum_{i=1}^{8} V_i(t, x_t)$$

where

$$
\begin{aligned}
V_1(t, x_t) &= e^{2\alpha t} x^T(t) P_1 x(t) = e^{2\alpha t} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},\\
V_2(t, x_t) &= \int_{t-h(t)}^{t} e^{2\alpha s} x^T(s) P_2 x(s) ds,\\
V_3(t, x_t) &= \int_{t-h_1}^{t} e^{2\alpha s} x^T(s) P_3 x(s) ds,\\
V_4(t, x_t) &= \int_{t-h_2}^{t} e^{2\alpha s} x^T(s) P_4 x(s) ds,\\
V_5(t, x_t) &= h_2 \int_{-h_2}^{0} \int_{t+\theta}^{t} e^{2\alpha s} \dot{x}^T(s) P_5 \dot{x}(s) ds d\theta,\\
V_6(t, x_t) &= [h_2 - h_1] \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} e^{2\alpha s} \dot{x}^T(s) P_6 \dot{x}(s) ds d\theta,\\
V_7(t, x_t) &= h_2 \int_{-h_2}^{0} \int_{t+\theta}^{t} e^{2\alpha s} y^T(s) P_7 y(s) ds d\theta,\\
V_8(t, x_t) &= [h_2 - h_1] \int_{-h_2}^{-h_1} \int_{t+\theta}^{t} e^{2\alpha s} y^T(s) P_8 y(s) ds d\theta.
\end{aligned}
$$

Taking the time derivative of $V(t, x(t))$ along the solution of system (3.11) gives that

$$
\begin{aligned}
\dot{V}_1(t, x_t) &= e^{2\alpha t}\big[2\alpha x^T(t) P x(t) + 2 \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} P_1 & Q_1^T \\ 0 & Q_2^T \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ 0 \end{bmatrix}\big]\\
&= e^{2\alpha t}\Big[2\alpha x^T(t) P_1 x(t) + 2x^T(t) P_1 y(t)\\
&\quad + 2x^T(t) Q_1^T \big[-y(t) + A(t)x(t) + B(t)x(t-h(t)) + f_t + g_{t-h}\big]\\
&\quad + 2y^T(t) Q_2^T \big[-y(t) + A(t)x(t) + B(t)x(t-h(t)) + f_t + g_{t-h}\big]\Big],
\end{aligned}
$$

where, for simplicity, $f(t,x(t)) := f_t, \quad g(t,x(t-h(t))) := g_{t-h}$.

$$\begin{aligned}
\dot{V}_2(t,x_t) &= e^{2\alpha t}\big[x^T(t)P_2x(t)-(1-\dot{h}(t))e^{-2\alpha h(t)}x^T(t-h(t))P_2x(t-h(t))\big]\\
&\leq e^{2\alpha t}\big[x^T(t)P_2x(t)-(1-\delta)e^{-2\alpha h_2}x^T(t-h(t))P_2x(t-h(t)),\\
\dot{V}_3(t,x_t) &= e^{2\alpha t}\big[x^T(t)P_3x(t)-e^{-2\alpha h_1}x^T(t-h_1)P_3x(t-h_1)\big],\\
\dot{V}_4(t,x_t) &= e^{2\alpha t}\big[x^T(t)P_4x(t)-e^{-2\alpha h_2}x^T(t-h_2)P_4x(t-h_2)\big],\\
\dot{V}_5(t,x_t) &= e^{2\alpha t}\big[h_2^2\dot{x}^T(t)P_5\dot{x}(t)-h_2\int_{t-h_2}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)P_5\dot{x}(s)ds\big],\\
\dot{V}_6(t,x_t) &= e^{2\alpha t}\big[h_{21}^2\dot{x}^T(t)P_6\dot{x}(t)-h_{21}\int_{t-h_2}^{t-h_1}e^{2\alpha(s-t)}\dot{x}^T(s)P_6\dot{x}(s)ds\big],\\
\dot{V}_7(t,x_t) &= e^{2\alpha t}\big[h_2^2y^T(t)P_7y(t)-h_2\int_{t-h_2}^{t}e^{2\alpha(s-t)}y^T(s)P_7y(s)ds\big],\\
\dot{V}_8(t,x_t) &= e^{2\alpha t}\big[h_{21}^2y^T(t)P_8y(t)-h_{21}\int_{t-h_2}^{t-h_1}e^{2\alpha(s-t)}y^T(s)P_8y(s)ds\big],
\end{aligned}$$

where $h_{21}=h_2-h_1$. For any a scalar $s\in[t-h_2,t]$, we obtain $e^{-2\alpha h_2}\leq e^{2\alpha(s-t)}\leq 1$. Thus,

$$\begin{aligned}
-h_2\int_{t-h_2}^{t}e^{2\alpha(s-t)}\dot{x}^T(s)P_5\dot{x}(s)ds &\leq -h(t)e^{-2\alpha h_2}\int_{t-h_2}^{t}\dot{x}^T(s)P_5\dot{x}(s)ds\\
&\leq -h(t)e^{-2\alpha h_2}\int_{t-h(t)}^{t}\dot{x}^T(s)P_5\dot{x}(s)ds,
\end{aligned}$$

$$-h_2\int_{t-h_2}^{t}e^{2\alpha(s-t)}y^T(s)P_7y(s)ds \leq e^{-2\alpha h_2}\int_{t-h(t)}^{t}y^T(s)dsP_7\int_{t-h(t)}^{t}y(s)ds,$$

$$-h_{21}\int_{t-h_2}^{t-h_1}e^{2\alpha(s-t)}y^T(s)P_8y(s)ds \leq e^{-2\alpha h_2}\int_{t-h_2}^{t-h_1}y^T(s)dsP_8\int_{t-h_2}^{t-h_1}y(s)ds.$$

We use to follow from Lemma 2.3.12, then we get

$$\begin{aligned}
&-h(t)e^{-2\alpha h_2}\int_{t-h(t)}^{t}\dot{x}^T(s)P_5\dot{x}(s)ds\\
&\leq e^{-2\alpha h_2}\begin{bmatrix}x(t)\\ x(t-h(t))\end{bmatrix}^T\begin{bmatrix}-P_5 & P_5\\ P_5 & -P_5\end{bmatrix}\begin{bmatrix}x(t)\\ x(t-h(t))\end{bmatrix}\\
&\leq e^{-2\alpha h_2}\Big[-x(t)P_5x(t)+2x(t)P_5x(t-h(t))-x(t-h(t))P_5x(t-h(t))\Big],
\end{aligned}$$

and

$$\begin{aligned}
& -h_{21}\int_{t-h_2}^{t-h_1} e^{2\alpha(s-t)}\dot{x}^T(s)P_6\dot{x}(s)ds \\
\leq\ & e^{-2\alpha h_2}\Big[-x(t-h_1)P_6x(t-h_1)+2x(t-h_1)P_6x(t-h_2) \\
& -x(t-h_2)P_6x(t-h_2)\Big].
\end{aligned}$$

By the existence of positive scalars ϵ_1 and ϵ_2, we obtain

$$\epsilon_1\eta x^T(t)x(t)-\epsilon_1 f_t^T f_t \geq 0, \tag{3.12}$$

$$\epsilon_2\gamma x^T(t-h(t))x(t-h(t))-\epsilon_2 g_{t-h}^T g_{t-h} \geq 0, \tag{3.13}$$

It follows from $\dot{V}(t,x(t))$ with respect to system (3.11), together with (3.12),(3.13),

$$\begin{aligned}
\dot{V}(t,x(t)) \ \leq\ & \sum_{i=1}^{8}\dot{V}_i(t,x_t) \\
& +e^{2\alpha t}\Big[\epsilon_1\eta x^T(t)x(t)-\epsilon_1 f_t^T f_t+\epsilon_2\gamma x^T(t-h(t))x(t-h(t))-\epsilon_2 g^T g\Big] \\
\leq\ & e^{2\alpha t}\zeta^T(t)\chi\zeta(t),
\end{aligned}$$

where $\zeta(t)=\Big[x^T(t) \quad x^T(t-h(t)) \quad x^T(t-h_1) \quad x^T(t-h_2) \quad y_t^T \quad \int_{t-h(t)}^{t} y^T(s)ds \quad \int_{t-h_2}^{t-h_1} y^T(s)ds \quad f_t^T \quad g_{t-h}^T\Big]^T$, with $\chi=\Omega+MF(t)N+N^TF(t)^TM^T$, where Ω, M, N are given by

$$\Omega = \begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & A_{15} & 0 & 0 & Q_1^T & Q_1^T \\
* & A_{22} & 0 & 0 & A_{25} & 0 & 0 & 0 & 0 \\
* & * & A_{33} & A_{34} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & A_{44} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & A_{55} & 0 & 0 & Q_2^T & Q_2^T \\
* & * & * & * & * & -e^{-2\alpha h_2}P_7 & 0 & 0 & 0 \\
* & * & * & * & * & * & -e^{-2\alpha h_2}P_8 & 0 & 0 \\
* & * & * & * & * & * & * & -\epsilon_1 I & 0 \\
* & * & * & * & * & * & * & * & -\epsilon_2 I
\end{bmatrix},$$

$$\begin{aligned}
M^T &= \Big[E^TQ_1 \quad 0 \quad 0 \quad 0 \quad E^TQ_2 \quad 0 \quad 0 \quad 0 \quad 0\Big], \\
N &= \Big[A1 \quad B1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\Big],
\end{aligned}$$

where

$$
\begin{aligned}
A_{11} &= 2\alpha P_1 + Q_1^T A + A^T Q_1 + \epsilon_1 \eta I + P_2 + P_3 + P_4 - e^{-2\alpha h_2} P_5, \\
A_{22} &= -(1-\delta) e^{-2\alpha h_2} P_2 - e^{-2\alpha h_2} P_5 + \epsilon_2 \gamma I, \\
A_{33} &= -e^{-2\alpha h_1} P_3 - e^{-2\alpha h_2} P_6, \\
A_{44} &= -e^{-2\alpha h_2} P_4 - e^{-2\alpha h_2} P_6, \\
A_{55} &= -Q_2^T - Q_2 + h_2^2[P_5 + P_7] + h_{21}^2[P_6 + P_8], \\
A_{12} &= Q_1^T B(t), \\
A_{15} &= P_1 - Q_1^T + A(t)^T Q_2, \\
A_{25} &= B(t)^T Q_2, \\
A_{34} &= e^{-2\alpha h_2} P_6.
\end{aligned}
$$

By using Schur complement lemma 2.3.16, (3.10) is equivalent to the following condition

$$\Omega + \varepsilon^{-1} M M^T + \varepsilon N N^T \leq 0. \tag{3.14}$$

By using Lemma 2.3.13, condition (3.14) is equivalent to the following condition

$$\chi \ := \ \Omega + MF(t)N + N^T F(t)^T M^T \leq 0.$$

By assumption (3.10), we obtain

$$\dot{V}(t, x(t)) \leq 0. \tag{3.15}$$

Integrating both sides of (3.15) form 0 to t, we find

$$V(t, x(t)) \leq V(0, x(0)),$$

and hence

$$e^{2\alpha t} \lambda_{min}(P_1) \|x(t)\|^2 \leq V_1(t, x(t)) \leq V(t, x(t)) \leq V(0, x(0)),$$

where

$$\begin{aligned}
V(0,x(0)) &= x^T(0)P_1x(0) + \int_{-h(0)}^{0} e^{2\alpha s}x^T(s)P_2x(s)ds \\
&\quad + \int_{-h_1}^{0} e^{2\alpha s}x^T(s)P_3x(s)ds + \int_{-h_2}^{0} e^{2\alpha s}x^T(s)P_4x(s)ds \\
&\quad + h_2\int_{-h_2}^{0}\int_{\theta}^{0} e^{2\alpha s}\dot{x}^T(s)P_5\dot{x}(s)dsd\theta \\
&\quad + h_{21}\int_{-h_2}^{-h_1}\int_{\theta}^{0} e^{2\alpha s}\dot{x}^T(s)P_6\dot{x}(s)dsd\theta \\
&\quad + h_2\int_{-h_2}^{0}\int_{\theta}^{0} e^{2\alpha s}\dot{x}^T(s)P_7\dot{x}(s)dsd\theta \\
&\quad + h_{21}\int_{-h_2}^{-h_1}\int_{\theta}^{0} e^{2\alpha s}\dot{x}^T(s)P_8\dot{x}(s)dsd\theta.
\end{aligned}$$

Let us set $\lambda = max\Big[\lambda_{max}P_1, h_2\lambda_{max}P_2, h_1\lambda_{max}P_3, h_2\lambda_{max}P_4, h_2^3\lambda_{max}P_5, h_{21}^3\lambda_{max}P_6, h_2^3\lambda_{max}P_7, h_{21}^3\lambda_{max}P_8\Big]$, we obtain that

$$e^{2\alpha t}\lambda_{min}(P_1)\|x(t)\|^2 \le V(0,x(0)) \le \lambda\big[\sup_{-h_2\le s\le 0}(\|\phi(s)\|, \|\psi(s)\|)\big]^2,$$

and

$$\|x(t)\| \le \sqrt{\frac{\lambda}{\lambda_{min}(P_1)}}\big[\sup_{-h_2\le s\le 0}(\|\phi\|, \|\psi\|)\big]e^{-\alpha t}.$$

Therefore, system (3.9) is robustly exponentially stable with a decay rate α. The proof of theorem is complete. □

Example 3.3.1 Consider uncertain linear time-varying delay system with nonlinear perturbations described by the following state equation of the form

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)x(t-h(t)) + f(t,x(t)) + g(t,x(t-h(t))), \\
& \quad A(t) = A + KF(t)A_1, B(t) = B + KF(t)B_1, \qquad (3.16)
\end{aligned}$$

where

$$A = \begin{bmatrix} -8 & 1 \\ 2 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} -0.2 & 0 \\ -0.1 & -0.3 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0.1 & 0.01 \\ 0.02 & -0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad K = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix},$$

$$f(t,x(t)) = \begin{bmatrix} 0.7037sin(t)x_1(t) \\ 0.7037cos(t)x_2(t) \end{bmatrix}, g(t,x(t-h(t))) = \begin{bmatrix} 0.7037cos(t)x_1(t-h(t)) \\ 0.7037sin(t)x_2(t-h(t)) \end{bmatrix},$$

By taking $\epsilon_1 = 0.0266, \epsilon_2 = 0.0157, \varepsilon = 0.4351, \alpha = 0.1$ and $h(t) = 0.1 + 1.2sin^2(\frac{1}{3}t)$, we obtain $\gamma = \eta = 0.7037$, $\delta = 0.5$ and $h_1 = 0.1$, $h_2 = 1.3$. By using LMI Toolbox in MATLAB, we use the condition (i) in Theorem 3.3.1 for this example. The solutions of LMI verify as follows of the form

$$P_1 = \begin{bmatrix} 0.0789 & -0.0156 \\ -0.0156 & 0.0811 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.7097 & -0.0270 \\ -0.0270 & 0.7336 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0.1172 & -0.0710 \\ -0.0710 & 0.1391 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0.1199 & -0.0738 \\ -0.0738 & 0.1428 \end{bmatrix},$$

$$P_5 = 10^{-3} \times \begin{bmatrix} 0.9295 & -0.0164 \\ -0.0164 & 0.9189 \end{bmatrix}, \quad P_6 = \begin{bmatrix} 0.0010 & 0.0000 \\ 0.0000 & 0.0010 \end{bmatrix},$$

$$P_7 = \begin{bmatrix} 0.0015 & 0.0000 \\ 0.0000 & 0.0015 \end{bmatrix}, \quad P_8 = \begin{bmatrix} 0.0017 & 0.0000 \\ 0.0000 & 0.0017 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.0809 & -0.0034 \\ -0.0034 & 0.0741 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.0091 & -0.0003 \\ -0.0003 & 0.0091 \end{bmatrix}.$$

Therefore, the system (3.16) is robust exponentially stable. □

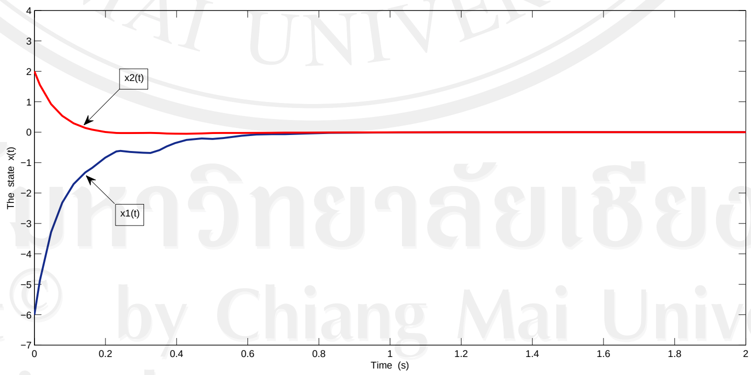

Figure 3.4: The simulation of solutions for the states $x_1(t)$ and $x_2(t)$ in the uncertain linear delay system (3.16) where $F(t) = I$ with initial conditions $x_1(t) = -6$ and $x_2(t) = 2$, $-1.3 \leq t \leq 0$ by using dde45 in Matlab.

3.4 Stability Criteria of Uncertain Linear Non-autonomous System with Time-varying Delays

Consider the uncertain linear non-autonomous system with time-varying delays of the form

$$\begin{cases} \dot{x}(t) = [A_0(t) + \Delta A_0(t)]x(t) + [A_1(t) + \Delta A_1(t)]x(t - h(t)), & t \geq 0 \; ; \\ x(t) = \phi(t), & t \in [-h, 0], \end{cases} \tag{3.17}$$

where $x(t) \in R^n$, $A_i(t), i = 0, 1$ are given matrix functions continuous on $[0, \infty), 0 \leq h(t) \leq h, h > 0$. Consider the initial function $\phi(t) \in C([-h, 0], R^n)$ with the norm $\|\phi\| = \sup_{t\in[-h,0]} \|\phi(t)\|$, and the admissible control $u(\cdot) \in L_2([0, t], R^m)$, for all $t \in \mathbb{R}^+$. The delay $h(t)$ is a continuously differentiable function satisfying

$$0 \leq h(t) \leq h, \quad \dot{h}(t) \leq \delta < 1.$$

The uncertainties $\Delta A_0, \Delta A_1$ are time-varying and satisfy the condition:

$$\Delta A_i(t) = G_i(t)F(t)H_i(t), \quad i = 0, 1,$$
$$\|F(t)\| \leq 1, \quad \forall t \in R^+,$$

where $G_i(t), H_i(t)$, $i = 0, 1$ are given matrix functions of appropriate dimensions. Given numbers $\epsilon > 0$, $\epsilon_0 > 0$, $\epsilon_1 > 0$, $\alpha > 0$, $h > 0$, we set

$$P_\epsilon(t) = P(t) + \epsilon I, \quad A_{0,\alpha}(t) = A_0(t) + \alpha I,$$
$$Q(t) = \epsilon_0 H_0^T(t)H_0(t) + I, \quad S(t) = \epsilon_1 I - H_1(t)H_1^T(t),$$
$$R(t) = \frac{e^{2\alpha h}}{1-\delta}[A_1(t)A_1^T(t) + \epsilon_1 G_1(t)G_1^T(t)$$
$$+ A_1(t)H_1^T(t)S^{-1}(t)H_1(t)A_1^T(t)] + \epsilon_0^{-1}G_0(t)G_0^T(t).$$

Consider the Riccati differential equation

$$\dot{P}_\epsilon(t) + P_\epsilon(t)A_{0,\alpha}(t) + A_{0,\alpha}^T(t)P_\epsilon(t) + P_\epsilon(t)R(t)P_\epsilon(t) + Q(t) = 0. \tag{3.18}$$

Theorem 3.4.1 *The uncertain linear non-autonomous system* (3.17)*, where* $u(t) = 0$*, is robustly exponentially stable if there exist positive real numbers* $\alpha, \epsilon, \epsilon_0, \epsilon_1$*, and* *a matrix function* $P(t) \in BM^+(0,\infty)$ *such that* $\epsilon_1 I - H_1(t)H_1^T(t) > 0$ *and the RDE* (3.18) *holds. Moreover, the solution* $x(t,\phi)$ *satisfies the inequality*

$$\|x(t,\phi)\| \leq N\|\phi\|e^{-\alpha t}, \quad t \in \mathbb{R}^+,$$

where

$$N = \sqrt{\frac{\lambda_{\max}(P(0))}{\epsilon} + \frac{1}{2\alpha\epsilon}(1 - e^{-2\alpha h}) + 1}.$$

Proof. Let $P_\epsilon(t) \in BM^+(0,\infty), t \in R^+$, be a solution of the RDE (3.18). We take the change of the state variable

$$y(t) = e^{\alpha t}x(t), \quad t \in \mathbb{R}^+, \tag{3.19}$$

then the linear delay system (3.17), where $u(t) = 0$, is transformed to the delay system

$$\begin{aligned} \dot{y}(t) = [A_{0,\alpha}(t) + \Delta A_0(t)]y(t) + e^{\alpha h(t)}[A_1(t) + \Delta A_1(t)]y(t-h(t)), \\ y(t) = e^{\alpha t}\phi(t), \quad t \in [-h, 0], \end{aligned} \tag{3.20}$$

Consider the following time-varying Lyapunov function for the system (3.20),

$$V(t,y_t) = \langle P(t)y(t), y(t)\rangle + \epsilon\|y(t)\|^2 + \int_{t-h(t)}^{t} \|y(s)\|^2 ds.$$

It is easy to see that

$$\epsilon\|y(t)\|^2 \leq V(t,y_t) \leq (p + \epsilon + h)\|y_t\|^2, \tag{3.21}$$

where $p = \max_{t\geq 0}|P(t)|$ which is a finite number because $P(t) \in BM^+(0,\infty)$ and hence $P(t)$ is a bounded function. Taking the derivative of $V(\cdot)$ in t along the solution of $y(t)$ of system (3.20), we have

$$\begin{aligned} \dot{V}(t,y_t) &= \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle P_\epsilon(t)\dot{y}(t), y(t)\rangle + \|y(t)\|^2 - (1 - \dot{h}(t))\|y(t-h(t))\|^2 \\ &= \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle P_\epsilon(t)A_{0,\alpha}(t)y(t), y(t)\rangle + 2\langle P_\epsilon(t)G_0F(t)H_0y(t), y(t)\rangle \\ &\quad + \|y(t)\|^2 - (1 - \dot{h}(t))\|y(t-h(t))\|^2 \\ &\quad + 2e^{\alpha h(t)}\langle P_\epsilon(t)[A_1(t) + G_1F(t)H_1]y(t-h), y(t)\rangle \\ &\leq \langle \dot{P}(t)y(t), y(t)\rangle + 2\langle P_\epsilon(t)A_{0,\alpha}(t)y(t), y(t)\rangle \\ &\quad + 2\langle P_\epsilon(t)G_0F(t)H_0y(t), y(t)\rangle + \|y(t)\|^2 - (1-\delta)\|y(t-h(t))\|^2 \\ &\quad + 2\langle e^{\alpha h(t)}P_\epsilon(t)[A_1(t) + G_1F(t)H_1]y(t-h(t)), y(t)\rangle. \end{aligned}$$

From Lemma 2.3.11, it follows that

$$2\langle e^{\alpha h(t)}P_\epsilon(t)[A_1(t)+G_1(t)F(t)H_1(t)]y(t-h(t)),y(t)\rangle-(1-\delta)\|y(t-h(t))\|^2$$
$$\le \frac{e^{2\alpha h(t)}}{1-\delta}\langle P_\epsilon(t)[A_1(t)+G_1(t)F(t)H_1(t)][A_1(t)+G_1(t)F(t)H_1(t)]^T P_\epsilon(t)y(t),y(t)\rangle.$$

Using Lemma 2.3.15, for numbers ϵ_0,ϵ_1 such that

$$2\langle P_\epsilon G_0FH_0y,y\rangle\le\frac{1}{\epsilon_0}\langle P_\epsilon G_0G_0^TP_\epsilon y,y\rangle+\epsilon_0\langle H_0^TH_0y,y\rangle,$$

$$[A_1+G_1FH_1][A_1+G_1FH_1]^T\le A_1A_1^T+A_1H_1^T(\epsilon_1I-H_1H_1^T)^{-1}H_1A_1^T+\epsilon_1G_1G_1^T,$$

provided $\epsilon_1I-H_1H_1^T>0$. Furthermore, note that $e^{2\alpha h(t)}\le e^{2\alpha h},\forall t\in\mathbb{R}^+$, we obtain

$$\begin{aligned}&\dot V(t,y(t))\\&\le\langle\dot P(t)y(t),y(t)\rangle+2\langle P_\epsilon(t)A_{0,\alpha}(t)y(t),y(t)\rangle\\&\quad+\frac{1}{\epsilon_0}\langle P_\epsilon G_0G_0^TP_\epsilon y,y\rangle+\epsilon_0\langle H_0^TH_0y(t),y(t)\rangle+\|y(t)\|^2\\&\quad+\frac{e^{2\alpha h}}{1-\delta}\langle\{P_\epsilon[A_1A_1^T+A_1H_1^T(\epsilon_1I-H_1H_1^T)^{-1}H_1A_1^T+\epsilon_1G_1^TG_1]P_\epsilon\}y(t),y(t)\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\dot V(t,y(t))\le\langle\{&\dot P_\epsilon+P_\epsilon A_{0,\alpha}+A_{0,\alpha}^TP_\epsilon+\epsilon_0^{-1}P_\epsilon G_0G_0^TP_\epsilon+\epsilon_1H_0^TH_0+I\\&+\frac{e^{2\alpha h}}{1-\delta}[P_\epsilon A_1A_1^TP_\epsilon+P_\epsilon A_1H_1^T(\epsilon_1I-H_1H_1^T)^{-1}H_1A_1^TP_\epsilon\\&+\epsilon_1P_\epsilon G_1^TG_1P_\epsilon]\}y(t),y(t)\rangle.\end{aligned}$$

Since $P(t)$ is the solution of (3.18), we obtain

$$\dot V(t,y(t))\le 0,\quad\forall t\in\mathbb{R}^+. \tag{3.22}$$

Thus, from (3.19), (3.20) and Lemma 2.3.7, it follows the boundedness of the solution $y(t,\phi)$ for the system (3.20); i.e., there exists $N>0$ such that

$$\|y(t,\phi)\|\le N\|\phi\|,\quad\forall t\ge0.$$

Returning to the solution $x(t,\phi)$ of the system (3.17) by the transformation (3.19), we obtain

$$\|x(t,\phi)\|\le N\|\phi\|e^{-\alpha t},\quad\forall t\ge0,$$

which gives the exponential stability of (3.17). To determine the stability factor N, we integrate both sides of (3.22) from 0 to t we find

$$V(t, y(t)) - V(0, y(0)) \leq 0, \quad \forall t \in \mathbb{R}^+,$$

and hence

$$\begin{aligned}&\langle P(t)y(t), y(t)\rangle + \epsilon\|y(t)\|^2 + \int_{t-h(t)}^{t} \|y(s)\|^2 ds \\ &\leq \langle P_0 y(0), y(0)\rangle + \epsilon\|y(0)\|^2 + \int_{-h(0)}^{0} \|y(s)\|^2 ds.\end{aligned}$$

Since

$$\langle P(t)y, y\rangle \geq 0, \quad \int_{t-h(t)}^{t} \|y(s)\|^2 ds \geq 0,$$

and

$$\int_{-h(0)}^{0} \|y(s)\|^2 ds \leq \|\phi\|^2 \int_{-h(0)}^{0} e^{2\alpha s} ds = \frac{1}{\alpha}(1 - e^{-2\alpha h(0)})\|\phi\|^2 \leq \frac{1}{2\alpha}(1 - e^{-2\alpha h})\|\phi\|^2,$$

we have

$$\epsilon\|y(t)\|^2 \leq \lambda_{\max}(P(0))\|y(0)\|^2 + \epsilon\|y(0)\|^2 + \frac{1}{2\alpha}(1 - e^{-2\alpha h})\|\phi\|^2.$$

Returning to the solution $x(t, \phi)$ of system (3.17) and noting that

$$\|y(0)\| = \|x(0)\| = \phi(0) \leq \|\phi\|,$$

we have

$$\|x(t, \phi)\| \leq N\|\phi\|e^{-\alpha t}, \quad \forall t \in \mathbb{R}^+,$$

where

$$N = \sqrt{\frac{\lambda_{\max}(P(0))}{\epsilon} + \frac{1}{2\alpha\epsilon}(1 - e^{-2\alpha h}) + 1}.$$

The proof of the theorem is complete. □

Example 3.4.1 Consider the uncertain linear non-autonomous unforced system with time-varying delay (3.17), where $u(t) = 0$, with any initial function $\phi(t)$ and time-delay function $h(t) = 3\sin^2(2/3)t$ and

$$A_0(t) = \begin{bmatrix} -\frac{1}{2} & -1 \\ 0 & e^{-2t} - \frac{3}{4} \end{bmatrix}, \quad A_1(t) = \begin{bmatrix} -\frac{\sqrt{2-e^{-2t}}}{e^3(e^{-2t}+1)} & 0 \\ 0 & -\frac{\sqrt{2-e^{-2t}}}{2e^3} \end{bmatrix},$$

$$G_0(t) = \begin{bmatrix} \frac{e^t}{e^{-2t}+1} & 0 \\ 0 & \frac{e^{-t}}{\sqrt{2}} \end{bmatrix}, \quad H_0(t) = \begin{bmatrix} e^{-t} & e^{-t} \\ e^t & e^{-t} \end{bmatrix},$$

$$G_1(t) = \begin{bmatrix} \frac{e^t}{e^3(e^{-2t}+1)} & 0 \\ 0 & \frac{e^{-t}}{e^3} \end{bmatrix}, \quad H_1(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

We see that $h = 3$, and $\dot{h}(t) = 2\sin(4/3t)$ and then $\delta = 2$. Taking $\alpha = \epsilon = \epsilon_0 = 1$ and $\epsilon_1 = 2$, we have

$$\epsilon_1 I - H_1(t)H_1^T(t) = \begin{bmatrix} 2 - e^{-2t} & 0 \\ 0 & 2 - e^{-2t} \end{bmatrix} > 0.$$

We can verify that the matrix $P(t) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & 1 \end{bmatrix}$ is a solution of (3.18). Therefore, by Theorem 3.4.1 the system is robustly exponentially stable and the solution satisfies

$$\|x(t,\phi)\| \le (3 - e^{-3})\|\phi\|e^{-t}, \quad t \in \mathbb{R}^+.$$

$\square$

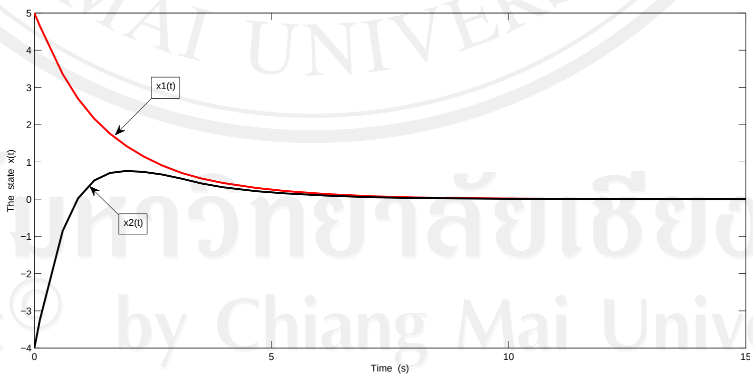

Figure 3.5: The simulation of solutions for the states $x_1(t)$ and $x_2(t)$ in the LPD delay system (3.15) with initial conditions $x_1(t) = 5$ and $x_2(t) = -4$, $-3 \le t \le 0$ by using dde45 in Matlab.