Chapter 4

Discrete Time Delay Systems

This chapter presents robust stability criteria of discret-time delay systems such as linear parameter dependent (LPD) system with time-delay, uncertain LPD system with time-varying delays and uncertain linear system with interval time-varying delays and nonlinear perturbations. We use appropriate Lyapunov functions and derive stability conditions in terms of linear matrix inequalities (LMIs). Numerical examples are presented to illustrate the effectiveness of the theoretical results.

4.1 Stability Criteria of LPD System with Time-delay

Consider the discrete-time LPD system with time-delay of the form

$$\begin{cases} x(k+1) = A(\alpha)x(k) + B(\alpha)x(k-h), & \forall k \in Z^+; \\ x(k) = \phi(k), & \forall k \in [-h, 0], \end{cases} \tag{4.1}$$

where $x(k) \in R^n$ represents the system state vector at time $k \in Z^+$ and h is a positive integer representing the time delay. $A(\alpha)$ and $B(\alpha)$ are uncertain $M^{n\times n}$ matrices belonging to the polytope of the form

$$\{A(\alpha), B(\alpha)\} = \Big\{ \sum_{i=1}^{N} \alpha_i A_i, \sum_{i=1}^{N} \alpha_i B_i \Big\},$$

$$\sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0, A_i, B_i \in M^{n\times n}, i = 1, ..., N.$$

Here $\phi(k)$, is a real valued initial function on $[-h, 0]$. Let us set

$$M_{i,j,l}[A,P,B] = \begin{bmatrix} A_i^T P_j A_l & A_i^T P_j B_l \\ B_i^T P_j A_l & B_i^T P_j B_l \end{bmatrix}, \quad N_i[P,Q] = \begin{bmatrix} -P_i + Q_i & 0 \\ 0 & -Q_i \end{bmatrix}.$$

Theorem 4.1.1 *The system* (4.1) *is robustly stable if there exist symmetric positive definite matrices* $P_i, Q_i,\ i = 1, 2, ..., N$ *satisfying the following conditions.*

$$(i)\quad M_{i,i,i}[A,P,B] + N_i[P,Q] < -I, \quad i = 1,2,...,N.$$

$$(ii)\quad M_{i,j,i}[A,P,B] + M_{i,i,j}[A,P,B] + M_{j,i,i}[A,P,B] + 2N_i[P,Q] + N_j[P,Q] < \frac{1}{(N-1)^2}I, \quad i = 1,2,...,N, i \neq j, j = 1,2,...,N.$$

$$(iii)\quad M_{i,j,l}[A,P,B] + M_{i,l,j}[A,P,B] + M_{j,i,l}[A,P,B] + M_{j,l,i}[A,P,B] + M_{l,i,j}[A,P,B] + M_{l,j,i}[A,P,B] + 2N_i[P,Q] + 2N_j[P,Q] + 2N_l[P,Q] < \frac{6}{(N-1)^2}I, \quad i = 1,2,...,N-2, j = i+1,...,N-1, l = j+1,...,N.$$

Proof. Consider a Lyapunov function for system (4.1) of the form

$$V(k,x(k)) = x^T(k)P(\alpha)x(k) + \sum_{i=k-h}^{k-1} x^T(i)Q(\alpha)x(i),$$

where $P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i$ and $Q(\alpha) = \sum_{i=1}^{N} \alpha_i Q_i$. Then, the difference of Lyapunov function along a trajectory of solution of (4.1) is given by

$$\begin{aligned}
\Delta V(k,x(k)) &= V(k+1,x(k+1)) - V(k,x(k)) \\
&= x^T(k)A^T(\alpha)P(\alpha)A(\alpha)x(k) + x^T(k-h)B^T(\alpha)P(\alpha)A(\alpha)x(k) \\
&\quad + x^T(k)A^T(\alpha)P(\alpha)B(\alpha)x(k-h) + x^T(k)Q(\alpha)x(k) \\
&\quad + x^T(k-h)B^T(\alpha)P(\alpha)B(\alpha)x(k-h) \\
&\quad - x^T(k)P(\alpha)x(k) - x^T(k-h)Q(\alpha)x(k-h).
\end{aligned}$$

By the definition of $A(\alpha), B(\alpha), P(\alpha)$ and $Q(\alpha)$, we obtain

$$\begin{aligned}
\Delta V(k,x(k)) &= \sum_{i=1}^{N}\alpha_i \sum_{j=1}^{N}\alpha_j \sum_{l=1}^{N}\alpha_l \left[x^T(k)\left(A_i^T P_j A_l - P_i + Q_i\right)x(k)\right] \\
&\quad + \sum_{i=1}^{N}\alpha_i \sum_{j=1}^{N}\alpha_j \sum_{l=1}^{N}\alpha_l \left[x^T(k-h)\left(B_i^T P_j A_l\right)x(k)\right] \\
&\quad + \sum_{i=1}^{N}\alpha_i \sum_{j=1}^{N}\alpha_j \sum_{l=1}^{N}\alpha_l \left[x^T(k)\left(A_i^T P_j B_l\right)x(k-h)\right] \\
&\quad + \sum_{i=1}^{N}\alpha_i \sum_{j=1}^{N}\alpha_j \sum_{l=1}^{N}\alpha_l \left[x^T(k-h)\left(B_i^T P_j B_l - Q_i\right)x(k-h)\right]
\end{aligned}$$

$$
\begin{aligned}
= \; & \sum_{i=1}^{N} \alpha_i^3 Y^T \Big[M_{i,i,i}[A,P,B] + N_i[P,Q] \Big] Y \\
& + \sum_{i=1}^{N} \sum_{i \neq j, j=1}^{N} \alpha_i^2 \alpha_j Y^T \begin{bmatrix} M_{i,j,i}[A,P,B] + M_{i,i,j}[A,P,B] \\ + M_{j,i,i}[A,P,B] + 2N_i[P,Q] + N_j[P,Q] \end{bmatrix} Y \\
& + \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^{N} \alpha_i \alpha_j \alpha_l Y^T \begin{bmatrix} M_{i,j,l}[A,P,B] + M_{i,l,j}[A,P,B] \\ + M_{j,i,l}[A,P,B] + M_{j,l,i}[A,P,B] \\ + M_{l,i,j}[A,P,B] + M_{l,j,i}[A,P,B] \\ + 2N_i[P,Q] + 2N_j[P,Q] + 2N_l[P,Q] \end{bmatrix} Y,
\end{aligned}
$$

where $Y = [x(k) \quad x(k-h)]^T$. By conditions $(i) - (iii)$, we have

$$
\begin{aligned}
\Delta V(k, x(k)) \; < \; & -Y^T \Big[\sum_{i=1}^{N} \alpha_i^3 I - \frac{1}{(N-1)^2} \sum_{i=1}^{N} \sum_{i \neq j, j=1}^{N} \alpha_i^2 \alpha_j I \\
& - \frac{6}{(N-1)^2} \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^{N} \alpha_i \alpha_j \alpha_l I \Big] Y.
\end{aligned}
$$

We define Φ and Λ as

$$
\Phi \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i (\alpha_i - \alpha_j)^2 = (N-1) \sum_{i=1}^{N} \alpha_i^3 - \sum_{i=1}^{N} \sum_{j \neq i; j=1}^{N} \alpha_i^2 \alpha_j \geq 0,
$$

$$
\begin{aligned}
\Lambda \; \equiv \; & \sum_{i=1}^{N} \sum_{j \neq i; j=1}^{N-1} \sum_{l \neq i; l=2}^{N} \alpha_i [\alpha_j - \alpha_j]^2 \\
= \; & (N-2) \sum_{i=1}^{N} \sum_{j \neq i; j=1}^{N-1} \alpha_i^2 \alpha_j - 6 \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^{N} \alpha_i \alpha_j \alpha_l \geq 0.
\end{aligned}
$$

From $(N-1)\Phi + \Lambda \geq 0$, we obtain

$$
\sum_{i=1}^{N} \alpha_i^3 - \frac{1}{(N-1)^2} \sum_{i=1}^{N} \sum_{i \neq j, j=1}^{N} \alpha_i^2 \alpha_j - \frac{6}{(N-1)^2} \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^{N} \alpha_i \alpha_j \alpha_l \geq 0.
$$

Therefore, we conclude $\Delta V(k) < 0$ which means that the system (4.1) is robustly stable. The proof of theorem is complete. □

Example 4.1.1 Consider the following discrete-time LPD systems with time-delay of the form

$$
x(k+1) = A(\alpha)x(k) + B(\alpha)x(k-h), \quad k \in Z^+, \tag{4.2}
$$

where h is any positive integer and

$$A(\alpha) = \alpha_1 A_1 + \alpha_2 A_2,$$

$$B(\alpha) = \alpha_1 B_1 + \alpha_2 B_2,$$

where

$$A_1 = \begin{bmatrix} 0.0233 & 0.4211 \\ -0.7472 & 0.1765 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.0198 & 0.5124 \\ -0.6389 & 0.2015 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0634 & 0.0036 \\ -0.0054 & 0.0523 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.0701 & 0.0041 \\ -0.0065 & 0.0621 \end{bmatrix}.$$

By using LMI Toolbox in MATLAB, we use the condition $(i), (ii)$ and (iii) in Theorem 4.1.1 for this example. The solutions of LMI verify as follows of the form

$$P_1 = 10^3 \times \begin{bmatrix} 1.6967 & -0.1306 \\ -0.1306 & 2.3893 \end{bmatrix}, \quad P_2 = 10^3 \times \begin{bmatrix} 1.7014 & -0.1525 \\ -0.1525 & 2.3556 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 521.0819 & 17.3861 \\ 17.3861 & 547.6851 \end{bmatrix}, Q_2 = \begin{bmatrix} 528.0290 & 11.8215 \\ 11.8215 & 559.8311 \end{bmatrix}.$$

Therefore, the system (4.2) is robustly stable. □

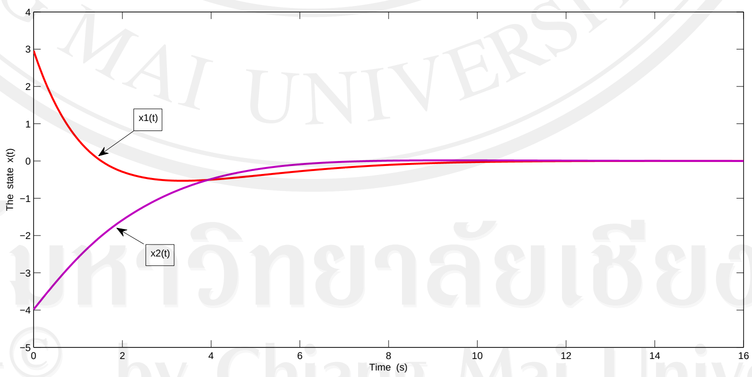

Figure 4.1: The simulation of solutions for the states $x_1(k)$ and $x_2(k)$ in the LPD delay system (4.2) where $h = 2, \alpha_1 = \alpha_2 = \frac{1}{2}$ with initial conditions $x_1(k) = 3$ and $x_2(k) = -4$, $k = -2, -1, 0$ by using method of Runge-Kutta order 4(h=0.01) with Matlab.

Theorem 4.1.2 *The system* (4.1) *is robustly stable if there exist symmetric positive* *definite matrices* $P_i, Q_i,\ i = 1,2,...,N; Z_{iii}, i = 1,2,...,N; Z_{iij} = Z_{jii}^T$ *and* $Z_{iji}, i = 1,2,...,N, j \neq i, j = 1,2,...,N; Z_{ijl} = Z_{lji}^T, Z_{ilj} = Z_{jli}^T, Z_{jil} = Z_{lij}^T; i = 1,2,...,N-2, j = i+1,...,N-1, l = j+1,...,N$ *satisfying the following conditions.*

$$(i) \quad M_{i,i,i}[A,P,B] + N_i[P,Q] < Z_{iii}, \quad i = 1,2,...,N,$$

$$(ii) \quad M_{i,j,i}[A,P,B] + M_{i,i,j}[A,P,B] + M_{j,i,i}[A,P,B] + 2N_i[P,Q] + N_j[P,Q] < Z_{iij} + Z_{jii}^T + Z_{iji},$$
$$i = 1,2,...,N, i \neq j, j = 1,2,...,N,$$

$$(iii) \quad M_{i,j,l}[A,P,B] + M_{i,l,j}[A,P,B] + M_{j,i,l}[A,P,B] + M_{j,l,i}[A,P,B] + M_{l,i,j}[A,P,B] + M_{l,j,i}[A,P,B] + 2N_i[P,Q] + 2N_j[P,Q] + 2N_l[P,Q]$$
$$< Z_{ijl} + Z_{lji}^T + Z_{ilj} + Z_{jli}^T + Z_{jil} + Z_{lij}^T,$$
$$i = 1,2,...,N-2, j = i+1,...,N-1, l = j+1,...,N;$$

$$(iv) \quad \begin{bmatrix} Z_{1i1} & Z_{1i2} & \cdots & Z_{1iN} \\ Z_{2i1} & Z_{2i2} & \cdots & Z_{2iN} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{Ni1} & Z_{Ni2} & \cdots & Z_{NiN} \end{bmatrix} \leq 0, i = 1,2,...,N.$$

Proof. Consider a Lyapunov function for system (4.1) of the form

$$V(k, x(k)) = x^T(k)P(\alpha)x(k) + \sum_{i=k-h}^{k-1} x^T(i)Q(\alpha)x(i)$$

where $P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i$ and $Q(\alpha) = \sum_{i=1}^{N} \alpha_i Q_i$. Then the difference of Lyapunov function along a trajectory of solution of (4.1) is given by

$$\begin{aligned} \Delta V(k, x(k)) &= \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j \sum_{l=1}^{N} \alpha_l \left[x^T(k) \left(A_i^T P_j A_l - P_i + Q_i \right) x(k) \right] \\ &+ \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j \sum_{l=1}^{N} \alpha_l \left[x^T(k-h) \left(B_i^T P_j A_l \right) x(k) \right] \\ &+ \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} \alpha_j \sum_{l=1}^{N} \alpha_l \left[x^T(k) \left(A_i^T P_j B_l \right) x(k-h) \right] \end{aligned}$$

$$+\sum_{i=1}^{N}\alpha_i\sum_{j=1}^{N}\alpha_j\sum_{l=1}^{N}\alpha_l\left[x^T(k-h)\left(B_i^TP_jB_l-Q_i\right)x(k-h)\right]$$

$$=\sum_{i=1}^{N}\alpha_i^3Y^T\left[M_{i,i,i}[A,P,B]+N_i[P,Q]\right]Y$$

$$+\sum_{i=1}^{N}\sum_{i\neq j,j=1}^{N}\alpha_i^2\alpha_jY^T\begin{bmatrix}M_{i,j,i}[A,P,B]+M_{i,i,j}[A,P,B]\\+M_{j,i,i}[A,P,B]+2N_i[P,Q]+N_j[P,Q]\end{bmatrix}Y$$

$$+\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\alpha_i\alpha_j\alpha_lY^T\begin{bmatrix}M_{i,j,l}[A,P,B]+M_{i,l,j}[A,P,B]\\+M_{j,i,l}[A,P,B]+M_{j,l,i}[A,P,B]\\+M_{l,i,j}[A,P,B]+M_{l,j,i}[A,P,B]\\+2N_i[P,Q]+2N_j[P,Q]+2N_l[P,Q]\end{bmatrix}Y,$$

where $Y=[x(k) \quad x(k-h)]^T$. By conditions $(i)-(iii)$, we have

$$\Delta V(k,x(k)) < \sum_{i=1}^{N}\alpha_i^3Y^TZ_{iii}Y+\sum_{i=1}^{N}\sum_{i\neq j,j=1}^{N}\alpha_i^2\alpha_jY^T[Z_{iij}+Z_{jii}^T+Z_{iji}]Y$$

$$+\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}\sum_{l=j+1}^{N}\alpha_i\alpha_j\alpha_lY^T[Z_{ijl}+Z_{lji}^T+Z_{ilj}+Z_{jli}^T+Z_{jil}+Z_{lij}^T]Y$$

$$=Y^T\alpha_1\begin{bmatrix}\alpha_1I\\\alpha_2I\\\vdots\\\alpha_NI\end{bmatrix}^T\begin{bmatrix}Z_{111}&Z_{112}&\cdots&Z_{11N}\\Z_{211}&Z_{212}&\cdots&Z_{21N}\\\vdots&\vdots&\ddots&\vdots\\Z_{N11}&Z_{N12}&\cdots&Z_{N1N}\end{bmatrix}\begin{bmatrix}\alpha_1I\\\alpha_2I\\\vdots\\\alpha_NI\end{bmatrix}Y$$

$$+Y^T\alpha_2\begin{bmatrix}\alpha_1I\\\alpha_2I\\\vdots\\\alpha_NI\end{bmatrix}^T\begin{bmatrix}Z_{121}&Z_{122}&\cdots&Z_{12N}\\Z_{221}&Z_{222}&\cdots&Z_{22N}\\\vdots&\vdots&\ddots&\vdots\\Z_{N21}&Z_{N22}&\cdots&Z_{N2N}\end{bmatrix}\begin{bmatrix}\alpha_1I\\\alpha_2I\\\vdots\\\alpha_NI\end{bmatrix}Y$$

$$+\cdots+Y^T\alpha_N\begin{bmatrix}\alpha_1I\\\alpha_2I\\\vdots\\\alpha_NI\end{bmatrix}^T\begin{bmatrix}Z_{1N1}&Z_{1N2}&\cdots&Z_{1NN}\\Z_{2N1}&Z_{2N2}&\cdots&Z_{2NN}\\\vdots&\vdots&\ddots&\vdots\\Z_{NN1}&Z_{NN2}&\cdots&Z_{NNN}\end{bmatrix}\begin{bmatrix}\alpha_1I\\\alpha_2I\\\vdots\\\alpha_NI\end{bmatrix}Y.$$

Thus, we have

$$\Delta V(k,x(k)) \quad < \quad Y^T \left(\begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_N I \end{bmatrix}^T \sum_{i=1}^{N} \alpha_i \begin{bmatrix} Z_{1i1} & Z_{1i2} & \cdots & Z_{1iN} \\ Z_{2i1} & Z_{2i2} & \cdots & Z_{2iN} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{Ni1} & Z_{Ni2} & \cdots & Z_{NiN} \end{bmatrix} \begin{bmatrix} \alpha_1 I \\ \alpha_2 I \\ \vdots \\ \alpha_N I \end{bmatrix} \right) Y.$$

By the condition (iv) we obtain $\Delta V(k) < 0$. Therefore, the system (4.1) is robustly stable. □

Example 4.1.2 Consider the following discrete-time LPD systems with time-delay of the form

$$x(k+1) = A(\alpha)x(k) + B(\alpha)x(k-h), \quad k \in Z^+, \tag{4.3}$$

where h is any positive integer and

$$A(\alpha) = \alpha_1 \begin{bmatrix} -0.52130 & 0.34646 \\ -0.21218 & -0.72940 \end{bmatrix} + \alpha_2 \begin{bmatrix} -0.63410 & 0.26354 \\ -0.25410 & -0.71280 \end{bmatrix},$$

$$B(\alpha) = \alpha_1 \begin{bmatrix} 0.0318 & -0.00234 \\ -0.00456 & 0.0350 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.0354 & -0.00364 \\ -0.00605 & 0.0834 \end{bmatrix}.$$

By using LMI Toolbox in MATLAB, we use the condition $(i), (ii), (iii)$ and (iv) in Theorem 4.1.2 for this example. The solutions of LMI verify as follows of the form

$$P_1 = 10^3 \times \begin{bmatrix} 3.8531 & 0.4311 \\ 0.4311 & 5.4663 \end{bmatrix}, \quad P_2 = 10^3 \times \begin{bmatrix} 4.4796 & 0.2519 \\ 0.2519 & 5.0000 \end{bmatrix},$$

$$Q_1 = 10^3 \times \begin{bmatrix} 1.3666 & 0.0713 \\ 0.0713 & 1.2885 \end{bmatrix}, \quad Q_2 = 10^3 \times \begin{bmatrix} 1.2795 & -0.0066 \\ -0.0066 & 1.2584 \end{bmatrix},$$

$$Z_{111} = 10^3 \times \begin{bmatrix} -2.9670 & 0 & 0 & 0 \\ 0 & -2.9670 & 0 & 0 \\ 0 & 0 & -2.9670 & 0 \\ 0 & 0 & 0 & -2.9670 \end{bmatrix},$$

$$Z_{222} = 10^3 \times \begin{bmatrix} -2.9770 & 0 & 0 & 0 \\ 0 & -2.9770 & 0 & 0 \\ 0 & 0 & -2.9770 & 0 \\ 0 & 0 & 0 & -2.9770 \end{bmatrix},$$

$$Z_{112} = Z_{121} = Z_{211} = 10^3 \times \begin{bmatrix} -0.4322 & 0 & 0 & 0 \\ 0 & -0.4322 & 0 & 0 \\ 0 & 0 & -0.4322 & 0 \\ 0 & 0 & 0 & -0.4322 \end{bmatrix},$$

and

$$Z_{212} = Z_{122} = Z_{221} = 10^3 \times \begin{bmatrix} -0.4162 & 0 & 0 & 0 \\ 0 & -0.4162 & 0 & 0 \\ 0 & 0 & -0.4162 & 0 \\ 0 & 0 & 0 & -0.4162 \end{bmatrix}.$$

Therefore, the system (4.3) is robustly stable. □

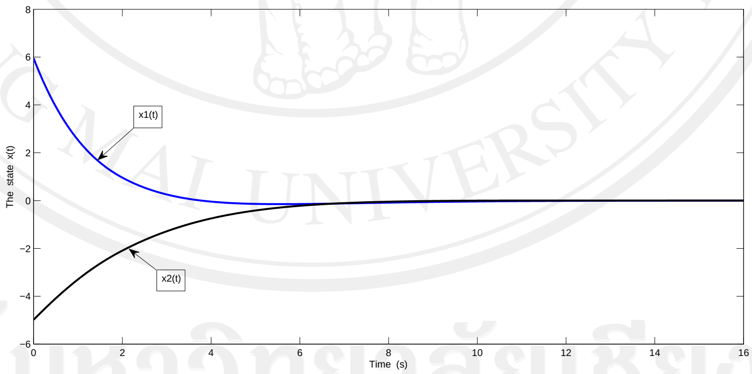

Figure 4.2: The simulation of solutions for the states $x_1(k)$ and $x_2(k)$ in the LPD delay system (4.3) where $h = 2, \alpha_1 = \alpha_2 = \frac{1}{2}$ with initial conditions $x_1(k) = 6$ and $x_2(k) = -5$, $k = -2, -1, 0$. by using method of Runge-Kutta order 4 with Matlab.

4.2 Stability Criteria of Uncertain LPD System with Time-Varying Delay

We consider the uncertain discrete-time linear parameter dependent (LPD) system with time-varying delay of the form

$$\begin{cases} x(k+1) = \Big[A(\alpha) + \Delta A(k)\Big]x(k) + \Big[B(\alpha) + \Delta B(k)\Big]x(k-h(k)), & \forall k \in Z^+; \\ x(k) = \phi(k), & \forall k \in [-h, 0], \end{cases} \tag{4.4}$$

where $x(k) \in R^n$ represents the system state vector at time $k \in Z^+ = \{0, 1, 2, \dots\}$ and $h(k) \in Z^+$ is a positive integer representing the time delay. $A(\alpha)$ and $B(\alpha)$ are uncertain $M^{n\times n}$ matrices belonging to the polytope of the form

$$\{A(\alpha), B(\alpha)\} = \Big\{\sum_{i=1}^{N} \alpha_i A_i, \sum_{i=1}^{N} \alpha_i B_i\Big\},$$

$$\sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0, A_i, B_i \in M^{n\times n}, i = 1, ..., N.$$

$\phi(k)$ is a real valued initial function on $[-h, 0]$. $\Delta A(k)$ and $\Delta B(k)$ are unknown matrices representing time-varying parameter uncertainties, we are assumed to be of the form

$$\Delta A(k) = K(\alpha)\Delta(k)A_1(\alpha), \Delta B(k) = K(\alpha)\Delta(k)B_1(\alpha),$$

where $A_1(\alpha) = \sum_{i=1}^{N} \alpha_i A_i^1$, $B_1(\alpha) = \sum_{i=1}^{N} \alpha_i B_i^1$, $K(\alpha) = \sum_{i=1}^{N} \alpha_i K_i$ and $\Delta(k)$ satisfies

$$\Delta(k) = F(k)[I - JF(k)]^{-1}, \quad I - JJ^T > 0.$$

The uncertain matrix $F(k)$ satisfies

$$F(k)^T F(k) \leq I.$$

In addition, we assume that the time-varying delay $h(k)$ is upper and lower bounded. It satisfies the following assumption of the form

$$h_1 \leq h(k) \leq h_2,$$

where h_1 and h_2 are known positive integer. Let us set

$$P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i, \quad Q(\alpha) = \sum_{i=1}^{N} \alpha_i Q_i,$$

$$\bar{A}^T(\alpha) = A(\alpha) + \Delta A(k), \quad \bar{B}(\alpha) = B(\alpha) + \Delta B(k).$$

Lemma 4.2.1 *Let $\bar{A}^T(\alpha), \bar{B}(\alpha)$ be given matrices as in (4.4). Let P_i, Q_i, $i = 1, 2, ..., N$, be symmetric positive matrices and positive real numbers ϵ, ζ, $\hat{h} = h_2 - h_1 + 1$. Then,*

$$\begin{bmatrix} \bar{A}^T(\alpha)P(\alpha)\bar{A}(\alpha) - P(\alpha) + \hat{h}Q(\alpha) & \bar{A}^T(\alpha)P(\alpha)\bar{B}(\alpha) \\ \bar{B}^T(\alpha)P(\alpha)\bar{A}(\alpha) & \bar{B}^T(\alpha)P(\alpha)\bar{B}(\alpha) - Q(\alpha) \end{bmatrix} < 0,$$

if and only if

$$(i) \quad \begin{bmatrix} -P_i + \hat{h}Q_i & 0 & A_i^T P_i & \epsilon^{-1} A_i^{1^T} & 0 \\ 0 & -Q_i & B_i^T P_i & \epsilon^{-1} B_i^{1^T} & 0 \\ A_i P_i & B_i P_i & -P_i & 0 & \epsilon P_i K_i \\ \epsilon^{-1} A_i^1 & \epsilon^{-1} B_i^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_i^T P_i & J^T & -I \end{bmatrix} < -\zeta I, \quad i = 1, 2, ...N,$$

$$(ii) \quad \begin{bmatrix} -P_i + \hat{h}Q_i & 0 & A_i^T P_j & \epsilon^{-1} A_i^{1^T} & 0 \\ 0 & -Q_i & B_i^T P_j & \epsilon^{-1} B_i^{1^T} & 0 \\ A_i P_j & B_i P_j & -P_i & 0 & \epsilon P_i K_j \\ \epsilon^{-1} A_i^1 & \epsilon^{-1} B_i^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_i^T P_j & J^T & -I \end{bmatrix}$$

$$+ \begin{bmatrix} -P_j + \hat{h}Q_j & 0 & A_j^T P_i & \epsilon^{-1} A_j^{1^T} & 0 \\ 0 & -Q_j & B_j^T P_i & \epsilon^{-1} B_j^{1^T} & 0 \\ A_j P_i & B_j P_i & -P_j & 0 & \epsilon P_j K_i \\ \epsilon^{-1} A_j^1 & \epsilon^{-1} B_j^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_j^T P_i & J^T & -I \end{bmatrix} < \frac{2\zeta I}{N-1},$$

$i = 1, ..., N-1, j = i+1, ..., N.$

Proof. We consider

$$\begin{bmatrix} \bar{A}^T(\alpha)P(\alpha)\bar{A}(\alpha) - P(\alpha) + \hat{h}Q(\alpha) & \bar{A}^T(\alpha)P(\alpha)\bar{B}(\alpha) \\ \bar{B}^T(\alpha)P(\alpha)\bar{A}(\alpha) & \bar{B}^T(\alpha)P(\alpha)\bar{B}(\alpha) - Q(\alpha) \end{bmatrix},$$

where $\hat{h} = h_2 - h_1 + 1$, $P(\alpha) = \sum_{i=1}^{N} \alpha_i P_i$ and $Q(\alpha) = \sum_{i=1}^{N} \alpha_i Q_i$. Then,

$$\begin{bmatrix} -P(\alpha) + \hat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \bar{A}^T(\alpha)P(\alpha)\bar{A}(\alpha) & \bar{A}^T(\alpha)P(\alpha)\bar{B}(\alpha) \\ \bar{B}^T(\alpha)P(\alpha)\bar{A}(\alpha) & \bar{B}^T(\alpha)P(\alpha)\bar{B}(\alpha) \end{bmatrix}$$
$$= \begin{bmatrix} -P(\alpha) + \hat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \bar{A}^T(\alpha) \\ \bar{B}^T(\alpha) \end{bmatrix} P(\alpha) \begin{bmatrix} \bar{A}(\alpha) & \bar{B}(\alpha) \end{bmatrix}.$$

We assume that

$$\begin{bmatrix} -P(\alpha) + \hat{h}Q(\alpha) & 0 \\ 0 & -Q(\alpha) \end{bmatrix} + \begin{bmatrix} \bar{A}^T(\alpha) \\ \bar{B}^T(\alpha) \end{bmatrix} P(\alpha) \begin{bmatrix} \bar{A}(\alpha) & \bar{B}(\alpha) \end{bmatrix} < 0.$$

Using the well-known Schur compliment lemma (Lemma 2.3.16), we change above inequality in this form

$$\begin{bmatrix} -P(\alpha) + \hat{h}Q(\alpha) & 0 & A(\alpha)^T + [K(\alpha)\Delta(k)A_1(\alpha)]^T \\ * & -Q(\alpha) & B(\alpha)^T + [K(\alpha)\Delta(k)B_1(\alpha)]^T \\ * & * & -P(\alpha)^{-1} \end{bmatrix} < 0.$$

We rewrite above inequality again as

$$\begin{bmatrix} -P(\alpha) + \hat{h}Q(\alpha) & 0 & A(\alpha)^T \\ 0 & -Q(\alpha) & B(\alpha)^T \\ A(\alpha) & B(\alpha) & -P(\alpha)^{-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ K(\alpha) \end{bmatrix} \Delta(k) \begin{bmatrix} A_1(\alpha) & B_1(\alpha) & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} A_1(\alpha) & B_1(\alpha) & 0 \end{bmatrix}^T \Delta(k)^T \begin{bmatrix} 0 \\ 0 \\ K(\alpha) \end{bmatrix}^T < 0. \tag{4.5}$$

By using Lemma 2.3.14, inequality (4.5) holds if and only if there exists $\epsilon > 0$ such that

$$\begin{bmatrix} -P(\alpha) + \hat{h}Q(\alpha) & 0 & A(\alpha)^T \\ 0 & -Q(\alpha) & B(\alpha)^T \\ A(\alpha) & B(\alpha) & -P(\alpha)^{-1} \end{bmatrix}$$
$$+ \begin{bmatrix} \epsilon^{-1}A_1(\alpha)^T & 0 \\ \epsilon^{-1}B_1(\alpha)^T & 0 \\ 0 & \epsilon K(\alpha) \end{bmatrix} \begin{bmatrix} I & -J \\ -J & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1}A_1(\alpha)^T & 0 \\ \epsilon^{-1}B_1(\alpha)^T & 0 \\ 0 & \epsilon K(\alpha) \end{bmatrix}^T < 0. \tag{4.6}$$

We apply Scher complement lemma in (4.6) , we obtain

$$\begin{bmatrix} -P(\alpha)+\hat{h}Q(\alpha) & 0 & A(\alpha)^T & \epsilon^{-1}A_1(\alpha)^T & 0 \\ 0 & -Q(\alpha) & B(\alpha)^T & \epsilon^{-1}B_1(\alpha)^T & 0 \\ A(\alpha) & B(\alpha) & -P(\alpha)^{-1} & 0 & \epsilon K(\alpha) \\ \epsilon^{-1}A_1(\alpha) & \epsilon^{-1}B_1(\alpha) & 0 & -I & J \\ 0 & 0 & \epsilon K(\alpha)^T & J^T & -I \end{bmatrix} < 0. \qquad (4.7)$$

Pre-multiplying (4.7) by $diag\{I, I, P(\alpha), I, I\}$ and post-multiplying by $diag\{I, I, P(\alpha), I, I\}$. We obtain

$$\begin{bmatrix} -P(\alpha)+\hat{h}Q(\alpha) & 0 & A(\alpha)^T P(\alpha) & \epsilon^{-1}A_1(\alpha)^T & 0 \\ 0 & -Q(\alpha) & B(\alpha)^T P(\alpha) & \epsilon^{-1}B_1(\alpha)^T & 0 \\ P(\alpha)A(\alpha) & P(\alpha)B(\alpha) & -P(\alpha) & 0 & \epsilon P(\alpha)K(\alpha) \\ \epsilon^{-1}A_1(\alpha) & \epsilon^{-1}B_1(\alpha) & 0 & -I & J \\ 0 & 0 & \epsilon K(\alpha)^T P(\alpha) & J^T & -I \end{bmatrix} < 0. (4.8)$$

The facts that $\sum_{i=1}^{N} \alpha_i = 1$, we obtain the following identities :

$$\sum_{i=1}^{N} \alpha_i A_i \sum_{i=1}^{N} \alpha_i B_i = \sum_{i=1}^{N} \alpha^2 A_i B_i + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_i [A_i B_j + A_j B_i],$$

$$(N-1)\sum_{i=1}^{N} \alpha_i^2 \zeta - 2\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \zeta = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} [\alpha_i - \alpha_j]^2 \zeta \geq 0.$$

Hence, the inequality (4.8) is equivalent to the following inequalities

$$\begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^T P_i & \epsilon^{-1}A_i^{1T} & 0 \\ 0 & -Q_i & B_i^T P_i & \epsilon^{-1}B_i^{1T} & 0 \\ A_i P_i & B_i P_i & -P_i & 0 & \epsilon P_i K_i \\ \epsilon^{-1}A_i^1 & \epsilon^{-1}B_i^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_i^T P_i & J^T & -I \end{bmatrix} < -\zeta I, \quad i = 1, 2, ...N,$$

and

$$\begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^TP_j & \epsilon^{-1}{A_i^1}^T & 0 \\ 0 & -Q_i & B_i^TP_j & \epsilon^{-1}{B_i^1}^T & 0 \\ A_iP_j & B_iP_j & -P_i & 0 & \epsilon P_iK_j \\ \epsilon^{-1}A_i^1 & \epsilon^{-1}B_i^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_i^TP_j & J^T & -I \end{bmatrix}$$

$$+\begin{bmatrix} -P_j+\hat{h}Q_j & 0 & A_j^TP_i & \epsilon^{-1}{A_j^1}^T & 0 \\ 0 & -Q_j & B_j^TP_i & \epsilon^{-1}{B_j^1}^T & 0 \\ A_jP_i & B_jP_i & -P_j & 0 & \epsilon P_jK_i \\ \epsilon^{-1}A_j^1 & \epsilon^{-1}B_j^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_j^TP_i & J^T & -I \end{bmatrix} < \frac{2\zeta I}{N-1},$$

$i=1,...,N-1, j=i+1,...,N.$

The proof of lemma is complete. □

Theorem 4.2.2 *The system* (4.4) *is robustly stable if there exist symmetric positive matrices $P_i > 0$ and $Q_i > 0$, $i = 1, 2, ..., N$, and positive real numbers ϵ, ζ, $\hat{h} = h_2 - h_1 + 1$ satisfying the LMI.*

$$(i) \quad \begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^TP_i & \epsilon^{-1}{A_i^1}^T & 0 \\ 0 & -Q_i & B_i^TP_i & \epsilon^{-1}{B_i^1}^T & 0 \\ A_iP_i & B_iP_i & -P_i & 0 & \epsilon P_iK_i \\ \epsilon^{-1}A_i^1 & \epsilon^{-1}B_i^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_i^TP_i & J^T & -I \end{bmatrix} < -\zeta I, \quad i=1,2,...N,$$

$$(ii) \quad \begin{bmatrix} -P_i+\hat{h}Q_i & 0 & A_i^TP_j & \epsilon^{-1}{A_i^1}^T & 0 \\ 0 & -Q_i & B_i^TP_j & \epsilon^{-1}{B_i^1}^T & 0 \\ A_iP_j & B_iP_j & -P_i & 0 & \epsilon P_iK_j \\ \epsilon^{-1}A_i^1 & \epsilon^{-1}B_i^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_i^TP_j & J^T & -I \end{bmatrix}$$

$$+\begin{bmatrix} -P_j+\hat{h}Q_j & 0 & A_j^TP_i & \epsilon^{-1}A_j^{1^T} & 0 \\ 0 & -Q_j & B_j^TP_i & \epsilon^{-1}B_j^{1^T} & 0 \\ A_jP_i & B_jP_i & -P_j & 0 & \epsilon P_jK_i \\ \epsilon^{-1}A_j^1 & \epsilon^{-1}B_j^1 & 0 & -I & J \\ 0 & 0 & \epsilon K_j^TP_i & J^T & -I \end{bmatrix} < \frac{2\zeta I}{N-1},$$

$i=1,...,N-1, j=i+1,...,N.$

Proof. We define the following Lyapunov function for system (4.4) of the form

$$V(x(k)) = V_1(x(k)) + V_2(x(k)) + V_3(x(k))$$

where

$$V_1(x(k)) = x^T(k)P(\alpha)x(k), \quad V_2(x(k)) = \sum_{i=k-h(k)}^{k-1} x^T(i)Q(\alpha)x(i),$$

$$V_3(x(k)) = \sum_{j=-h_2+2}^{-h_1+1}\sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l), \quad P(\alpha)=\sum_{i=1}^{N}\alpha_iP_i, \quad Q(\alpha)=\sum_{i=1}^{N}\alpha_iQ_i.$$

A Lyapunov difference for the system (4.4) is defined as

$$\Delta V(x(k)) = \Delta V_1(x(k)) + \Delta V_2(x(k)) + \Delta V_3(x(k)).$$

Thus, we have

$$\begin{aligned}
\Delta V_1(x(k)) &= V_1(x(k+1)) - V_1(x(k)) \\
&= x^T(k+1)P(\alpha)x(k+1) - x^T(k)P(\alpha)x(k) \\
&= x^T(k)\bar{A}^T(\alpha)P(\alpha)\bar{A}(\alpha)x(k) + x^T(k-h(k))\bar{B}^T(\alpha)P(\alpha)\bar{A}(\alpha)x(k) \\
&\quad + x^T(k-h(k))\bar{B}^T(\alpha)P(\alpha)\bar{B}(\alpha)x(k-h(k)) \\
&\quad + x^T(k)\bar{A}^T(\alpha)P(\alpha)\bar{B}(\alpha)x(k-h(k)) - x^T(k)P(\alpha)x(k).
\end{aligned}$$

Next, we consider

$$\begin{aligned}
\Delta V_2(x(k)) &= V_2(x(k+1)) - V_2(x(k)) \\
&= \sum_{i=k+1-h(k+1)}^{k} x^T(i)Q(\alpha)x(i) - \sum_{i=k-h(k)}^{k-1} x^T(i)Q(\alpha)x(i)
\end{aligned}$$

$$= \; x^T(k)Q(\alpha)x(k) - x^T(k-h(k))Q(\alpha)x(k-h(k)) + \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h(k+1)}^{k-1} x^T(i)Q(\alpha)x(i) + \sum_{i=k+1-h_1}^{k-1} x^T(i)Q(\alpha)x(i).$$

Form $h(k) \geq h_1$, we have that

$$\sum_{i=k+1-h_1}^{k-1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h(k+1)}^{k-1} x^T(i)Q(\alpha)x(i) \leq 0.$$

Thus, we obtain

$$\Delta V_2(x(k)) \;\leq\; x^T(k)Q(\alpha)x(k) - x^T(k-h(k))Q(\alpha)x(k-h(k)) + \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i).$$

Finally, we know that

$$\begin{aligned}\Delta V_3(x(k)) &= V_3(x(k+1)) - V_3(x(k)) \\ &= \sum_{j=-h_2+2}^{-h_1+1} \Big[x^T(k)Q(\alpha)x(k) + \sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l) - \sum_{l=k+j-1}^{k-1} x^T(l)Q(\alpha)x(l) \Big] \\ &= (h_2-h_1)x^T(k)Q(\alpha)x(k) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i). \qquad (4.9)\end{aligned}$$

It is easy to see that

$$\Delta V_2(x(k)) + \Delta V_3(x(k)) \;\leq\; (h_2-h_1+1)x^T(k)Q(\alpha)x(k) - x^T_{k-h}Q(\alpha)x_{k-h} + \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i),$$

for simplicity, we let $x(k-h(k)) = x_{k-h}$. Since, $h(k) \leq h_2$, we obtain that

$$\sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Q(\alpha)x(i) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Q(\alpha)x(i) \leq 0.$$

Therefore, we conclude that

$$\begin{aligned}\Delta V(x(k)) &\le x^T(k)\bar{A}^T(\alpha)P(\alpha)\bar{A}(\alpha)x(k) + x^T(k-h)\bar{B}^T(\alpha)P(\alpha)\bar{A}(\alpha)x(k)\\ &+ x^T(k)\bar{A}^T(\alpha)P(\alpha)\bar{B}(\alpha)x_{k-h} + x^T_{k-h}\bar{B}^T(\alpha)P(\alpha)\bar{B}(\alpha)x_{k-h}\\ &- x^T(k)P(\alpha)x(k) + (h_2 - h_1 + 1)x^T(k)Q(\alpha)x(k) - x^T_{k-h}Q(\alpha)x_{k-h}.\end{aligned}$$

Then, it follows form above inequality and we rewrite this inequality as

$$\Delta V(x(k)) \le Y^T \begin{bmatrix} \Delta_{11} & \bar{A}^T(\alpha)P(\alpha)\bar{B}(\alpha)\\ \bar{B}^T(\alpha)P(\alpha)\bar{A}(\alpha) & \bar{B}^T(\alpha)P(\alpha)\bar{B}(\alpha) - Q(\alpha)\end{bmatrix} Y, \quad (4.10)$$

where $\Delta_{11} = \bar{A}^T(\alpha)P(\alpha)\bar{A}(\alpha) - P(\alpha) + \hat{h}Q(\alpha)$, $\hat{h} = h_2 - h_1 + 1$ and $Y^T = [x(k)^T \quad x(k-h(k))^T]$. From (4.10), we assume $(i), (ii)$ and by Lemma 4.2.1 then we obtain that $\Delta V(x(k)) < 0$. Therefore, this means that the system (4.4) is robustly stable. The proof of theorem is complete. □

Example 4.2.2 Consider the following the uncertain discrete-time LPD system with time-varying delays (4.4) where $h(k) = 2 + cos(\frac{k\pi}{2})$ i.e., $h_1 = 1, h_2 = 3$ and

$$A_1 = \begin{bmatrix} -0.6 & 0.02\\ 0.02 & -0.6\end{bmatrix}, \quad A_2 = \begin{bmatrix} -0.7 & 0.03\\ 0.03 & -0.7\end{bmatrix}, \quad B_1 = \begin{bmatrix} -0.6 & 0.02\\ 0.02 & -0.08\end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.8 & 0.03\\ 0.03 & -0.09\end{bmatrix}, A_1^1 = \begin{bmatrix} 0.005 & 0.0001\\ 0.0001 & 0.005\end{bmatrix}, \quad A_2^1 = \begin{bmatrix} 0.006 & 0.0002\\ 0.0002 & 0.006\end{bmatrix},$$

$$B_1^1 = \begin{bmatrix} -0.007 & 0.0005\\ 0.0005 & -0.007\end{bmatrix}, B_2^1 = \begin{bmatrix} -0.004 & 0.0002\\ 0.0002 & -0.004\end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.01 & 0.003\\ 0.003 & 0.01\end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.02 & 0.001\\ 0.001 & 0.02\end{bmatrix},$$

and $J = \begin{bmatrix} 0.001 & 0\\ 0 & 0.001\end{bmatrix}$. By using LMI Toolbox in MATLAB, we use the condition (i) and (ii) in Theorem 4.2.2 for this example. The solutions of LMI verify as follows of the form $\epsilon = 1$, $P_1 = \begin{bmatrix} 31.3635 & 1.2365\\ 1.2365 & 29.4763\end{bmatrix}$, $P_2 = \begin{bmatrix} 37.6354 & 0.2543\\ 0.2543 & 41.3745\end{bmatrix}$, $Q_1 = \begin{bmatrix} 9.4325 & 0.5587\\ 0.5587 & 11.4534\end{bmatrix}$, and $Q_2 = \begin{bmatrix} 10.8564 & 1.3856\\ 1.3856 & 11.9781\end{bmatrix}$. □

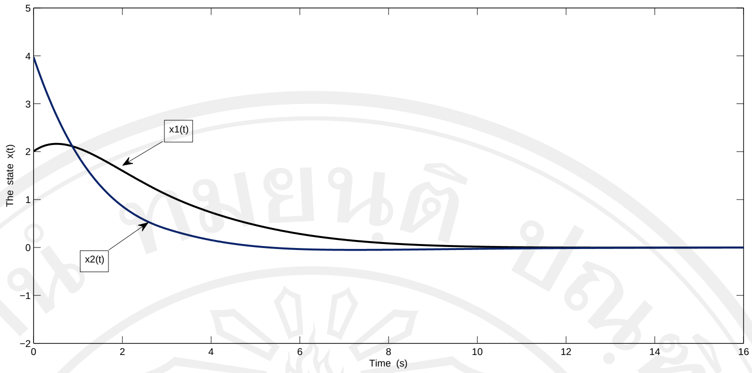

Figure 4.3: The simulation of solutions for the states $x_1(k)$ and $x_2(k)$ in the this example for Uncertain discrete LPD delay system with initial conditions $x_1(k) = 2$ and $x_2(k) = 4$, $k = -3, -2, -1, 0$ by using method of Runge-Kutta order 4(h=0.01) with Matlab.

4.3 Stability Criteria of Uncertain Linear Time-Varying Delay System with Nonlinear Perturbation

Consider the uncertain linear time-varying delay system with nonlinear perturbations described by the following state equation of the form

$$\begin{cases} x(k+1) = \big[A + \Delta A\big]x(k) + \big[B + \Delta B\big]x(k - h(k)) \\ \qquad\qquad + f(k, x(k)) + g(t, x(k - h(k))), & \forall k \in Z^+; \\ x(k) = \phi(k), & \forall k \in [-h_2, 0], \end{cases} \tag{4.11}$$

where $x(k) \in R^n$, $h(k)$ is a positive integer representing the time delay. A and B are given matrix of appropriate dimensions. $\phi(k)$ is a real valued initial function on $[-h_2, 0]$ with the norm $\|\phi\| = \sup_{t\in[-h_2,0]} \|\phi(k)\|$. The uncertainties $f(\cdot), g(\cdot)$ represent the nonlinear parameter perturbations with respect to the current state $x(k)$ and the delayed state $x(k-h(k))$, respectively, and are bounded in magnitude:

$$f^T(k, x(k))f(k, x(k)) \le \eta x^T(k)x(k),$$

$$g^T(k, x(k - h(k)))g(k, x(k - h(k))) \le \rho x^T(k - h(k))x(k - h(k)),$$

where η, ρ are given nonnegative integers. The uncertainties $\Delta A(k)$ and $\Delta B(k)$ are unknown matrices representing time-varying parameter uncertainties, we are assumed to be of the form

$$\Delta A(k) = K\Delta(k)A_1, \quad \Delta B(k) = K\Delta(k)B_1,$$

where $K, A_1, B_1 \in M^{n\times n}$ and $\Delta(k)$ satisfies

$$\Delta(k) = F(k)[I - JF(k)]^{-1}, \quad I - JJ^T > 0.$$

The uncertain matrix $F(k)$ satisfies

$$F(k)^T F(k) \leq I.$$

In addition, we assume that the time-varying delays $h(k)$ are upper and lower bounded. It satisfies the following assumption of the form

$$h_1 \leq h(k) \leq h_2,$$

where h_1 and h_2 are known positive integer. Let us set

$$\bar{A} = A + \Delta A, \quad \bar{B} = B + \Delta B.$$

Lemma 4.3.1 *Let $\bar{A}, \bar{B}$ be given matrices as in (4.11). Let P, Q be symmetric positive matrices and positive real numbers $\epsilon, \epsilon_1, \epsilon_2, \eta, \gamma$ and $\hat{h} = h_2 - h_1 + 1$. Then,*

$$\begin{bmatrix} \bar{A}^T P\bar{A} - P + \hat{h}Q + \epsilon_1\eta I & \bar{A}^T P\bar{B} & \bar{A}^T P & \bar{A}^T P \\ \bar{B}^T P\bar{A} & \bar{B}^T P\bar{B} - Q + \epsilon_2\gamma I & \bar{B}^T P & \bar{B}^T P \\ P\bar{A} & P\bar{B} & P - \epsilon_1 I & P \\ P\bar{A} & P\bar{B} & P & P - \epsilon_2 I \end{bmatrix} < 0,$$

if and only if

$$\begin{bmatrix} -P + \hat{h}Q + \epsilon_1\eta I & 0 & 0 & 0 & A^T P & \epsilon^{-1}A_1^T & 0 \\ 0 & -Q + \epsilon_2\gamma I & 0 & 0 & B^T P & \epsilon^{-1}B_1^T & 0 \\ 0 & 0 & -\epsilon_1 I & 0 & P & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 I & P & 0 & \epsilon K \\ PA & PB & P & P & -P & 0 & \epsilon PK \\ \epsilon^{-1}A_1 & \epsilon^{-1}B_1 & 0 & 0 & 0 & -I & J \\ 0 & 0 & 0 & \epsilon K^T & \epsilon K^T P & J^T & -I \end{bmatrix} < 0.$$

Proof. We consider

$$\begin{bmatrix} \bar{A}^T P\bar{A} - P + \hat{h}Q + \epsilon_1\eta I & \bar{A}^T P\bar{B} & \bar{A}^T P & \bar{A}^T P \\ \bar{B}^T P\bar{A} & \bar{B}^T P\bar{B} - Q + \epsilon_2\gamma I & \bar{B}^T P & \bar{B}^T P \\ P\bar{A} & P\bar{B} & P - \epsilon_1 I & P \\ P\bar{A} & P\bar{B} & P & P - \epsilon_2 I \end{bmatrix}$$

where $\hat{h} = h_2 - h_1 + 1$. Then, we rewrite above matrix as

$$\begin{bmatrix} -P + \hat{h}Q + \epsilon_1\eta I & 0 & 0 & 0 \\ 0 & -Q + \epsilon_2\gamma I & 0 & 0 \\ 0 & 0 & -\epsilon_1 I & 0 \\ 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} + \begin{bmatrix} \bar{A}^T P\bar{A} & \bar{A}^T P\bar{B} & \bar{A}^T P & \bar{A}^T P \\ \bar{B}^T P\bar{A} & \bar{B}^T P\bar{B} & \bar{B}^T P & \bar{B}^T P \\ P\bar{A} & P\bar{B} & P & P \\ P\bar{A} & P\bar{B} & P & P \end{bmatrix}$$

$$= \begin{bmatrix} -P + \hat{h}Q + \epsilon_1\eta I & 0 & 0 & 0 \\ 0 & -Q + \epsilon_2\gamma I & 0 & 0 \\ 0 & 0 & -\epsilon_1 I & 0 \\ 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \\ I \\ I \end{bmatrix} P \begin{bmatrix} \bar{A} & \bar{B} & I & I \end{bmatrix}.$$

We assume that

$$\begin{bmatrix} -P + \hat{h}Q + \epsilon_1\eta I & 0 & 0 & 0 \\ 0 & -Q + \epsilon_2\gamma I & 0 & 0 \\ 0 & 0 & -\epsilon_1 I & 0 \\ 0 & 0 & 0 & -\epsilon_2 I \end{bmatrix} + \begin{bmatrix} \bar{A}^T \\ \bar{B}^T \\ I \\ I \end{bmatrix} P \begin{bmatrix} \bar{A} & \bar{B} & I & I \end{bmatrix} < 0.$$

Using the well-known Schur complement lemma (Lemma 2.3.16), the above inequality is equivalent to below inequality as

$$\begin{bmatrix} -P+\hat{h}Q+\epsilon_1\eta I & 0 & 0 & 0 & \bar{A}^T \\ 0 & -Q+\epsilon_2\gamma I & 0 & 0 & \bar{B}^T \\ 0 & 0 & -\epsilon_1 I & 0 & I \\ 0 & 0 & 0 & -\epsilon_2 I & I \\ \bar{A} & \bar{B} & I & I & P^{-1} \end{bmatrix} < 0.$$

We rewrite above inequality again as

$$\begin{bmatrix} -P+\hat{h}Q+\epsilon_1\eta I & 0 & 0 & 0 & A^T \\ 0 & -Q+\epsilon_2\gamma I & 0 & 0 & B^T \\ 0 & 0 & -\epsilon_1 I & 0 & I \\ 0 & 0 & 0 & -\epsilon_2 I & I \\ A & B & I & I & P^{-1} \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ 0 \\ 0 \\ K \\ K \end{bmatrix} \Delta(k) \begin{bmatrix} A_1 & B_1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_1 & B_1 & 0 & 0 & 0 \end{bmatrix}^T \Delta(k)^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \\ K \end{bmatrix}^T < 0.$$

By using Lemma 2.3.15, above inequality holds if and only if there exists $\epsilon > 0$ such that

$$\begin{bmatrix} -P+\hat{h}Q+\epsilon_1\eta I & 0 & 0 & 0 & A^T \\ 0 & -Q+\epsilon_2\gamma I & 0 & 0 & B^T \\ 0 & 0 & -\epsilon_1 I & 0 & I \\ 0 & 0 & 0 & -\epsilon_2 I & I \\ A & B & I & I & P^{-1} \end{bmatrix} +$$

$$\begin{bmatrix} \epsilon^{-1}A_1^T & 0 \\ \epsilon^{-1}B_1^T & 0 \\ 0 & 0 \\ 0 & \epsilon K \\ 0 & \epsilon K \end{bmatrix} \begin{bmatrix} I & -J \\ -J^T & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1}A_1^T & 0 \\ \epsilon^{-1}B_1^T & 0 \\ 0 & 0 \\ 0 & \epsilon K \\ 0 & \epsilon K \end{bmatrix}^T < 0. \qquad (4.12)$$

We use Schur complement lemma (Lemma 2.3.16) again for (4.12), it becomes that

$$\begin{bmatrix} -P+\hat{h}Q+\epsilon_1\eta I & 0 & 0 & 0 & A^T & \epsilon^{-1}A_1^T & 0 \\ 0 & -Q+\epsilon_2\gamma I & 0 & 0 & B^T & \epsilon^{-1}B_1^T & 0 \\ 0 & 0 & -\epsilon_1 I & 0 & I & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 I & I & 0 & \epsilon K \\ A & B & I & I & P^{-1} & 0 & \epsilon K \\ \epsilon^{-1}A_1 & \epsilon^{-1}B_1 & 0 & 0 & 0 & -I & J \\ 0 & 0 & 0 & \epsilon K^T & \epsilon K^T & J^T & -I \end{bmatrix} < 0 \quad (4.13)$$

Pre-multiplying (4.12) by $diag\{I, I, I, I, P, I, I\}$ and post-multiplying by $diag\{I, I, I, I, P, I, I\}$. We obtain the result

$$\begin{bmatrix} -P+\hat{h}Q+\epsilon_1\eta I & 0 & 0 & 0 & A^TP & \epsilon^{-1}A_1^T & 0 \\ 0 & -Q+\epsilon_2\gamma I & 0 & 0 & B^TP & \epsilon^{-1}B_1^T & 0 \\ 0 & 0 & -\epsilon_1 I & 0 & P & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 I & P & 0 & \epsilon K \\ PA & PB & P & P & -P & 0 & \epsilon PK \\ \epsilon^{-1}A_1 & \epsilon^{-1}B_1 & 0 & 0 & 0 & -I & J \\ 0 & 0 & 0 & \epsilon K^T & \epsilon K^TP & J^T & -I \end{bmatrix} < 0.$$

The proof of lemma is complete. □

Theorem 4.3.2 *The system* (4.11) *is robustly stable if there exist symmetric positive matrices* P, Q *and positive real numbers* $\epsilon, \epsilon_1, \epsilon_2, \eta, \gamma$ *and* $\hat{h} = h_2 - h_1 + 1$ *satisfying* *the LMI.*

$$\begin{bmatrix} -P+\hat{h}Q+\epsilon_1\eta I & 0 & 0 & 0 & A^TP & \epsilon^{-1}A_1^T & 0 \\ 0 & -Q+\epsilon_2\gamma I & 0 & 0 & B^TP & \epsilon^{-1}B_1^T & 0 \\ 0 & 0 & -\epsilon_1 I & 0 & P & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 I & P & 0 & \epsilon K \\ PA & PB & P & P & -P & 0 & \epsilon PK \\ \epsilon^{-1}A_1 & \epsilon^{-1}B_1 & 0 & 0 & 0 & -I & J \\ 0 & 0 & 0 & \epsilon K^T & \epsilon K^T P & J^T & -I \end{bmatrix} < 0. \tag{4.14}$$

Proof. We define the following Lyapunov function for system (4.11) of the form

$$V(x(k)) = V_1(x(k)) + V_2(x(k)) + V_3(x(k)),$$

where

$$V_1(x(k)) = x^T(k)Px(k), V_2(x(k)) = \sum_{i=k-h(k)}^{k-1} x^T(i)Qx(i),$$

$$V_3(x(k)) = \sum_{j=-h_2+2}^{-h_1+1} \sum_{l=k+j-1}^{k-1} x^T(l)Qx(l).$$

A Lyapunov difference for the system (4.11) is defined as

$$\Delta V(x(k)) = \Delta V_1(x(k)) + \Delta V_2(x(k)) + \Delta V_3(x(k)).$$

Thus, we have

$$\begin{aligned} \Delta V_1(x(k)) &= V_1(x(k+1)) - V_1(x(k)) \\ &= x^T(k+1)Px(k+1) - x^T(k)Px(k) \\ &= x^T(k)\bar{A}^TP\bar{A}x(k) + x^T(k)\bar{A}^TP\bar{B}x(k-h(k)) + x^T(k)\bar{A}^TPf(k,x(k)) \\ &\quad +x^T(k)\bar{A}^TPg(t,x(k-h(k))) + x^T(k-h(k))\bar{B}^TP\bar{A}x(k) \\ &\quad +x^T(k-h(k))\bar{B}^TP\bar{B}x(k-h(k)) + x^T(k-h(k))\bar{B}^TPf(k,x(k)) \\ &\quad +x^T(k-h(k))\bar{B}^TPg(t,x(k-h(k))) + f^T(k,x(k))PAx(k) \\ &\quad +f^T(k,x(k))PBx(k-h(k)) + f^T(k,x(k))Pf(k,x(k)) \end{aligned}$$

$$+f^T(k,x(k))Pg(k,x(k-h(k))) + g^T(k,x(k-h(k)))PAx(k)$$
$$+g^T(k,x(k-h(k)))PBx(k-h(k)) + g^T(k,x(k-h(k)))Pf(k,x(k))$$
$$+g^T(k,x(k-h(k)))Pg(k,x(k-h(k))) - x^T(k)Px(k)$$

Next, we consider

$$\begin{aligned}\Delta V_2(x(k)) &= V_2(x(k+1)) - V_2(x(k)) \\ &= \sum_{i=k+1-h(k+1)}^{k} x^T(i)Qx(i) - \sum_{i=k-h(k)}^{k-1} x^T(i)Qx(i) \\ &= x^T(k)Qx(k) - x^T(k-h(k))Qx(k-h(k)) \\ &\quad + \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Qx(i) - \sum_{i=k+1-h(k+1)}^{k-1} x^T(i)Qx(i) \\ &\quad + \sum_{i=k+1-h_1}^{k-1} x^T(i)Qx(i).\end{aligned}$$

Form $h(k) \geq h_1$, we have that

$$\sum_{i=k+1-h_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-h(k+1)}^{k-1} x^T(i)Qx(i) \leq 0.$$

Thus, we obtain

$$\Delta V_2(k) \leq x^T(k)Qx(k) - x^T(k-h(k))Qx(k-h(k)) + \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Qx(i).$$

Finally, we have

$$\begin{aligned}\Delta V_3(x(k)) &= V_3(x(k+1)) - V_3(x(k)) \\ &= \sum_{j=-h_2+2}^{-h_1+1}\Big[x^T(k)Qx(k) + \sum_{l=k+j-1}^{k-1} x^T(l)Qx(l) - \sum_{l=k+j-1}^{k-1} x^T(l)Qx(l)\Big] \\ &= (h_2-h_1)x^T(k)Qx(k) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Qx(i).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\Delta V_2(k) + \Delta V_3(k) &\leq (h_2-h_1+1)x^T(k)Qx(k) - x^T(k-h(k))Qx(k-h(k)) \\ &\quad + \sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Qx(i) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Qx(i).\end{aligned}$$

Since, $h(k) \leq h_2$, we obtain that

$$\sum_{i=k+1-h(k+1)}^{k-h_1} x^T(i)Qx(i) - \sum_{i=k+1-h_2}^{k-h_1} x^T(i)Qx(i) \leq 0.$$

Therefore, we conclude that

$$\begin{aligned}
\Delta V(k) \leq\ & x^T(k)\bar{A}^T P\bar{A}x(k) + x^T(k)\bar{A}^T P\bar{B}x(k-h(k)) + x^T(k)\bar{A}^T Pf(k,x(k)) \\
& + x^T(k)\bar{A}^T Pg(t,x(k-h(k))) + x^T(k-h(k))\bar{B}^T P\bar{A}x(k) \\
& + x^T(k-h(k))\bar{B}^T P\bar{B}x(k-h(k)) + x^T(k-h(k))\bar{B}^T Pf(k,x(k)) \\
& + x^T(k-h(k))\bar{B}^T Pg(t,x(k-h(k))) + f^T(k,x(k))PAx(k) \\
& + f^T(k,x(k))PBx(k-h(k)) + f^T(k,x(k))Pf(k,x(k)) \\
& + f^T(k,x(k))Pg(k,x(k-h(k))) + g^T(k,x(k-h(k)))PAx(k) \\
& + g^T(k,x(k-h(k)))PBx(k-h(k)) + g^T(k,x(k-h(k)))Pf(k,x(k)) \\
& + g^T(k,x(k-h(k)))Pg(k,x(k-h(k))) - x^T(k)Px(k) \\
& + (h_2 - h_1 + 1)x^T(k)Qx(k) - x^T(k-h(k))Qx(k-h(k)) \\
& + \epsilon_1\eta x^T(t)x(t) - \epsilon_1 f^T(k,x(k))^T f(k,x(k)) \\
& + \epsilon_2\gamma x^T(t-h(t))x(t-h(t)) - \epsilon_2 g^T(k,x(k-h(k)))g^T(k,x(k-h(k))).
\end{aligned}$$

Since,

$$\epsilon_1\eta x^T(t)x(t) - \epsilon_1 f^T(k,x(k))^T f(k,x(k)) \geq 0,$$

$$\epsilon_2\gamma x^T(t-h(t))x(t-h(t)) - \epsilon_2 g^T(k,x(k-h(k)))g^T(k,x(k-h(k))) \geq 0.$$

Then, it follows from above inequality as

$$\Delta V(k) \leq Y^T \begin{bmatrix} \Delta_{11} & \bar{A}^T P\bar{B} & \bar{A}^T P & \bar{A}^T P \\ \bar{B}^T P\bar{A} & \bar{B}^T P\bar{B} - Q + \epsilon_2\gamma I & \bar{B}^T P & \bar{B}^T P \\ P\bar{A} & P\bar{B} & P - \epsilon_1 I & P \\ P\bar{A} & P\bar{B} & P & P - \epsilon_2 I \end{bmatrix} Y, \tag{4.15}$$

where $\hat{h} = h_2 - h_1 + 1$, $\Delta_{11} = \bar{A}^T P\bar{A} - P + \hat{h}Q + \epsilon_1\eta I$

$$Y = \begin{bmatrix} x^T(k) & x^T(k-h(k)) & f(k,x(k)) & g^T(k,x(k-h(k))) \end{bmatrix}^T.$$

From (4.15), assumption (4.14) and Lemma 4.3.1, we conclude that

$$\Delta V(x(k)) < 0.$$

Therefore, this means that the system (4.11) is robustly stable. The proof of theorem is complete. □

Example 4.3.2 Consider uncertain linear time-varying delay system with nonlinear perturbations described by the following state equation of the form

$$\begin{aligned} x(k+1) &= \big[A+\Delta A\big]x(k) + \big[B+\Delta B\big]x(k-h(k)) + f(k,x(k)) \\ &\quad + g(t,x(k-h(k))), \quad k \in Z^{+} \end{aligned} \tag{4.16}$$

where

$$A = \begin{bmatrix} -0.062110 & 0.026464 \\ -0.024400 & -0.061080 \end{bmatrix}, \quad B = \begin{bmatrix} 0.001482 & -0.001284 \\ -0.002704 & 0.002664 \end{bmatrix},$$

$$f(k,x(k)) = \begin{bmatrix} 0.1sin(k)x_1(k) \\ 0.1cos(k)x_2(k) \end{bmatrix}, \quad g(t,x(t-h(t))) = \begin{bmatrix} 2cos(k)x_1(k-h(k)) \\ 2sin(k)x_2(k-h(k)) \end{bmatrix},$$

$$K = \begin{bmatrix} 0.01288 & -0.01484 \\ -0.01486 & 0.01480 \end{bmatrix}, A_1 = \begin{bmatrix} -0.0560 & 0.00846 \\ -0.0024208 & -0.062900 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.0288 & -0.00484 \\ -0.00486 & 0.0480 \end{bmatrix}, \quad J = \begin{bmatrix} -0.0071 & 0.0117 \\ 0.0117 & -0.0155 \end{bmatrix}, \quad F(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By taking $\epsilon_1 = \epsilon_2 = 2, \epsilon = 1$ and $h(k) = 3sin^2(\frac{k\pi}{2}t)$, we obtain $\gamma = \eta = 0.1$, and $h_2 - h_1 = 3$. By using LMI Toolbox in MATLAB, we use the assumption (4.14) in Theorem 4.3.2 for this example. The solutions of LMI verify as follows of the form

$$P = \begin{bmatrix} 0.9176 & 0.0009 \\ 0.0009 & 0.9205 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.2127 & 0.0002 \\ 0.0002 & 0.2137 \end{bmatrix}.$$

Therefore, the system (4.16) is robustly stable. □

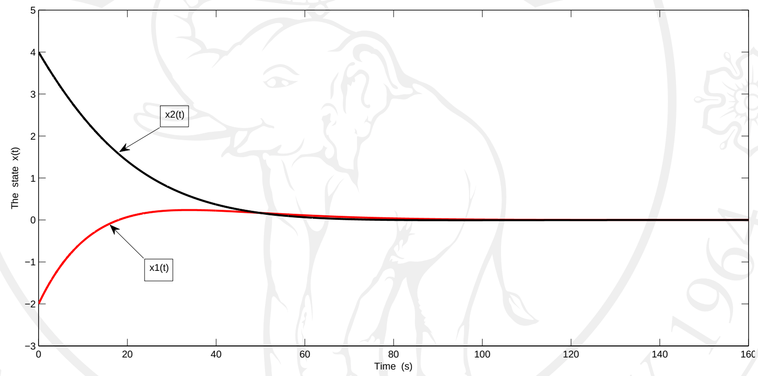

Figure 4.4: The simulation of solutions for the states $x_1(t)$ and $x_2(t)$ in the LPD delay system (4.16) with initial conditions $x_1(t) = -2$ and $x_2(t) = 4$, $k = -4, -3, -2, -1, 0$ by using method of Runge-Kutta order 4(h=0.01) with Matlab.