Chapter 5

Discontinuous Time Delay Systems

In this chapter, we consider the problems of the robust exponential stability for uncertain impulsive switched system and uncertain impulsive switched linear parameter dependent (LPD) system with time-varying delays and nonlinear perturbations. We use appropriate Lyapunov function and derive exponential stability condition in terms of linear matrix inequalities (LMIs) for uncertain impulsive switched system with time-varying delays and nonlinear perturbations. We apply the Halanay lemma to study the robust exponential stability for uncertain impulsive switched LPD system with time-varying delays and nonlinear perturbations. The new stability condition is less conservative and is more general than some existing results. Numerical examples are presented to illustrate the effectiveness of the theoretical results.

5.1 Stability Criteria of Uncertain Impulsive Switched System with Time-varying Delays and Nonlinear Perturbations

Consider the uncertain impulsive switched system with time-varying delays and nonlinear perturbations of the form

$$\begin{cases} \dot{x}(t) = \Big[A_{i_k} + \Delta A_{i_k}(t)\Big]x(t) + \Big[B_{i_k} + \Delta B_{i_k}(t)\Big]x(t - h_{i_k}(t)) \\ \qquad + f_{i_k}(t, x(t)) + g_{i_k}(t, x(t - h_{i_k}(t))), & t \neq t_k; \\ \Delta x(t) = I_k(x(t)) = D_k x(t), & t = t_k; \\ x(t) = \phi(t), \dot{x}(t) = \psi(t), & \forall t \in [-h, 0], \end{cases} \tag{5.1}$$

where $x(t) \in R^n$ is the state, $0 \leq h_{i_k}(t) \leq h$ and $\phi(t), \psi(t)$ are a piecewise continuous vector-valued initial function. A_{i_k}, B_{i_k} and D_k are given real matrices of appropriate dimensions. The uncertainties $f_{i_k}(.), g_{i_k}(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h(t))$, respectively, and are bounded in magnitude:

$$f_{i_k}^T(t, x(t)) f_{i_k}(t, x(t)) \leq \eta x^T(t) x(t),$$

$$g_{i_k}^T(t, x(t-h_{i_k}(t))) g_{i_k}(t, x(t-h_{i_k}(t))) \leq \rho x^T(t-h(t)) x(t-h(t)),$$

where η, ρ are given nonnegative constants. $\Delta x(t) = x(t_k^+) - x(t_k^-)$, $\lim_{\nu \to 0^+} x(t_k + \nu) = x(t_k^+)$, $x(t_k^-) = \lim_{\nu \to 0^+} x(t - \nu)$. we assume that the solution of the impulsive switched system (5.1) is right continuous i.e., $x(t_k^+) = x(t_k)$. $i_k \in \{1, 2, ..., m\}$, $k, m \in Z^+$, t_k is an impulsive switching time point and $t_0 < t_1 < t_2 < \cdots < t_k < \cdots, t_k \to +\infty$ as $k \to +\infty$. Under the switching law of system (5.1), at the time t_k, the system switches to the i_k subsystem from the i_{k-1} subsystem. The delay $h_{i_k}(t)$ is a time varying bounded continuous function satisfying

$$0 \leq h_{i_k}(t) \leq h, \quad \dot{h_{i_k}}(t) \leq \delta < 1,$$

for all i_k and $t > 0$. The uncertainty $\Delta A_{i_k}(t)$ and $\Delta B_{i_k}(t)$ are time varying matrices and satisfy the condition

$$\Delta A_{i_k}(t) = E_{i_k} \Delta_{i_k}(t) H_{i_k}, \Delta B_{i_k}(t) = E_{i_k} \Delta_{i_k}(t) M_{i_k},$$

where $\Delta_{i_k}(t)$ satisfies

$$\Delta_{i_k}(t) = F_{i_k}(t)[I - J F_{i_k}(t)]^{-1}, \quad I - JJ^T > 0.$$

The uncertain matrix $F_{i_k}(t)$ satisfies

$$F_{i_k}(t)^T F_{i_k}(t) \leq I.$$

Theorem 5.1.1 *The system* (5.1) *is robust exponentially stable, if there exist symmetric positive definite matrices P_{i_k} and Q_{i_k} for all $i_k \in \{1, 2, ..., m\}$, $k, m \in Z^+$ and positive real numbers $\delta, \beta, \eta, \rho, \epsilon_1, \epsilon_2$ such that the following conditions hold.*

$$(i) \begin{bmatrix} A_{11} & P_{i_k} \hat{B}_{i_k} & P_{i_k} & P_{i_k} \\ \hat{B}_{i_k}^T P_{i_k} & -(1-\delta) e^{-2\beta h} Q & 0 & 0 \\ P_{i_k} & 0 & -\epsilon_1 I & 0 \\ P_{i_k} & 0 & 0 & -\epsilon_2 I \end{bmatrix} \leq 0,$$

$$(ii) \begin{bmatrix} P_{i_{k-1}} & (I+D_k)^T P_{i_k} \\ P_{i_k}(I+D_k) & P_{i_k} \end{bmatrix} > 0,$$

where $A_{11} = 2\beta P_{i_k} + \hat{A}_{i_k}^T P_{i_k} + P_{i_k}\hat{A}_{i_k} + Q + (\epsilon_1\eta + \epsilon_2\rho)I$, $\hat{A}_{i_k} = A_{i_k} + \Delta A_{i_k}(t)$, $\hat{B}_{i_k} = B_{i_k} + \Delta B_{i_k}(t)$.

Proof. We consider for $t \in (t_k, t_{k+1}]$ and define the following Lyapunov function for system (5.1) of the form

$$V(t,x(t)) = e^{2\beta t}x^T(t)P_{i_k}x(t) + \int_{t-h_{i_k}(t)}^{t} e^{2\beta s}x^T(s)Qx(s)ds.$$

The derivative of V along the trajectories of system (5.1) is given by

$$\begin{aligned} D^+V &\leq e^{2\beta t}\Big[2\beta x^T(t)P_{i_k}x(t) + \dot{x}^T(t)P_{i_k}x(t) + x^T(t)P_{i_k}\dot{x}(t)\Big] \\ &+ e^{2\beta t}\Big[x^T(t)Qx(t) - (1-\dot{h}_{i_k}(t))e^{-2\beta h}x^T(t-h_{i_k}(t))Qx(t-h_{i_k}(t))\Big]. \end{aligned}$$

Thus, we obtain that

$$\begin{aligned} D^+V \leq\ & e^{2\beta t}\Big[2\beta x^T(t)P_{i_k}x(t) + x^T(t)\hat{A}_{i_k}^T P_{i_k}x(t) + x^T(t-h_{i_k}(t))\hat{B}_{i_k}^T P_{i_k}x(t) \\ +\ & x^T(t)P_{i_k}\hat{A}_{i_k}x(t) + x^T(t)P_{i_k}\hat{B}_{i_k}x(t-h_{i_k}(t)) + g_{i_k}(t,x(t-h_{i_k}(t)))P_{i_k}x(t) \\ +\ & f_{i_k}^T(t,x(t))P_{i_k}x(t) + x^T(t)P_{i_k}f_{i_k}(t,x(t)) + x^T(t)P_{i_k}g_{i_k}(t,x(t-h_{i_k}(t)))\Big] \\ +\ & e^{2\beta t}\Big[x^T(t)Qx(t) - (1-\delta)e^{-2\beta h}x^T(t-h_{i_k}(t))Qx(t-h_{i_k}(t))\Big] \\ +\ & e^{2\beta t}\Big[\epsilon_1\eta x^T(t)x(t) - \epsilon_1 f_{i_k}^T(t,x(t))f_{i_k}(t,x(t))\Big] \\ +\ & e^{2\beta t}\Big[\epsilon_2\rho x^T(t)x(t) - \epsilon_2 g_{i_k}^T(t,x(t-h_{i_k}(t)))g_{i_k}(t,x(t-h_{i_k}(t)))\Big]. \end{aligned}$$

Then, we have $D^+V \leq e^{2\beta t}y^T(t)\Phi y(t)$, where

$$\Phi = \begin{bmatrix} A_{11} & P_{i_k}\hat{B}_{i_k} & P_{i_k} & P_{i_k} \\ \hat{B}_{i_k}^T P_{i_k} & -(1-\delta)e^{-2\beta h}Q & 0 & 0 \\ P_{i_k} & 0 & -\epsilon_1 I & 0 \\ P_{i_k} & 0 & 0 & -\epsilon_2 I \end{bmatrix}$$

where $A_{11} = 2\beta P_{i_k} + \hat{A}_{i_k}^T P_{i_k} + P_{i_k}\hat{A}_{i_k} + Q + (\epsilon_1\eta + \epsilon_2\rho)I$ and $y^T(t) = [x^T(t) \quad x^T(t-h_{i_k}(t)) \quad f_{i_k}^T(t,x(t)) \quad g_{i_k}^T(t,x(t-h_{i_k}(t)))]$. By the condition (i), we conclude that $D^+V(t,x(t)) \leq 0$. Integrating both sides of this inequality from 0 to t, we find

$$V(t,x(t)) - V(0,x(0)) \leq 0,$$

and hence

$$\begin{aligned} V(t,x(t)) &\leq x^T(0)P_{i_k}x(0) + \int_{0-h_{i_k}(0)}^{0} e^{2\beta s}x^T(s)Qx(s)ds \\ &\leq \lambda\|\phi\|, \end{aligned}$$

where

$$\|x(0)\| = \|\phi(0)\| \leq \|\phi\|,$$

$$\begin{aligned} \int_{-h}^{0} e^{2\beta s}x^T(s)Qx(s)ds &\leq \lambda_{max}(Q)\|\phi\|^2 \int_{-h}^{0} e^{2\beta s}ds \\ &= \frac{\lambda_{max}(Q)}{2\beta}(1-e^{-2\beta h})\|\phi\|^2, \end{aligned}$$

$\lambda = max\{\lambda_{max}(P_{i_k}), \frac{\lambda_{max}(Q)}{2\beta}(1-e^{-2\beta h})\}$. Therefore, the solution $x(t,\phi)$ of the system (5.1) is bounded and it is easy to see that

$$\|x(t,\phi)\| \leq \frac{\lambda}{\eta}\|\phi\|e^{-\beta t},$$

where $\eta = \lambda_{min}(P_{i_k})$. This means that the system (5.1) is exponentially stable. But it is except at the impulsive and switching points. We consider the time points $t_k, k = 1,2,3,...$, when the system switches form the t_{k-1} subsystem to the t_k subsystem. To ensure the exponentially stable with a decay rate β, the following condition is required to be satisfied

$$\begin{aligned} V(t_k^+, x(t_k^+)) - V(t_k, x(t_k)) &= x^T(t_k^+)P_{i_k}x(t_k^+) - x^T(t_k)P_{i_{k-1}}x(t_k) \\ &= x(t_k)^T\Big[(I+D_k)^T P_{i_k}(I+D_k) - P_{i_{k-1}}\Big]x(t_k) \\ &< 0. \end{aligned}$$

Since we use the assumption (ii). This means that

$$(I+D_k)^T P_{i_k}(I+D_k) - P_{i_{k-1}} < 0,$$

or, equivalently,

$$P_{i_{k-1}} - (I+D_k)^T P_{i_k}(I+D_k) > 0.$$

We see that the above inequality is equivalent to this inequality

$$\begin{bmatrix} P_{i_{k-1}} & (I+D_k)^T P_{i_k} \\ P_{i_k}(I+D_k) & P_{i_k} \end{bmatrix} > 0.$$

The proof of the theorem is complete. □

Theorem 5.1.2 *The system* (5.1) *is robust exponentially stable, if there exist symmetric positive definite matrices* P_{i_k} *and* Q_{i_k} *for all* $i_k \in \{1, 2, ..., m\}$, $k, m \in Z^+$ *and positive real numbers* $\delta, \beta, \eta, \rho, \epsilon, \epsilon_1, \epsilon_2$ *such that the following LMIs hold.*

$$(i) \quad \begin{bmatrix} \Theta_{11} & P_{i_k}B_{i_k} & P_{i_k} & P_{i_k} & \epsilon^{-1}H_{i_k}^T & \epsilon P_{i_k}E_{i_k} \\ B_{i_k}^T P_{i_k} & -(1-\delta)e^{-2\beta h}Q & 0 & 0 & \epsilon^{-1}M_{i_k}^T & 0 \\ P_{i_k} & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_{i_k} & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}H_{i_k} & \epsilon^{-1}M_{i_k} & 0 & 0 & -I & J \\ \epsilon E_{i_k}^T P_{i_k} & 0 & 0 & 0 & J & -I \end{bmatrix} \leq 0,$$

$$(ii) \quad \begin{bmatrix} P_{i_{k-1}} & (I+D_k)^T P_{i_k} \\ P_{i_k}(I+D_k) & P_{i_k} \end{bmatrix} > 0,$$

where $\Theta_{11} = 2\beta P_{i_k} + A_{i_k}^T P_{i_k} + P_{i_k}A_{i_k} + Q + (\epsilon_1\eta + \epsilon_2\rho)I$.

Proof. Let us set

$$\Phi_{i_k} = \begin{bmatrix} \Delta_{11} & P_{i_k}\hat{B}_{i_k} & P_{i_k} & P_{i_k} \\ \hat{B}_{i_k}^T P_{i_k} & -(1-\delta)e^{-2\beta h}Q & 0 & 0 \\ P_{i_k} & 0 & -\epsilon_1 I & 0 \\ P_{i_k} & 0 & 0 & -\epsilon_2 I \end{bmatrix},$$

where $\Delta_{11} = 2\beta P_{i_k} + \hat{A}_{i_k}^T P_{i_k} + P_{i_k}\hat{A}_{i_k} + Q + (\epsilon_1\eta + \epsilon_2\rho)I$, $\hat{A}_{i_k} = A_{i_k} + \Delta A_{i_k}(t)$, $\hat{B}_{i_k} = B_{i_k} + \Delta B_{i_k}(t)$ and

$$\Omega_{i_k} = \begin{bmatrix} \Theta_{11} & P_{i_k}B_{i_k} & P_{i_k} & P_{i_k} \\ B_{i_k}^T P_{i_k} & -(1-\delta)e^{-2\beta h}Q & 0 & 0 \\ P_{i_k} & 0 & -\epsilon_1 I & 0 \\ P_{i_k} & 0 & 0 & -\epsilon_2 I \end{bmatrix}.$$

Then, we obtain

$$\begin{aligned}\Phi_{i_k} &= \Omega_{i_k} + \begin{bmatrix} \Delta A_{i_k}^T(t)P_{i_k} + P_{i_k}\Delta A_{i_k}(t) & P_{i_k}\Delta B_{i_k}(t) & 0 & 0 \\ \Delta B_{i_k}^T(t)P_{i_k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \Omega_{i_k} + \begin{bmatrix} H_{i_k}^T\Delta_{i_k}^T(t)E_{i_k}^T P_{i_k} + P_{i_k}E_{i_k}\Delta_{i_k}(t)H_{i_k} & P_{i_k}E_{i_k}\Delta_{i_k}(t)M_{i_k} & 0 & 0 \\ M_{i_k}^T\Delta_{i_k}^T(t)E_{i_k}^T P_{i_k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \Omega_{i_k} + \begin{bmatrix} P_{i_k}E_{i_k} \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta_{i_k}(t) \begin{bmatrix} H_{i_k} & M_{i_k} & 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} H_{i_k} & M_{i_k} & 0 & 0 \end{bmatrix}^T \Delta_{i_k}^T(t) \begin{bmatrix} P_{i_k}E_{i_k} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T .\end{aligned}$$

By Lemma 2.3.14, we assume $\Phi_{i_k} \leq 0$ if and only if there exists $\epsilon > 0$ such that

$$\Omega_{i_k} + \begin{bmatrix} \epsilon^{-1} \begin{bmatrix} H_{i_k}^T \\ M_{i_k}^T \\ 0 \\ 0 \end{bmatrix} & \epsilon \begin{bmatrix} P_{i_k}E_{i_k} \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} I & -J \\ -J & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1} \begin{bmatrix} H_{i_k}^T \\ M_{i_k}^T \\ 0 \\ 0 \end{bmatrix} & \epsilon \begin{bmatrix} P_{i_k}E_{i_k} \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix}^T \leq 0.$$

On using Schur complement Lemma (Lemma 2.3.16) in above inequality, it becomes that

$$\begin{bmatrix} \Theta_{11} & P_{i_k}B_{i_k} & P_{i_k} & P_{i_k} & \epsilon^{-1}H_{i_k}^T & \epsilon P_{i_k}E_{i_k} \\ B_{i_k}^T P_{i_k} & -(1-\delta)e^{-2\beta h}Q & 0 & 0 & \epsilon^{-1}M_{i_k}^T & 0 \\ P_{i_k} & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_{i_k} & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}H_{i_k} & \epsilon^{-1}M_{i_k} & 0 & 0 & -I & J \\ \epsilon E_{i_k}^T P_{i_k} & 0 & 0 & 0 & J & -I \end{bmatrix} \leq 0.$$

The proof of the theorem is complete. □

Example 5.1.2 We consider the following uncertain impulsive switched system with Time-varying delays (5.1) under a given switching law. That is, the switching status alternates as $i_1 \to i_2 \to i_1 \to i_2 \to \cdots$. We consider robust performance of the system (5.1) by using Theorem 5.1.2. The system (5.1) is specified as follows:

$$A_1 = \begin{bmatrix} -9 & 1 \\ 2 & -8 \end{bmatrix}, B_1 = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}, H_1 = \begin{bmatrix} -0.21 & 0.001 \\ 0 & -0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -11 & 2 \\ 1 & -8.2 \end{bmatrix}, B_2 = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, H_2 = \begin{bmatrix} -0.3 & 0.01 \\ 0 & -0.3 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} -0.3 & 0 \\ 0.005 & -0.2 \end{bmatrix}, E_2 = \begin{bmatrix} -0.3 & 0 \\ 0.003 & -0.4 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} -0.1 & 0.003 \\ 0 & -0.1 \end{bmatrix}, M_2 = \begin{bmatrix} -0.3 & 0.005 \\ 0 & -0.3 \end{bmatrix}, J = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$D_1 = D_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The nonlinear functions $f_i(\cdot), g_i(\cdot)$, $i = 1, 2$ of the form

$$f_1(t, x(t)) = \begin{bmatrix} 1.1529\cos(t)x_1(t) \\ 1.1529\cos(t)x_2(t) \end{bmatrix}, f_2(t, x(t)) = \begin{bmatrix} 1.1529\sin(t)x_1(t) \\ 1.1529\cos(t)x_2(t) \end{bmatrix},$$

and

$$g_1(t, x(t-h_1(t))) = \begin{bmatrix} 1.1529\sin(t)x_1(t-h_1(t)) \\ 1.1529\sin(t)x_2(t-h_1(t)) \end{bmatrix},$$

$$g_2(t, x(t-h_2(t))) = \begin{bmatrix} 1.1529\cos(t)x_1(t-h_2(t)) \\ 1.1529\sin(t)x_2(t-h_2(t)) \end{bmatrix}.$$

We choose that $\delta = 0.5$, $F_1(t) = F_2(t) = I$, $\epsilon = 1, \epsilon_1 = \epsilon_2 = 6.2715$, $\delta = 0.1$, $\beta = 0.1$, $h_1(t)) = 8.0259\sin^2(\frac{0.5}{8.0259})t$ and $h_2(t)) = 7.8259\sin^2(\frac{0.5}{7.8259})t$, that is, $h = 8.0259$. By using LMI Toolbox in MATLAB, we use the asumptions (i) and (ii) in Theorem 5.1.2 to this example. The solutions of LMI are as follows:

$$P_1 = \begin{bmatrix} 1.6636 & 0.3223 \\ 0.3223 & 1.8792 \end{bmatrix}, P_2 = \begin{bmatrix} 1.3194 & 0.2114 \\ 0.2114 & 1.8478 \end{bmatrix}, Q = \begin{bmatrix} 7.7471 & -0.1482 \\ -0.1482 & 8.2042 \end{bmatrix}.$$

We conclude the relation between δ and $h_{\max}$.

δ	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
$h_{\max}$	3.4444	5.4718	6.9102	8.0259	8.9375	9.7082	10.3759	10.9648

Therefore, the system (5.1) is 0.1−stable. □

Numerical Simulations

Numerical experiments are carried out to investigate dynamical system by using dde45lin in Matlab. In Fig. 5.1, the parameters of the system are specified as in Example 5.1.2 and the initial condition is $x(t) = [4 \quad -5]^T, t \in [-8.0259, 0]$,

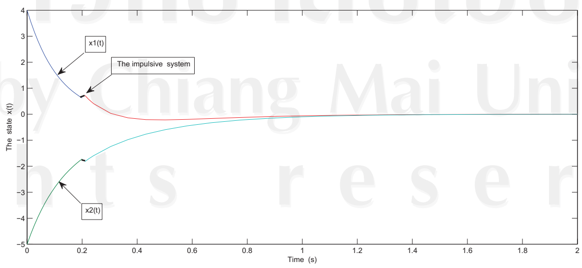

Figure 5.1: The simulation of solutions for the impulsive switched time-varying delay system in example 5.1.2.

Consider the uncertain impulsive switched systems with time-varying delays of the form

$$\begin{cases} \dot{x}(t) = \Big[A_{i_k} + \Delta A_{i_k}(t)\Big]x(t) + \Big[B_{i_k} + \Delta B_{i_k}(t)\Big]x(t - h_{i_k}(t)), & t \neq t_k; \\ \Delta x(t) = I_k(x(t)) = D_k x(t), & t = t_k; \\ x(t) = \phi(t), \quad \dot{x}(t) = \psi(t), & \forall t \in [-h, 0], \end{cases} \tag{5.2}$$

From system (5.2) where , we take the change of the state variable

$$y(t) = e^{\beta t}x(t), \quad t \in R^+, \tag{5.3}$$

then the linear delay system (5.2) is transformed to the delay system

$$\begin{aligned} \dot{y}(t) &= \overline{A}_{i_k}(t,\beta)y(t) + \overline{B}_{i_k}(t,\beta)y(t - h_{i_k}(t)), \\ y(t) = e^{\beta t}\phi(t), &\quad \dot{y}(t) = \beta e^{\beta t}\phi(t) + e^{\beta t}\psi(t) \quad t \in [-h, 0], \end{aligned} \tag{5.4}$$

where

$$\overline{A}_{i_k}(t,\beta) = \widetilde{A}_{i_k}(t) + \beta I = A_{i_k} + \Delta A_{i_k}(t) + \beta I,$$
$$\overline{B}_{i_k}(t,\beta) = e^{\beta h_{i_k}(t)}\widetilde{B}_{i_k}(t) = e^{\beta h_{i_k}(t)}\Big[B_{i_k} + \Delta B_{i_k}(t)\Big],$$

that is, $A_{i_k}(\beta) = A_{i_k} + \beta I$.

We introduce the following notations for using in Lemma 2.1,

$$\Phi_{i_k}(t) = \begin{bmatrix} \Xi_{11} & P_{i_k}\widetilde{B}_{i_k}(t) + e^{\beta h}W_{i_k} & K_{i_k}\overline{A}_{i_k}(t,\beta) \\ \widetilde{B}^T_{i_k}(t)P_{i_k} + e^{\beta h}W_{i_k} & -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k} & K_{i_k}\widetilde{B}_{i_k}(t) \\ \overline{A}^T_{i_k}(t,\beta)K_{i_k} & \widetilde{B}^T_{i_k}(t)K_{i_k} & h^2e^{2\beta h}W_{i_k} - 2K_{i_k} \end{bmatrix},$$

where

$$\Xi_{11} = \overline{A}^T_{i_k}(t,\beta)P_{i_k} + P_{i_k}\overline{A}_{i_k}(t,\beta) + Q_{i_k} - e^{\beta h}W_{i_k}.$$

Lemma 5.1.3 *Let $\overline{A}_{i_k}(t), \widetilde{B}_{i_k}(t) \in R^{n\times n}$ be given matrices for all $i_k \in \{1, 2, ..., m\}$, for all $k, m \in N$ as in system (5.4). Let P_{i_k}, Q_{i_k}, W_{i_k} and K_{i_k} for all $i_k \in \{1, 2, ..., m\}$, for all $k, m \in Z^+$ be symmetric positive definite matrices and positive* *real numbers δ, β, ϵ, ϵ_1 and h. Then $\Phi_{i_k}(t) \leq 0$ if and only if*

$$\begin{bmatrix} \Delta_{11} & P_{i_k}B_{i_k} + e^{\beta h}W_{i_k} & K_{i_k}A_{i_k}(\beta) & \epsilon^{-1}H_{i_k}^T & \epsilon P_{i_k}E_{i_k} & \epsilon_1^{-1}H_{i_k}^T & 0 \\ \star & \Delta_{22} & K_{i_k}B_{i_k} & \epsilon^{-1}M_{i_k}^T & 0 & \epsilon_1^{-1}M_{i_k}^T & 0 \\ \star & \star & \Delta_{33} & 0 & 0 & 0 & \epsilon_1 K_{i_k}E_{i_k} \\ \star & \star & \star & -I & J & 0 & 0 \\ \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & -I & J \\ \star & \star & \star & \star & \star & \star & -I \end{bmatrix} \leq 0, \quad (5.5)$$

where $\Delta_{11} = A_{i_k}^T(\beta)P_{i_k} + P_{i_k}A_{i_k}(\beta) + Q_{i_k} - e^{\beta h}W_{i_k}$, $\Delta_{22} = -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k}$ and $\Delta_{33} = h^2e^{2\beta h}W_{i_k} - 2K_{i_k}$.

Proof. We consider $\Phi_{i_k}(t)$ for $t \in (t_k, t_{k+1}]$, $i_k \in \{1, 2, ..., m\}$, for all $m, k \in Z^+$ as

$$\Phi_{i_k}(t) = \begin{bmatrix} \Xi_{11} & P_{i_k}\widetilde{B}_{i_k}(t) + e^{\beta h}W_{i_k} & K_{i_k}\overline{A}_{i_k}(t,\beta) \\ \widetilde{B}_{i_k}^T(t)P_{i_k} + e^{\beta h}W_{i_k} & -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k} & K_{i_k}\widetilde{B}_{i_k}(t) \\ \overline{A}_{i_k}^T(t,\beta)K_{i_k} & \widetilde{B}_{i_k}^T(t)K_{i_k} & h^2e^{2\beta h}W_{i_k} - 2K_{i_k} \end{bmatrix},$$

where

$$\Xi_{11} = \overline{A}_{i_k}^T(t,\beta)P_{i_k} + P_{i_k}\overline{A}_{i_k}(t,\beta) + Q_{i_k} - e^{\beta h}W_{i_k}.$$

Then, we have

$$\begin{aligned} \Phi_{i_k}(t) &= \begin{bmatrix} \Delta_{11} & P_{i_k}B_{i_k} + e^{\beta h}W_{i_k} & K_{i_k}A_{i_k}(\beta) \\ B_{i_k}^TP_{i_k} + e^{\beta h}W_{i_k} & -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k} & K_{i_k}B_{i_k} \\ A_{i_k}^T(\beta)K_{i_k} & B_{i_k}^TK_{i_k} & h^2e^{2\beta h}W_{i_k} - 2K_{i_k} \end{bmatrix} \\ &+ \begin{bmatrix} \Theta_{11} & P_{i_k}E_{i_k}\Delta_{i_k}(t)M_{i_k} & K_{i_k}E_{i_k}\Delta_{i_k}(t)H_{i_k} \\ M_{i_k}^T\Delta_{i_k}^T(t)E_{i_k}^TP_{i_k} & 0 & K_{i_k}E_{i_k}\Delta_{i_k}(t)M_{i_k} \\ H_{i_k}^T\Delta_{i_k}^T(t)E_{i_k}^TK_{i_k} & M_{i_k}^T\Delta_{i_k}^T(t)E_{i_k}^TK_{i_k} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \Delta_{11} & P_{i_k}B_{i_k} + e^{\beta h}W_{i_k} & K_{i_k}\overline{A}_{i_k}(t,\beta) \\ B_{i_k}^TP_{i_k} + e^{\beta h}W_{i_k} & -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k} & K_{i_k}\widetilde{B}_{i_k}(t) \\ \overline{A}_{i_k}^T(t,\beta)K_{i_k} & \widetilde{B}_{i_k}^T(t)K_{i_k} & h^2e^{\beta h}W_{i_k} - 2K_{i_k} \end{bmatrix} \end{aligned}$$

$$+\begin{bmatrix} P_{i_k}E_{i_k} \\ 0 \\ 0 \end{bmatrix} \Delta_{i_k}(t) \begin{bmatrix} H_{i_k} & M_{i_k} & 0 \end{bmatrix} + \begin{bmatrix} H_{i_k}^T \\ M_{i_k}^T \\ 0 \end{bmatrix} \Delta_{i_k}^T(t) \begin{bmatrix} E_{i_k}^T P_{i_k} & 0 & 0 \end{bmatrix},$$

where $\Delta_{11} = A_{i_k}^T(\beta)P_{i_k} + P_{i_k}A_{i_k}(\beta) + Q_{i_k} - e^{\beta h}W_{i_k}$ and $\Theta_{11} = P_{i_k}E_{i_k}\Delta_{i_k}(t)H_{i_k} + H_{i_k}^T\Delta_{i_k}^T(t)E_{i_k}^T P_{i_k}$. By Lemma 2.3.14, $\Phi_{i_k}(t) \leq 0$ is equivalent to this inequality

$$\begin{bmatrix} \Delta_{11} & P_{i_k}B_{i_k} + e^{\beta h}W_{i_k} & K_{i_k}\overline{A}_{i_k}(t,\beta) \\ B_{i_k}^T P_{i_k} + e^{\beta h}W_{i_k} & -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k} & K_{i_k}\widetilde{B}_{i_k}(t) \\ \overline{A}_{i_k}^T(t,\beta)K_{i_k} & \widetilde{B}_{i_k}^T(t)K_{i_k} & h^2 e^{\beta h}W_{i_k} - 2K_{i_k} \end{bmatrix}$$
$$+ \begin{bmatrix} \epsilon^{-1}H_{i_k}^T & \epsilon P_{i_k}E_{i_k} & 0 \\ \epsilon^{-1}M_{i_k}^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & -J \\ -J^T & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1}H_{i_k}^T & \epsilon P_{i_k}E_{i_k} & 0 \\ \epsilon^{-1}M_{i_k}^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T \leq 0.$$

By using Scher complement Lemma (Lemma 2.3.16) in above inequality, the above inequality is equivalent to

$$\begin{bmatrix} \Delta_{11} & P_{i_k}B_{i_k} + e^{\beta h}W_{i_k} & K_{i_k}\overline{A}_{i_k}(t,\beta) & \epsilon^{-1}H_{i_k}^T & \epsilon P_{i_k}E_{i_k} \\ B_{i_k}^T P_{i_k} + e^{\beta h}W_{i_k} & \Delta_{22} & K_{i_k}\widetilde{B}_{i_k}(t) & \epsilon^{-1}M_{i_k}^T & 0 \\ \overline{A}_{i_k}^T(t,\beta)K_{i_k} & \widetilde{B}_{i_k}^T(t)K_{i_k} & \Delta_{33} & 0 & 0 \\ \epsilon^{-1}H_{i_k} & \epsilon^{-1}M_{i_k} & 0 & -I & J \\ \epsilon E_{i_k}^T P_{i_k} & 0 & 0 & J^T & -I \end{bmatrix} \leq 0,$$

Using Lemma 2.3.14 and Schur complement Lemma (Lemma 2.3.16) again, the above inequality is equivalent to

$$\begin{bmatrix} \Delta_{11} & P_{i_k}B_{i_k} + e^{\beta h}W_{i_k} & K_{i_k}A_{i_k}(\beta) & \epsilon^{-1}H_{i_k}^T & \epsilon P_{i_k}E_{i_k} & \epsilon_1^{-1}H_{i_k}^T & 0 \\ \star & \Delta_{22} & K_{i_k}B_{i_k} & \epsilon^{-1}M_{i_k}^T & 0 & \epsilon_1^{-1}M_{i_k}^T & 0 \\ \star & \star & \Delta_{33} & 0 & 0 & 0 & \epsilon_1 K_{i_k}E_{i_k} \\ \star & \star & \star & -I & J & 0 & 0 \\ \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & -I & J \\ \star & \star & \star & \star & \star & \star & -I \end{bmatrix} \leq 0,$$

where $\Delta_{11} = A_{i_k}^T(\beta)P_{i_k} + P_{i_k}A_{i_k}(\beta) + Q_{i_k} - e^{\beta h}W_{i_k}$, $\Delta_{22} = -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k}$ *and* $\Delta_{33} = h^2e^{2\beta h}W_{i_k} - 2K_{i_k}$. The proof of the lemma is complete. □

Theorem 5.1.4 *The system* (5.2) *is robustly* $\beta-$ *stable, if there exist* P_{i_k}, Q_{i_k}, W_{i_k} *and* K_{i_k} *be positive definite symmetric matrices for all* $i_k \in \{1, 2, ..., m\}$, $m, k \in Z^+$ *and positive real numbers* δ, β, ϵ, ϵ_1 *and* h *such that the following LMI hold.*

$$(i)\quad \begin{bmatrix} \Delta_{11} & \Delta_{12} & K_{i_k}A_{i_k}(\beta) & \epsilon^{-1}H_{i_k}^T & \epsilon P_{i_k}E_{i_k} & \epsilon_1^{-1}H_{i_k}^T & 0 \\ \star & \Delta_{22} & K_{i_k}B_{i_k} & \epsilon^{-1}M_{i_k}^T & 0 & \epsilon_1^{-1}M_{i_k}^T & 0 \\ \star & \star & \Delta_{33} & 0 & 0 & 0 & \epsilon_1 K_{i_k}E_{i_k} \\ \star & \star & \star & -I & J & 0 & 0 \\ \star & \star & \star & \star & -I & 0 & 0 \\ \star & \star & \star & \star & \star & -I & J \\ \star & \star & \star & \star & \star & \star & -I \end{bmatrix} < 0,$$

$$(ii)\quad \begin{bmatrix} P_{i_{k-1}} & (I + D_k)^T P_{i_k} \\ P_{i_k}(I + D_k) & P_{i_k} \end{bmatrix} > 0,$$

$$(iii)\quad Q_{i_k} - Q_{i_{k-1}} < 0,$$

$$(iv)\quad W_{i_k} - W_{i_{k-1}} < 0,$$

where $\Delta_{11} = A_{i_k}^T(\beta)P_{i_k} + P_{i_k}A_{i_k}(\beta) + Q_{i_k} - e^{\beta h}W_{i_k}$, $\Delta_{22} = -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k}$, $\Delta_{12} = P_{i_k}B_{i_k} + e^{\beta h}W_{i_k}$ *and* $\Delta_{33} = h^2e^{2\beta h}W_{i_k} - 2K_{i_k}$.

Proof. For $t \in (t_k, t_{k+1}]$, we define the following Lyapunov function for system (5.4) of the form

$$V(t, y(t)) = y^T(t)P_{i_k}y(t) + \int_{t-h_{i_k}(t)}^{t} y^T(s)Q_{i_k}y(s)ds + he^{2\beta h}\int_{-h}^{0}\int_{t+s}^{t} \dot{y}^T(\alpha)W_{i_k}\dot{y}(\alpha)d\alpha ds.$$

The derivative of V along the trajectories of system (5.4) is given by

$$\begin{aligned} D^+V &= \dot{y}^T(t)P_{i_k}y(t) + y^T(t)P_{i_k}\dot{y}(t) \\ &\quad + y^T(t)Q_{i_k}y(t) - (1 - \dot{h}_{i_k}(t))y^T(t - h_{i_k}(t))Q_{i_k}y(t - h_{i_k}(t)) \\ &\quad + h^2e^{2\beta h}\dot{y}^T(t)W_{i_k}\dot{y}(t) - he^{2\beta h}\int_{-h}^{0}\dot{y}^T(s + t)W_{i_k}\dot{y}(s + t)ds \\ &\le y^T(t)\overline{A}_{i_k}^T(t, \beta)P_{i_k}y(t) + y^T(t - h_{i_k}(t))\overline{B}_{i_k}^T(t, \beta)P_{i_k}y(t) \\ &\quad + y^T(t)P_{i_k}(t)\overline{A}_{i_k}(t, \beta)y(t) + y^T(t)P_{i_k}(t)\overline{B}_{i_k}(t, \beta)y(t - h_{i_k}(t)) \\ &\quad + y^T(t)Q_{i_k}y(t) - (1 - \delta)y^T(t - h_{i_k}(t))Q_{i_k}y(t - h_{i_k}(t)) \end{aligned}$$

$$
\begin{aligned}
&+h^2e^{2\beta h}\dot{y}^T(t)W_{i_k}\dot{y}(t) - he^{2\beta h}\int_{-h}^{0}\dot{y}^T(s+t)W_{i_k}\dot{y}(s+t)ds \\
&+2\dot{y}^T(t)K_{i_k}\Big[-\dot{y}(t) + \overline{A}_{i_k}(t,\beta)y(t) + \overline{B}_{i_k}^T(t,\beta)y(t-h_{i_k}(t))\Big] \\
\leq\ & \begin{bmatrix} y(t) \\ e^{\beta h_{i_k}(t)}y(t-h_{i_k}(t)) \\ \dot{y}(t) \end{bmatrix}^T \Phi_{i_k}(t) \begin{bmatrix} y(t) \\ e^{\beta h_{i_k}(t)}y(t-h_{i_k}(t)) \\ \dot{y}(t) \end{bmatrix}.
\end{aligned}
$$

By estimation the last term for above inequality by using Jensen's Inequality, it follows that

$$
\begin{aligned}
&-he^{2\beta h}\int_{-h}^{0}\dot{y}^T(s+t)W_{i_k}\dot{y}(s+t)ds \\
\leq\ & -h(t)e^{2\beta h}\int_{-h(t)}^{0}\dot{y}^T(s+t)W_{i_k}\dot{y}(s+t)ds \\
=\ & -h(t)e^{2\beta h}\int_{t-h(t)}^{t}\dot{y}^T(s)W_{i_k}\dot{y}(s)ds \\
\leq\ & -e^{2\beta h}\Big[y(t) - y(t-h_{i_k}(t))\Big]^T W_{i_k}\Big[y(t) - y(t-h_{i_k}(t))\Big], \\
\leq\ & -e^{\beta h}e^{\beta h_{i_k}(t)}\Big[y(t) - y(t-h_{i_k}(t))\Big]^T W_{i_k}\Big[y(t) - y(t-h_{i_k}(t))\Big], \\
\leq\ & -e^{\beta h}y^T(t)W_{i_k}y(t) + 2e^{\beta h}e^{\beta h_{i_k}(t)}y^T(t)W_{i_k}y(t-h_{i_k}(t)) \\
& -e^{2\beta h_{i_k}(t)}y^T(t-h_{i_k}(t))W_{i_k}y(t-h_{i_k}(t))
\end{aligned} \tag{5.6}
$$

and

$$
\Phi_{i_k}(t) = \begin{bmatrix} \Xi_{11} & P_{i_k}\widetilde{B}_{i_k}(t) + e^{\beta h}W_{i_k} & K_{i_k}\overline{A}_{i_k}(t,\beta) \\ \widetilde{B}_{i_k}^T(t)P_{i_k} + e^{\beta h}W_{i_k} & -e^{-2\beta h}Q_{i_k} + \delta Q_{i_k} - W_{i_k} & K_{i_k}\widetilde{B}_{i_k}(t) \\ \overline{A}_{i_k}^T(t,\beta)K_{i_k} & \widetilde{B}_{i_k}^T(t)K_{i_k} & h^2e^{2\beta h}W_{i_k} - 2K_{i_k} \end{bmatrix},
$$

where

$$
\begin{aligned}
&\Xi_{11} = \overline{A}_{i_k}^T(t,\beta)P_{i_k} + P_{i_k}\overline{A}_{i_k}(t,\beta) + Q_{i_k} - e^{\beta h}W_{i_k}, \\
&\overline{A}_{i_k}(t,\beta) = A_{i_k} + \Delta A_{i_k}(t) + \beta I, \\
&\overline{B}_{i_k}(t,\beta) = e^{\beta h_{i_k}(t)}\widetilde{B}_{i_k}(t) = e^{\beta h_{i_k}(t)}\Big[B_{i_k} + \Delta B_{i_k}(t)\Big],
\end{aligned}
$$

that is, $A_{i_k}^T(\beta) = A_{i_k} + \beta I$. By (i) and Lemma 5.1.3, we conclude that

$$
D^+V(t,y(t)) < 0. \tag{5.7}
$$

Integrating both sides of (5.7) from 0 to t, we obtain

$$V(t, y(t)) - V(0, y(0)) \leq 0,$$

and hence

$$\begin{aligned} y^T(t)P_{i_k}y(t) &+ \int_{t-h_{i_k}(t)}^{t} y^T(s)Q_{i_k}y(s)ds + he^{2\beta h}\int_{-h}^{0}\int_{t+s}^{t} \dot{y}^T(\alpha)W_{i_k}\dot{y}(\alpha)d\alpha ds \\ &\leq y^T(0)P_{i_k}y(0) + \int_{-h_{i_k}(0)}^{0} y^T(s)Q_{i_k}y(s)ds + he^{2\beta h}\int_{-h}^{0}\int_{s}^{0} \dot{y}^T(\alpha)W_{i_k}\dot{y}(\alpha)d\alpha ds \end{aligned}$$

$$\int_{-h}^{0} y^T(s)Q_{i_k}y(s)ds \leq \lambda_{max}(Q)\|\phi\|^2\int_{-h}^{0} e^{2\beta s}ds = \frac{\lambda_{max}(Q)}{2\beta}(1 - e^{-2\beta h})\|\phi\|^2,$$

we have

$$\begin{aligned} \lambda_{min}(P_{i_k})\|y(t)\|^2 \quad &\leq \quad \lambda_{max}(P_{i_k})\|y(0)\|^2 + \frac{\lambda_{max}(Q_{i_k})}{2\beta}(1 - e^{-2\beta h})\|\phi\|^2 \\ &\quad +2h^3e^{2\beta h}\lambda_{max}(W_{i_k})\sup\{\|\phi\|, \|\psi\|\}^2. \qquad (5.8) \end{aligned}$$

Therefore, the solution $y(t, \phi, \psi)$ is bounded. Returning to the solution $x(t, \phi, \psi)$ of system (5.2), it is easy to see that

$$\|y(0)\| = \|x(0)\| = \phi(0) \leq \|\phi\|,$$

we have

$$\|x(t, \phi, \psi)\| \leq \xi(\|\phi\|)e^{-\beta t},$$

where

$$\xi(\|\phi, \psi\|) := \Big\{\frac{\lambda_{max}(P_{i_k})\|\phi\|^2 + \frac{\lambda_{max}(Q_{i_k})}{2\beta}(1 - e^{-2\beta h})\|\phi\|^2}{\lambda_{min}(P_{i_k})} \frac{+2h^3e^{2\beta h}\lambda_{max}(W_{i_k})\sup\{\|\phi\|, \|\psi\|\}^2}{\lambda_{min}(P_{i_k})}\Big\}^{\frac{1}{2}}$$

This means that the system (5.2) is robustly $\beta-$ stable. We will consider the case at the time point $t_k, k = 1, 2, 3, ...$ when the system switches form the t_{k-1} subsystem to the t_k subsystem. To ensure the $\beta-$ stability, we need to show that $V(t_k^+, y(t_k^+)) - V(t_k, y(t_k)) < 0$. We have

$$
\begin{aligned}
& V(t_k^+, y(t_k^+)) - V(t_k, y(t_k)) \\
= \;& y^T(t_k^+)P_{i_k}y(t_k^+) - y^T(t_k)P_{i_{k-1}}y(t_k) \\
& + \int_{t-h_{i_k}(t)}^{t} e^{2\beta s}x^T(s)\big[Q_{i_k} - Q_{i_{k-1}}\big]x(s)ds \\
& + he^{2\beta h}\int_{-h}^{0}\int_{t+s}^{s} \dot{y}^T(\alpha)\Big[W_{i_k} - W_{i_k-1}\Big]\dot{y}(\alpha)d\alpha ds \\
= \;& x(t_k)^T e^{\beta t_k}\Big[(I+D_k)^T P_{i_k}(I+D_k) - P_{i_{k-1}}\Big]e^{\beta t_k}x(t_k) \\
& + \int_{t-h_{i_k}(t)}^{t} e^{2\beta s}x^T(s)\big[Q_{i_k} - Q_{i_{k-1}}\big]x(s)ds \\
& + he^{2\beta h}\int_{-h}^{0}\int_{t+s}^{t} \dot{y}^T(\alpha)\Big[W_{i_k} - W_{i_k-1}\Big]\dot{y}(\alpha)d\alpha ds \\
= \;& x(t_k)^T e^{2\beta t_k}\Big[(I+D_k)^T P_{i_k}(t_k)(I+D_k) - P_{i_{k-1}}\Big]x(t_k) \\
& + \int_{t-h_{i_k}(t)}^{t} e^{2\beta s}x^T(s)\big[Q_{i_k} - Q_{i_{k-1}}\big]x(s)ds \\
& + he^{2\beta h}\int_{-h}^{0}\int_{t+s}^{t} \dot{y}^T(\alpha)\Big[W_{i_k} - W_{i_k-1}\Big]\dot{y}(\alpha)d\alpha ds
\end{aligned}
$$

By assumptions (ii), (iii) and (iv). We have

$$
\Big[(I+D_k)^T P_{i_k}(I+D_k) - P_{i_{k-1}}\Big] < 0, \quad Q_{i_k} - Q_{i_{k-1}} < 0, \quad W_{i_k} - W_{i_k-1} < 0.
$$

Therefore, $V(t_k^+, y(t_k^+)) - V(t_k, y(t_k)) < 0$. The proof of the theorem is complete.□

Example 5.1.4 We consider the following uncertain impulsive switched system with Time-varying delays (5.2) under a given switching law. That is, the switching status alternates as $i_1 \to i_2 \to i_1 \to i_2 \to \cdots$. We consider robust performance of the system (5.2) by using Theorem 2.2. The system (5.2) is specified as follows:

$$
A_1 = \begin{bmatrix} -7 & 1 \\ -1 & -6 \end{bmatrix}, B_1 = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}, H_1 = \begin{bmatrix} 0.4 & 0.1 \\ 0 & -0.4 \end{bmatrix},
$$

$$
A_2 = \begin{bmatrix} -6 & 1 \\ 2 & -8 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, H_2 = \begin{bmatrix} -0.5 & 0.2 \\ 0 & 0.7 \end{bmatrix},
$$

$$
E_1 = \begin{bmatrix} 0.3 & 0 \\ 0.1 & -0.3 \end{bmatrix}, E_2 = \begin{bmatrix} -0.3 & 0 \\ 0.2 & 0.4 \end{bmatrix},
$$

$$M_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.1 & 0.4 \end{bmatrix}, M_2 = \begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}, J = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

We choose $F_1(t) = F_2(t) = I$, $\epsilon = \epsilon_1 = \epsilon_2 = 1$, $\delta = 0.4$, $\beta = 0.1$, $h_1(t)) = 1.8518\sin^2(\frac{0.4}{1.8518})t$ and $h_2(t)) = 1.8518\sin^2(\frac{0.4}{1.8518})t$, i.e., , $h = 1.8518$. By using LMI Toolbox in MATLAB, the solutions of LMI are as follows:

$$P_1 = \begin{bmatrix} 10.2589 & -0.9181 \\ -0.9181 & 9.3092 \end{bmatrix}, P_2 = \begin{bmatrix} 8.7370 & 0.6651 \\ 0.6651 & 6.5525 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 46.6969 & -2.1673 \\ -2.1673 & 44.4882 \end{bmatrix}, Q_2 = \begin{bmatrix} 31.8035 & -1.5450 \\ -1.5450 & 29.6337 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.3578 & -0.0632 \\ -0.0632 & 0.3406 \end{bmatrix}, W_2 = \begin{bmatrix} 0.1553 & -0.0059 \\ -0.0059 & 0.1334 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 1.4810 & -0.1887 \\ -0.1887 & 1.2210 \end{bmatrix}, K_2 = \begin{bmatrix} 1.6788 & 0.1774 \\ 0.1774 & 0.9244 \end{bmatrix}, D_1 = D_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

and we find the maximum of time-varying delays ($h_{\max} = 1.8518$). We conclude the relation between δ and $h_{\max}$.

δ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$h_{\max}$	3.5619	2.6341	1.8518	1.1756	0.6312	0.3716	0.3168	0.2343

Therefore, the system (5.2) is 0.1−stable. □

Numerical Simulations

Numerical experiments are carried out to investigate dynamical system by using dde45lin in Matlab. In Fig. 5.2, the parameters of the system are specified as in Example 5.1.4 and the initial condition is $x(t) = [3 \quad -3]^T, t \in [-1.5, 0]$,

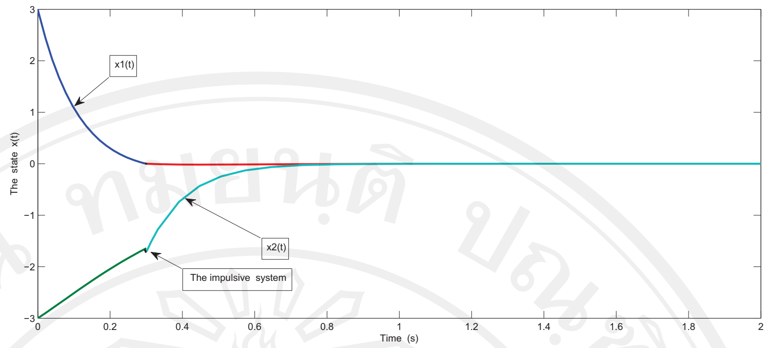

Figure 5.2: The simulation of solutions for the impulsive switched time-varying delay system in example 5.1.4

Next, we consider the linear system with time-varying delays of the form

$$\dot{x}(t) = Ax(t) + Bx(t - h(t)). \tag{5.9}$$

Corollary 5.1.5 *The system (5.9) is $\beta-$ stable , if there exist P, Q, W and K be positive definite symmetric matrices and positive real number δ, β, h such that the following LMI hold.*

$$\begin{bmatrix} A^T(\beta)P + PA(\beta) + Q - e^{\beta h}W & PB + e^{\beta h}W & KA(\beta) \\ B^TP + e^{\beta h}W & -e^{-2\beta h}Q + \delta Q - W & KB \\ A^T(\beta)K & B^TK & h^2e^{2\beta h}W_{i_k} - 2K \end{bmatrix} < 0. \tag{5.10}$$

Example 5.1.5.1 We consider the linear system (5.9) with time-varying delays in the form

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(t - h(t)). \tag{5.11}$$

We use the MATLAB LMI Toolbox for this example, we can compare with the results of other researchers, a summary is given in the following table by using the conditions in Corollary 2.3:

Methods	$h_{\max}$ (δ is unknown)	$h_{\max}$ (δ=0.1)	$h_{\max}$ (δ=2)	$h_{\max}$ ($\dot{h}$(t)=0)
Su [42]	0.405	-	-	-
Kim [18]	-	0.945	Not defined	1
Yue [55]	-	0.972	Not defined	1
Park [34]	-	-	-	4.359
Fridman [9]	-	-	-	4.470
Yan [54]	0.999	3.604	0.999	4.472
Our results	0.999	3.604	0.999	4.473

Our results use the convergent rate ($\beta = 0.000001$) for the condition in Corollary 5.1.5

Example 5.1.5.2 We consider the linear system (5.9) with time-varying delays in this form

$$\dot{x}(t) = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -0.5 & 0.1 \\ 0.3 & 0 \end{bmatrix} x(t-h). \tag{5.12}$$

We use the MATLAB LMI Toolbox for this example, we can compare with the results of other researchers, a summary is given in the following table by using the conditions in Corollary 5.1.5:

Methods	$h_{\max}$ (β=0)	$h_{\max}$ (β=0.4)	$h_{\max}$ (β=0.6)	$h_{\max}$(β=0.8)
Liu[22]	0.964	0.281	0.124	0.048
Kwon[21]	∞	2.649	1.765	1.345
Our results	∞	2.723	1.831	1.361

5.2 Stability Criteria of Uncertain Impulsive Switched LPD Time-Delay System with Nonlinear Perturbation

Consider the uncertain impulsive switched linear parameter dependent (LPD) control system with time-varying delays and nonlinear perturbations of the form

$$\begin{cases} \dot{x}(t) = \hat{A}_{i_k}(\alpha)x(t) + \hat{B}_{i_k}(\alpha)x(t-h_{i_k}(t)) + f_{i_k}(t,x(t)) & \\ \qquad +g_{i_k}(t,x(t-h_{i_k}(t))), & t \neq t_k; \\ \Delta x(t) = x(t) - x(t^-) = G_k(\alpha)x(t-h_{i_k}(t)), & t = t_k; \\ x(t) = \phi(t), & \forall t \in [-h,0], \\ \hat{A}_{i_k}(\alpha) = \Big[A_{i_k}(\alpha) + \Delta A_{i_k}(t)\Big], \hat{B}_{i_k}(\alpha) = \Big[B_{i_k}(\alpha) + \Delta B_{i_k}(t)\Big], & \end{cases} \tag{5.13}$$

where $x(t) \in R^n$ is the state, $n \in Z^+$ and $h_{i_k}(t)$ is a positive function representing the time-varying delays. $\phi(t)$ is a piecewise continuous vector-valued initial function. $A_{i_k}(\alpha)$, $B_{i_k}(\alpha)$ and $G_k(\alpha)$ are uncertain $M^{n\times n}$ matrices belonging to the polytope of the form

$$[A_{i_k}(\alpha), B_{i_k}(\alpha)] = \Big[\sum_{j=1}^{N} \alpha_j A_{i_k,j}, \sum_{j=1}^{N} \alpha_j B_{i_k,j}\Big],$$

$$\sum_{j=1}^{N} \alpha_j = 1, \alpha_j \geq 0, \quad A_{i_k,j}, B_{i_k,j} \in M^{n\times n}, j = 1,...,N,$$

and

$$G_k(\alpha) = \sum_{j=1}^{N} \alpha_j G_{k,j}, \quad G_{k,j} \in M^{n\times n}, j = 1,...,N.$$

The uncertainties $f(.), g(.)$ represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-h_{i_k}(t))$, respectively, and are bounded in magnitude:

$$f_{i_k}^T(t,x(t))f_{i_k}(t,x(t)) \leq \eta x^T(t)x(t),$$

$$g_{i_k}^T(t,x(t-h(t)))g_{i_k}(t,x_{i_k}(t-h(t))) \leq \rho x^T(t-h_{i_k}(t))x(t-h_{i_k}(t)),$$

where η, ρ are given nonnegative constants. $\Delta x(t) = x(t_k^+) - x(t_k^-)$, $x(t_k^-) = \lim_{\nu \to 0^+} x(t-\nu)$, $\lim_{\nu \to 0^+} x(t_k+\nu) = x(t_k^+) = x(t_k)$ means that the solution of the impulsive switched system (5.13) is right continuous. $i_k \in \{1, 2, ..., m\}$, $k \in N$, $m \in N$, t_k is an impulsive switching time point and $t_0 < t_1 < t_2 < \cdots < t_\infty$. Under the switching law of system (5.13), at the time point t_k, the system switches to the i_k subsystem from the i_{k-1} subsystem. The delay $h_{i_k}(t)$ is any time varying bounded continuous nonnegative function satisfying

$$0 \le h_{i_k}(t) \le h,$$

for all i_k and $t > 0$. The uncertainties $\Delta A_{i_k}(t)$, $\Delta B_{i_k}(t)$ and $\Delta C_{i_k}(t)$ are time varying matrices of the form

$$\Delta A_{i_k}(t) = E_{i_k}(\alpha)\Delta_{i_k}(t)M_{i_k}(\alpha), \quad \Delta B_{i_k}(t) = E_{i_k}(\alpha)\Delta_{i_k}(t)N_{i_k}(\alpha),$$

where $\Delta_{i_k}(t)$ satisfies

$$\Delta_{i_k}(t) = F_{i_k}(t)[I - JF_{i_k}(t)]^{-1}, \quad I - JJ^T > 0.$$

The uncertain matrix $F_{i_k}(t)$ satisfies

$$F_{i_k}(t)^T F_{i_k}(t) \le I. \tag{5.14}$$

We introduce the following notations for later use,

$$\Phi_{i_k}(\alpha) = \begin{bmatrix} \hat{\psi}_{i_k}(\alpha) & P(\alpha)\hat{B}_{i_k}(\alpha) & P(\alpha) & P(\alpha) \\ \hat{B}_{i_k}^T(\alpha)P(\alpha) & -bP(\alpha) & 0 & 0 \\ P(\alpha) & 0 & -\epsilon_1 I & 0 \\ P(\alpha) & 0 & 0 & -\epsilon_2 I \end{bmatrix}, \tag{5.15}$$

where

$$\hat{\psi}_{i_k}(\alpha) = \hat{A}_{i_k}^T(\alpha)P(\alpha) + P(\alpha)\hat{A}_{i_k}(\alpha) + (\epsilon_1\eta + \epsilon_2\rho)I + aP(\alpha),$$

$$P(\alpha) = \sum_{j=1}^{N} \alpha_j P_j,$$

and

$$\psi_{i_k}(j,l) = A_{i_k,j}^T P_l + P_j A_{i_k,l} + (\epsilon_1\eta + \epsilon_2\rho)I + aP_j. \tag{5.16}$$

Lemma 5.2.1 *Let $\hat{A}_{i_k}(\alpha), \hat{B}_{i_k}(\alpha) \in R^{n\times n}$ be given matrices for all $i_k \in \{1, 2, ..., m\}$, for all $m, k \in Z^+$ as in (5.13). Let $P_i \in R^{n\times n}$, $i = 1, 2, ..., N$, be symmetric positive definite matrices and positive real numbers $\eta, \rho, \epsilon, \epsilon_1, \epsilon_2, \zeta$ and $0 < b < a$. Then, $\Phi_{i_k}(\alpha) \le 0$ if and only if*

$$(i)\ \begin{bmatrix} \psi_{i_k}(j,j) & P_jB_{i_k,j} & P_j & P_j & \epsilon^{-1}M^T_{i_k,j} & \epsilon P_jE_{i_k,j} \\ B^T_{i_k,j}P_j & -bP_j & 0 & 0 & \epsilon^{-1}N^T_{i_k,j} & 0 \\ P_j & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_j & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,j} & \epsilon^{-1}N_{i_k,j} & 0 & 0 & -I & J \\ \epsilon E^T_{i_k,j}P_j & 0 & 0 & 0 & J & -I \end{bmatrix} \le -\zeta I, j = 1, ..., N.$$

$$(ii)\ \begin{bmatrix} \psi_{i_k}(j,l) & P_jB_{i_k,l} & P_j & P_j & \epsilon^{-1}M^T_{i_k,j} & \epsilon P_jE_{i_k,l} \\ B^T_{i_k,j}P_l & -bP_j & 0 & 0 & \epsilon^{-1}N^T_{i_k,j} & 0 \\ P_j & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_j & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,j} & \epsilon^{-1}N_{i_k,j} & 0 & 0 & -I & J \\ \epsilon E^T_{i_k,j}P_l & 0 & 0 & 0 & J & -I \end{bmatrix}$$

$$+ \begin{bmatrix} \psi_{i_k}(l,j) & P_lB_{i_k,j} & P_l & P_l & \epsilon^{-1}M^T_{i_k,l} & \epsilon P_lE_{i_k,j} \\ B^T_{i_k,l}P_j & -bP_l & 0 & 0 & \epsilon^{-1}N^T_{i_k,l} & 0 \\ P_l & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_l & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,l} & \epsilon^{-1}N_{i_k,l} & 0 & 0 & -I & J \\ \epsilon E^T_{i_k,l}P_l & 0 & 0 & 0 & J & -I \end{bmatrix} \le \frac{2\zeta I}{N-1},$$

$j = 1, ..., N-1, l = j+1, ..., N.$

Proof. We consider $\Phi_{i_k}(\alpha) \le 0$ define as

$$\Phi_{i_k}(\alpha) = \begin{bmatrix} \hat{\psi}_{i_k}(\alpha) & P(\alpha)\hat{B}_{i_k}(\alpha) & P(\alpha) & P(\alpha) \\ \hat{B}^T_{i_k}(\alpha)P(\alpha) & -bP(\alpha) & 0 & 0 \\ P(\alpha) & 0 & -\epsilon_1 I & 0 \\ P(\alpha) & 0 & 0 & -\epsilon_2 I \end{bmatrix} \le 0. \tag{5.17}$$

Then, (5.17) transforms to the below inequality

$$\Phi_{i_k}(\alpha) = \begin{bmatrix} \psi_{i_k}(\alpha) & P(\alpha)B_{i_k}(\alpha) & P(\alpha) & P(\alpha) \\ B_{i_k}^T(\alpha)P(\alpha) & -bP(\alpha) & 0 & 0 \\ P(\alpha) & 0 & -\epsilon_1 I & 0 \\ P(\alpha) & 0 & 0 & -\epsilon_2 I \end{bmatrix}$$

$$+ \begin{bmatrix} \Lambda_{i_k}(\alpha) & P(\alpha)E_{i_k}(\alpha)\Delta_{i_k}(t)N_{i_k}(\alpha) & 0 & 0 \\ N_{i_k}^T(\alpha)\Delta_{i_k}^T(t)E_{i_k}^T(\alpha)P(\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \psi_{i_k}(\alpha) & P(\alpha)B_{i_k}(\alpha) & P(\alpha) & P(\alpha) \\ B_{i_k}^T(\alpha)P(\alpha) & -bP(\alpha) & 0 & 0 \\ P(\alpha) & 0 & -\epsilon_1 I & 0 \\ P(\alpha) & 0 & 0 & -\epsilon_2 I \end{bmatrix}$$

$$+ \begin{bmatrix} P(\alpha)E_{i_k}(\alpha) \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta_{i_k}(t) \begin{bmatrix} M_{i_k}(\alpha) & N_{i_k}(\alpha) & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} M_{i_k}(\alpha) & N_{i_k}(\alpha) & 0 & 0 \end{bmatrix}^T \Delta_{i_k}^T(t) \begin{bmatrix} P(\alpha)E_{i_k}(\alpha) \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \leq 0,$$

where

$$\psi_{i_k}(\alpha) = A_{i_k}^T(\alpha)P(\alpha) + P(\alpha)A_{i_k}(\alpha) + (\epsilon_1\eta + \epsilon_2\rho)I + aP(\alpha),$$

$$\Lambda_{i_k}(\alpha) = M_{i_k}^T(\alpha)\Delta_{i_k}^T(t)E_{i_k}^T(\alpha)P(\alpha) + P(\alpha)E_{i_k}(\alpha)\Delta_{i_k}(t)M_{i_k}(\alpha).$$

By using Lemma 2.3.14, the above inequality holds if and only if there exists $\epsilon > 0$ such that

$$\begin{bmatrix} \psi_{i_k}(\alpha) & P(\alpha)B_{i_k}(\alpha) & P(\alpha) & P(\alpha) \\ B_{i_k}^T(\alpha)P(\alpha) & -bP(\alpha) & 0 & 0 \\ P(\alpha) & 0 & -\epsilon_1 I & 0 \\ P(\alpha) & 0 & 0 & -\epsilon_2 I \end{bmatrix} +$$

$$\begin{bmatrix} \epsilon^{-1}M_{i_k}^T(\alpha) & \epsilon P(\alpha)E_{i_k}(\alpha) \\ \epsilon^{-1}N_{i_k}^T(\alpha) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & -J \\ -J^T & I \end{bmatrix}^{-1} \begin{bmatrix} \epsilon^{-1}M_{i_k}^T(\alpha) & \epsilon P(\alpha)E_{i_k}(\alpha) \\ \epsilon^{-1}N_{i_k}^T(\alpha) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T \leq 0. \tag{5.18}$$

By Schur complement lemma, (5.18) is equivalent to

$$\begin{bmatrix} \psi_{i_k}(\alpha) & P(\alpha)B_{i_k}(\alpha) & P(\alpha) & P(\alpha) & \epsilon^{-1}M_{i_k}^T(\alpha) & \epsilon P(\alpha)E_{i_k}(\alpha) \\ B_{i_k}^T(\alpha)P(\alpha) & -bP(\alpha) & 0 & 0 & \epsilon^{-1}N_{i_k}^T(\alpha) & 0 \\ P(\alpha) & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P(\alpha) & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k}(\alpha) & \epsilon^{-1}N_{i_k}(\alpha) & 0 & 0 & -I & J \\ \epsilon E_{i_k}^T(\alpha)P(\alpha) & 0 & 0 & 0 & J & -I \end{bmatrix} \leq 0. \tag{5.19}$$

From (5.19), we obtain

$$\sum_{j=1}^{N} \alpha_j \sum_{l=1}^{N} \alpha_l \begin{bmatrix} \psi_{i_k}(j,l) & P_j B_{i_k,l} & P_j & P_j & \epsilon^{-1}M_{i_k,j}^T & \epsilon P_j E_{i_k,l} \\ B_{i_k,j}^T P_l & -bP_j & 0 & 0 & \epsilon^{-1}N_{i_k,j}^T & 0 \\ P_j & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_j & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,j} & \epsilon^{-1}N_{i_k,j} & 0 & 0 & -I & J \\ \epsilon E_{i_k,j}^T P_l & 0 & 0 & 0 & J & -I \end{bmatrix} \leq 0, \tag{5.20}$$

where $\psi_{i_k}(j,l) = A^T_{i_k,j}P_l + P_jA_{i_k,l} + (\epsilon_1\eta + \epsilon_2\rho)I + aP_j$. The facts that $\sum_{i=1}^{N}\alpha_i = 1$, we obtain the following identities :

$$\sum_{i=1}^{N}\alpha_iA_i\sum_{i=1}^{N}\alpha_iB_i = \sum_{i=1}^{N}\alpha^2A_iB_i + \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_i[A_iB_j + A_jB_i],$$

$$(N-1)\sum_{i=1}^{N}\alpha_i^2\zeta - 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\alpha_i\alpha_j\zeta = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}[\alpha_i-\alpha_j]^2\zeta \geq 0.$$

Hence, the inequality (5.20) is equivalent to the following inequalities

$$\begin{bmatrix} \psi_{i_k}(j,j) & P_jB_{i_k,j} & P_j & P_j & \epsilon^{-1}M^T_{i_k,j} & \epsilon P_jE_{i_k,j} \\ B^T_{i_k,j}P_j & -bP_j & 0 & 0 & \epsilon^{-1}N^T_{i_k,j} & 0 \\ P_j & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_j & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,j} & \epsilon^{-1}N_{i_k,j} & 0 & 0 & -I & J \\ \epsilon E^T_{i_k,j}P_l & 0 & 0 & 0 & J & -I \end{bmatrix} \leq -\zeta I, \quad j = 1,...,N. \tag{5.21}$$

and

$$\begin{bmatrix} \psi_{i_k}(j,l) & P_jB_{i_k,l} & P_j & P_j & \epsilon^{-1}M^T_{i_k,j} & \epsilon P_jE_{i_k,l} \\ B^T_{i_k,j}P_l & -bP_j & 0 & 0 & \epsilon^{-1}N^T_{i_k,j} & 0 \\ P_j & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_j & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,j} & \epsilon^{-1}N_{i_k,j} & 0 & 0 & -I & J \\ \epsilon E^T_{i_k,j}P_l & 0 & 0 & 0 & J & -I \end{bmatrix}$$
$$+\begin{bmatrix} \psi_{i_k}(l,j) & P_lB_{i_k,j} & P_l & P_l & \epsilon^{-1}M^T_{i_k,l} & \epsilon P_lE_{i_k,j} \\ B^T_{i_k,l}P_j & -bP_l & 0 & 0 & \epsilon^{-1}N^T_{i_k,l} & 0 \\ P_l & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_l & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,l} & \epsilon^{-1}N_{i_k,l} & 0 & 0 & -I & J \\ \epsilon E^T_{i_k,l}P_l & 0 & 0 & 0 & J & -I \end{bmatrix} \leq \frac{2\zeta I}{N-1},$$
$$j = 1,...,N-1, l = j+1,...,N. \tag{5.22}$$

Therefore, the inequality (5.17) is equivalent to (5.21) and (5.22). The proof of lemma is complete. □

Lemma 5.2.2 *Let $G_k(\alpha) \in R^{n\times n}$ be given matrices for all $k \in Z^+$ as in* (5.13)*. Let $P_i \in R^{n\times n}$, $i = 1, 2, ..., N$ be symmetric positive definite matrix and positive real numbers δ_k, ζ for all $k \in Z^+$. Then*

$$\begin{bmatrix} P(\alpha) & P(\alpha)G_k(\alpha) \\ G_k^T(\alpha)P(\alpha) & G_k^T(\alpha)P(\alpha)G_k(\alpha) \end{bmatrix} \leq \delta_k I_{2n}, \tag{5.23}$$

if and only if

$$(i) \quad \begin{bmatrix} -\delta_k I & 0 & P_j \\ 0 & -\delta_k I & G_{k,j}^T P_j \\ P_j & P_j G_{k,j} & -P_j \end{bmatrix} \leq -\zeta I, \quad j = 1, ..., N.$$

$$(ii) \quad \begin{bmatrix} -\delta_k I & 0 & P_j \\ 0 & -\delta_k I & G_{k,j}^T P_l \\ P_j & P_j G_{k,l} & -P_j \end{bmatrix} + \begin{bmatrix} -\delta_k I & 0 & P_l \\ 0 & -\delta_k I & G_{k,l}^T P_j \\ P_l & P_j G_{k,j} & -P_l \end{bmatrix} \leq \frac{2\zeta I}{N-1},$$

$$j = 1, ..., N-1, l = j+1, ..., N.$$

Proof. Consider inequality (5.23), we have

$$\begin{bmatrix} P(\alpha) & P(\alpha)G_k(\alpha) \\ G_k^T(\alpha)P(\alpha) & G_k^T(\alpha)P(\alpha)G_k(\alpha) \end{bmatrix} \leq \delta_k I_{2n}.$$

Equivalently,

$$\begin{bmatrix} -\delta_k I & 0 \\ 0 & -\delta_k I \end{bmatrix} + \begin{bmatrix} I \\ G_k^T(\alpha) \end{bmatrix} P(\alpha) \begin{bmatrix} I & G_k(\alpha) \end{bmatrix} \leq 0.$$

By using Schur complement Lemma in the above inequality, we get

$$\begin{bmatrix} -\delta_k I & 0 & I \\ 0 & -\delta_k I & G_k^T(\alpha) \\ I & G_k(\alpha) & -P^{-1}(\alpha) \end{bmatrix} \leq 0. \tag{5.24}$$

Pre-multiplying (5.24) by $diag\{I, I, P(\alpha)\}$ and post-multiplying by $diag\{I, I, P(\alpha)\}$. We obtain

$$\begin{bmatrix} -\delta_k I & 0 & P(\alpha) \\ 0 & -\delta_k I & G_k^T(\alpha)P(\alpha) \\ P(\alpha) & P(\alpha)G_k(\alpha) & -P(\alpha) \end{bmatrix} \leq 0. \tag{5.25}$$

The facts that $\sum_{i=1}^{N} \alpha_i = 1$, we have the following identities :

$$\sum_{i=1}^{N} \alpha_i A_i \sum_{i=1}^{N} \alpha_i B_i = \sum_{i=1}^{N} \alpha^2 A_i B_i + \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_i [A_i B_j + A_j B_i],$$

$$(N-1) \sum_{i=1}^{N} \alpha_i^2 \zeta - 2 \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \alpha_i \alpha_j \zeta = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} [\alpha_i - \alpha_j]^2 \zeta \geq 0.$$

Hence, the inequality (5.25) is equivalent to the following inequalities

$$\begin{bmatrix} -\delta_k I & 0 & P_j \\ 0 & -\delta_k I & G_{k,j}^T P_j \\ P_j & P_j G_{k,j} & -P_j \end{bmatrix} \leq -\zeta I, \quad j = 1, ..., N,$$

and

$$\begin{bmatrix} -\delta_k I & 0 & P_j \\ 0 & -\delta_k I & G_{k,j}^T P_l \\ P_j & P_j G_{k,l} & -P_j \end{bmatrix} + \begin{bmatrix} -\delta_k I & 0 & P_l \\ 0 & -\delta_k I & G_{k,l}^T P_j \\ P_l & P_j G_{k,j} & -P_l \end{bmatrix} \leq \frac{2\zeta I}{N-1},$$
$$j = 1, ..., N-1, l = j+1, ..., N.$$

The proof of lemma is complete. □

Theorem 5.2.3 *The system (5.13) is robustly exponentially stable, if there exist symmetric positive definite matrices P_i, $i = 1, 2, ..., N$, and positive real numbers $\alpha, \eta, \rho, \epsilon, \epsilon_1, \epsilon_2, \zeta$, $0 < b < a$, $\mu > 1$, $\delta_k > 0$ for all $k \in Z^+$ such that the following conditions hold.*

$$(i) \quad \begin{bmatrix} \psi_{i_k}(j,j) & P_j B_{i_k,j} & P_j & P_j & \epsilon^{-1} M_{i_k,j}^T & \epsilon P_j E_{i_k,j} \\ B_{i_k,j}^T P_j & -bP_j & 0 & 0 & \epsilon^{-1} N_{i_k,j}^T & 0 \\ P_j & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_j & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1} M_{i_k,j} & \epsilon^{-1} N_{i_k,j} & 0 & 0 & -I & J \\ \epsilon E_{i_k,j}^T P_j & 0 & 0 & 0 & J & -I \end{bmatrix} \leq -\zeta I, j = 1, ..., N.$$

$$(ii)\quad \begin{bmatrix} \psi_{i_k}(j,l) & P_jB_{i_k,l} & P_j & P_j & \epsilon^{-1}M_{i_k,j}^T & \epsilon P_jE_{i_k,l} \\ B_{i_k,j}^TP_l & -bP_j & 0 & 0 & \epsilon^{-1}N_{i_k,j}^T & 0 \\ P_j & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_j & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,j} & \epsilon^{-1}N_{i_k,j} & 0 & 0 & -I & J \\ \epsilon E_{i_k,j}^TP_l & 0 & 0 & 0 & J & -I \end{bmatrix}$$

$$+\begin{bmatrix} \psi_{i_k}(l,j) & P_lB_{i_k,j} & P_l & P_l & \epsilon^{-1}M_{i_k,l}^T & \epsilon P_lE_{i_k,j} \\ B_{i_k,l}^TP_j & -bP_l & 0 & 0 & \epsilon^{-1}N_{i_k,l}^T & 0 \\ P_l & 0 & -\epsilon_1 I & 0 & 0 & 0 \\ P_l & 0 & 0 & -\epsilon_2 I & 0 & 0 \\ \epsilon^{-1}M_{i_k,l} & \epsilon^{-1}N_{i_k,l} & 0 & 0 & -I & J \\ \epsilon E_{i_k,l}^TP_l & 0 & 0 & 0 & J & -I \end{bmatrix} \leq \frac{2\zeta I}{N-1},$$

$j = 1, ..., N-1, l = j+1, ..., N.$

$$(iii)\quad \begin{bmatrix} -\delta_k I & 0 & P_j \\ 0 & -\delta_k I & G_{k,j}^TP_j \\ P_j & P_jG_{k,j} & -P_j \end{bmatrix} \leq -\zeta I, \quad j = 1, ..., N.$$

$$(iv)\quad \begin{bmatrix} -\delta_k I & 0 & P_j \\ 0 & -\delta_k I & G_{k,j}^TP_l \\ P_j & P_jG_{k,l} & -P_j \end{bmatrix} + \begin{bmatrix} -\delta_k I & 0 & P_l \\ 0 & -\delta_k I & G_{k,l}^TP_j \\ P_l & P_jG_{k,j} & -P_l \end{bmatrix} \leq \frac{2\zeta I}{N-1},$$

$j = 1, ..., N-1, l = j+1, ..., N.$

$(v)\quad \mu h \leq \inf\limits_{k\in N}\{t_k - t_{k-1}\}.$

$(vi)\quad 1 \leq \max\{\bar{\delta}_k + \bar{\delta}_k e^{\lambda h}\} \leq M < e^{\lambda\tau}$, $\bar{\delta}_k = \frac{\delta_k}{\min\{\lambda_{\min}(P_i)\}}$, $i \in \{1, 2, ..., N\}$, *$k \in \mathbb{Z}^+$ and $\lambda > 0$ is the unique positive root of the equation $\lambda - a + be^{\lambda h} = 0$.*

Proof. We consider, for $t \in [t_{k-1}, t_k)$, the following Lyapunov function

$$V(x(t)) = x^T(t)P(\alpha)x(t).$$

It is easy to see that

$$\lambda_1\|x\|^2 \leq V(x(t)) \leq \lambda_2\|x\|^2,$$

where $\lambda_1 = \min(\lambda_{\min}(P_i))$ and $\lambda_2 = \max(\lambda_{\max}(P_i))$, $i = 1, 2, ..., N$. The Dini derivative of $V(x(t))$ along the trajectories of system (5.13) is given by

$$\begin{aligned} D^+V(x(t)) &= \dot{x}^T(t)P(\alpha)x(t) + x^T(t)P(\alpha)\dot{x}(t) \\ &= \Big[\hat{A}_{i_k}(\alpha)x(t) + \hat{B}_{i_k}(\alpha)x(t - h_{i_k}(t)) + f_{i_k}(t, x(t)) \\ &\quad + g_{i_k}(t, x(t - h_{i_k}(t)))\Big]^T P(\alpha)x(t) + x(t)P(\alpha)\Big[\hat{A}_{i_k}(\alpha)x(t) \\ &\quad + \hat{B}_{i_k}(\alpha)x(t - h_{i_k}(t)) + f_{i_k}(t, x(t)) + g_{i_k}(t, x(t - h_{i_k}(t)))\Big]. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} D^+V(x(t)) &\leq x^T(t)\hat{A}^T_{i_k}(\alpha)P(\alpha)x(t) + x^T(t - h_{i_k}(t))\hat{B}^T_{i_k}(\alpha)P(\alpha)x(t) \\ &\quad + x^T(t)P(\alpha)\hat{A}_{i_k}(\alpha)x(t) + x^T(t)P(\alpha)\hat{B}_{i_k}(\alpha)x(t - h_{i_k}(t)) \\ &\quad + g_{i_k}(t, x(t - h_{i_k}(t)))P(\alpha)x(t) + f^T_{i_k}(t, x(t))P(\alpha)x(t) \\ &\quad + x^T(t)P(\alpha)f_{i_k}(t, x(t)) + x^T(t)P(\alpha)g_{i_k}(t, x(t - h_{i_k}(t))) \\ &\quad + \epsilon_1\eta x^T(t)x(t) - \epsilon_1 f^T_{i_k}(t, x(t))f_{i_k}(t, x(t)) \\ &\quad + \epsilon_2\rho x^T(t)x(t) - \epsilon_2 g^T_{i_k}(t, x(t - h_{i_k}(t)))g_{i_k}(t, x(t - h_{i_k}(t))) \\ &\quad + ax^T(t)P(\alpha)x(t) - bx^T(t - h_{i_k}(t)))P(\alpha)x(t - h_{i_k}(t))) \\ &\quad - ax^T(t)P(\alpha)x(t) + bx^T(t - h_{i_k}(t)))P(\alpha)x(t - h_{i_k}(t))). \end{aligned}$$

Since,

$$\begin{aligned} &\hat{A}_{i_k}(\alpha) = \Big[A_{i_k}(\alpha) + \Delta A_{i_k}(t)\Big], \hat{B}^T_{i_k}(\alpha) = \Big[B_{i_k}(\alpha) + \Delta B_{i_k}(t)\Big], \\ &\epsilon_1\eta x^T(t)x(t) - \epsilon_1 f^T_{i_k}(t, x(t))f_{i_k}(t, x(t)) \geq 0, \\ &\epsilon_2\rho x^T(t)x(t) - \epsilon_2 g^T_{i_k}(t, x(t - h_{i_k}(t)))g_{i_k}(t, x(t - h_{i_k}(t))) \geq 0, \end{aligned}$$

we obtain

$$\begin{aligned} D^+V(x(t)) &\leq y^T(t)\Phi_{i_k}(\alpha)y(t) - aV(x(t)) + bV(x(t)) \\ &\leq y^T(t)\Phi_{i_k}(\alpha)y(t) - aV(x(t)) + b\bar{V}(x(t)), \end{aligned} \tag{5.26}$$

where

$$\bar{V}(x(t)) = \sup_{t-h\leq s\leq t}\{V(x(s))\},$$

and

$$\Phi_{i_k}(\alpha) = \begin{bmatrix} \hat{\psi}_{i_k}(\alpha) & P(\alpha)\hat{B}_{i_k}(\alpha) & P(\alpha) & P(\alpha) \\ \hat{B}_{i_k}^T(\alpha)P(\alpha) & -bP(\alpha) & 0 & 0 \\ P(\alpha) & 0 & -\epsilon_1 I & 0 \\ P(\alpha) & 0 & 0 & -\epsilon_2 I \end{bmatrix},$$

$$\hat{\psi}_{i_k}(\alpha) = \hat{A}_{i_k}^T(\alpha)P(\alpha) + P(\alpha)\hat{A}_{i_k}(\alpha) + (\epsilon_1\eta + \epsilon_2\rho)I + aP(\alpha),$$

and $y^T(t) = [x^T(t) \quad x^T(t-h_{i_k}(t)) \quad f_{i_k}^T(t,x(t)) \quad g_{i_k}^T(t,x(t-h_{i_k}(t)))]$. From (5.26), assumption $(i),(ii)$ and Lemma 5.2.1 then we conclude that

$$D^+V(x(t)) \leq -aV(x(t)) + b\bar{V}(x(t)). \tag{5.27}$$

From $a > b > 0$ and Lemma 2.3.17, there exist $\gamma \geq 1, \lambda > 0$ such that for all $t \in [t_{k-1}, t_k), k \in N$,

$$V(x(t)) \leq \gamma\bar{V}(x(t_{k-1}))e^{-\lambda(t-t_{k-1})}, \tag{5.28}$$

where

$$\bar{V}(x(t_{k-1})) = \sup_{t_{k-1}-h \leq s \leq t_{k-1}} \{V(x(s))\},$$

and $\lambda > 0$ satisfying $\lambda - a + be^{\lambda h} = 0$. We consider the case when $t = t_k$. In this case, we have

$$\begin{aligned} V(x(t_k)) &= x^T(t_k)P(\alpha)x(t_k) \\ &= [x(t_k^-) + G_k(\alpha)x(t_k - h(t_k))]^T P(\alpha)[x(t_k^-) + G(\alpha)x(t_k - h(t_k))] \\ &= x^T(t_k^-)P(\alpha)x(t_k^-) + 2x^T(t_k^-)P(\alpha)G_k(\alpha)x(t_k - h(t_k)) \\ &\quad + x^T(t_k - h(t_k))G_k^T(\alpha)P(\alpha)G_k(\alpha)x(t_k - h(t_k)) \\ &= \begin{bmatrix} x^T(t_k^-) \\ x(t_k - h(t_k)) \end{bmatrix}^T \begin{bmatrix} P(\alpha) & P(\alpha)G_k(\alpha) \\ G_k^T(\alpha)P(\alpha) & G_k^T(\alpha)P(\alpha)G_k(\alpha) \end{bmatrix} \begin{bmatrix} x^T(t_k^-) \\ x(t_k - h(t_k)) \end{bmatrix}. \end{aligned}$$

By assumption $(iii),(iv)$ and Lemma 5.2.2, we get

$$V(x(t_k)) \leq \bar{\delta}_k V(x(t_k^-)) + \bar{\delta}_k V(x(t_k - h(t_k))) \tag{5.29}$$

where $\bar{\delta_k} = \frac{\delta_k}{\min(\lambda_{\min}(P_i))}$, $i = 1, 2, 3, ..., N$. For $x(s) = \phi(s)$, with $s \in [t_0 - h, t_0]$, we will show that

$$V(x(t)) \leq \gamma^k M^{k-1} \max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t-t_0)}, t \in [t_{k-1}, t_k), k \in N. \quad (5.30)$$

We will prove inequality (5.30) by mathematical induction. Indeed, when $k = 1$, we have

$$V(x(t)) \leq \max(\lambda_{\max}(P_i))\|x(t)\|^2 = \max(\lambda_{\max}(P_i))\|\phi(t)\|^2, i = 1, 2, 3, ..., N.$$

Since, $\|\phi\|^2 = sup_{t_0-h \leq t \leq t_0}\|\phi(t)\|^2$, we have

$$\bar{V}(x(t_0)) \leq \max(\lambda_{\max}(P_i))\|\phi\|^2.$$

Thus, we conclude that

$$\begin{aligned} V(x(t)) &\leq \gamma \bar{V}(x(t_0)e^{-\lambda(t-t_0)} \leq \gamma \max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t-t_0)} \\ &\leq \gamma M^0 \max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t-t_0)}, t \in [t_0, t_1), i = 1, 2, 3, ..., N. \end{aligned}$$

Therefore, (5.30) holds for $k = 1$.

Next, we assume that (5.30) holds for $k \leq m, m \geq 1$. Then, we need to show that (5.30) holds when $k = m + 1$. By the above induction assumption, (5.27) and (5.28), we have

$$\begin{aligned} V(x(t_m)) &\leq \bar{\delta}_m V(x(t_m^-)) + \bar{\delta}_m V(x(t_m - h(t_m))), \\ &\leq \gamma^m M^{m-1} \bar{\delta}_m \max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t_m-t_0)} \\ &+ \gamma^m M^{m-1} \bar{\delta}_m \max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t_m-h(t_m)-t_0)} \\ &\leq \gamma^m M^{m-1} (\bar{\delta}_m + \bar{\delta}_m e^{\lambda h}) \max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t_m-t_0)} \\ &\leq \gamma^m M^m \max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t_m-t_0)}. \quad (5.31) \end{aligned}$$

Hence, it follows from conditions (vi), (5.27) and (5.30) that

$$\begin{aligned} V(x(t)) &\leq \gamma\bar{V}(x(t_m)e^{-\lambda(t-t_m)} = \gamma \max_{t_m-h\leq t\leq t_m}\{V(x(t))\}e^{-\lambda(t-t_m)} \\ &= \gamma \max\Big\{\sup_{t_{m-1}-h\leq s<t_m}\{V(x(t))\},\{V(x(t_m))\}\Big\}e^{-\lambda(t-t_m)} \\ &\leq \gamma \max\Big\{\gamma^m M^{m-1}\max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t_m-h-t_0)}, \\ &\quad \gamma^m M^m \max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t_m-t_0)}\Big\}e^{-\lambda(t-t_m)} \\ &= \gamma^{m+1}\max\Big\{M^{m-1}e^{\lambda h}, M^m\Big\}\max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t_m-t_0)}e^{-\lambda(t-t_m)} \\ &\leq \gamma^{m+1}M^m\max(\lambda_{\max}(P_i))\|\phi\|^2 e^{-\lambda(t-t_0)}. \end{aligned}$$

Therefore, (5.30) holds for all $k\in N$. Finally, we have to show that

$$\|x(t)\| \leq K\|\phi\|e^{-\alpha(t-t_0)}, t\geq t_0, \tag{5.32}$$

where $\alpha = \frac{1}{2}[\lambda - \frac{ln(\gamma M)}{\mu h}] > 0, K = \sqrt{\gamma\cdot\frac{\max(\lambda_{\max}(P_i))}{\min(\lambda_{\min}(P_i))}} \geq 1.$
From $\mu h \leq \inf_{k\in N}[t_k - t_{k-1}]$, we get that $k-1\leq \frac{t_{k-1}-t_0}{\mu h}$, which implies

$$(\gamma M)^{k-1} \leq e^{\frac{(t_{k-1}-t_0)ln(\gamma M)}{\mu h}} \leq e^{\frac{(t-t_0)ln(\gamma M)}{\mu h}},$$

for $t\in[t_{k-1},t_k)$. We obtain that

$$\begin{aligned} \|x(t)\|^2 &\leq \frac{V(x(t))}{\min(\lambda_{\min}(P_i))} \leq \gamma\frac{\max(\lambda_{\max}(P_i))}{\min(\lambda_{\min}(P_i))}\|\phi\|^2(\gamma M)^{k-1}e^{-\lambda(t-t_0)} \\ &\leq \gamma\frac{\max(\lambda_{\max}(P_i))}{\min(\lambda_{\min}(P_i))}\|\phi\|^2 e^{(-\lambda+\frac{ln\gamma M}{\mu h})(t-t_0)}, i=1,2,3,...,N. \end{aligned}$$

Therefore, we conclude that (5.31) holds by above inequality. This means that the system (5.13) is robustly exponentially stable. The proof of theorem is complete.□

Example 5.2.3 We consider the following uncertain impulsive switched LPD system with time-varying delays (5.13) where $u(t)=0$ under a given switching law. That is, the switching status alternates as $i_1\to i_2\to i_1\to i_2\to\cdots$. We consider robust performance of the system (5.13) by using Theorem 5.2.3. The system (5.13) is specified as follows:

$$A_1(\alpha) = \alpha_1\begin{bmatrix}-8.59 & 0.4\\ 0.2 & -8.045\end{bmatrix} + \alpha_2\begin{bmatrix}-8.35 & 0.2\\ 0.2 & -8.23\end{bmatrix},$$

$$A_2(\alpha) = \alpha_1 \begin{bmatrix} -8.65 & 0.1 \\ 0.1 & -8.48 \end{bmatrix} + \alpha_2 \begin{bmatrix} -8.85 & 0.2 \\ 0.4 & -8.34 \end{bmatrix},$$

$$B_1(\alpha) = \alpha_1 \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.2 & 0 \\ 0 & 0.4 \end{bmatrix}, B_2(\alpha) = \alpha_1 \begin{bmatrix} 0.3 & 0 \\ 0 & 0.5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$E_1(\alpha) = \alpha_1 \begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}, E_2(\alpha) = \alpha_1 \begin{bmatrix} 0.4 & 0 \\ 0 & 0.4 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.4 & 0 \\ 0 & 0.3 \end{bmatrix},$$

$$M_1(\alpha) = \alpha_1 \begin{bmatrix} 0.6 & 0 \\ 0 & 0.5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix},$$

$$M_2(\alpha) = \alpha_1 \begin{bmatrix} 0.4 & 0 \\ 0 & 0.6 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.6 & 0 \\ 0 & 0.4 \end{bmatrix},$$

$$N_1(\alpha) = \alpha_1 \begin{bmatrix} 0.9 & 0 \\ 0 & 1.1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, N_2(\alpha) = \alpha_1 \begin{bmatrix} 1.0 & 0 \\ 0 & 1.1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0.9 & 0 \\ 0 & 1.2 \end{bmatrix},$$

$$G_k(\alpha) = \alpha_1 \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad k \in N,$$

and nonlinear functions $f_i(.), g_i(.)$, $i = 1, 2$ are of the form

$$f_1(t, x(t)) = \begin{bmatrix} 1.1529\cos(t)x_1(t) \\ 1.1529\cos(t)x_2(t) \end{bmatrix}, f_2(t, x(t)) = \begin{bmatrix} 1.1529\sin(t)x_1(t) \\ 1.1529\cos(t)x_2(t) \end{bmatrix},$$

and

$$g_1(t, x(t - h_1(t))) = \begin{bmatrix} 1.1529\sin(t)x_1(t - h_1(t)) \\ 1.1529\sin(t)x_2(t - h_1(t)) \end{bmatrix},$$

$$g_2(t, x(t - h_2(t))) = \begin{bmatrix} 1.1529\cos(t)x_1(t - h_2(t)) \\ 1.1529\sin(t)x_2(t - h_2(t)) \end{bmatrix}.$$

We choose $F_1(t) = F_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, $\epsilon = 1, \epsilon_1 = \epsilon_2 = 6.5345$, $\alpha_1 = \alpha_2 = \frac{1}{2}$, $h_1(t)) = 0.1 + \sin^2(t)$ and $h_2(t)) = 0.1 + \cos^2(t)$. By using LMI

Toolbox in MATLAB, we apply the condition $(i), ..., (vi)$ in Theorem 3.3 to this example shows that the solutions of LMI are as follows:

$$P(\alpha) = \alpha_1 \begin{bmatrix} 5.0885 & 0.0848 \\ 0.0848 & 5.3186 \end{bmatrix} + \alpha_2 \begin{bmatrix} 4.5709 & 0.1627 \\ 0.1627 & 4.6779 \end{bmatrix},$$

$\delta_k = 160.6620$, $a = 10, b = 2$,$\zeta = 0.0711$ and $\eta = \rho = 1.1529$. The following parameters and matrices are required to be satisfied $(i), ..., (iv)$. By $(v), (vi)$, we can find the parameters constants $\mu = 3.7, \lambda \approx 1.33306$. Therefore, this example of the system (5.13) is robustly exponentially stable. □

Numerical Simulations

Numerical experiments are carried out to investigate dynamical system by using dde45lin in Matlab. In Fig. 5.3, the parameters of the system are specified as in Example 5.2.3 and the initial condition is $x(t) = [9 \quad -7]^T, t \in [-1.1, 0]$,

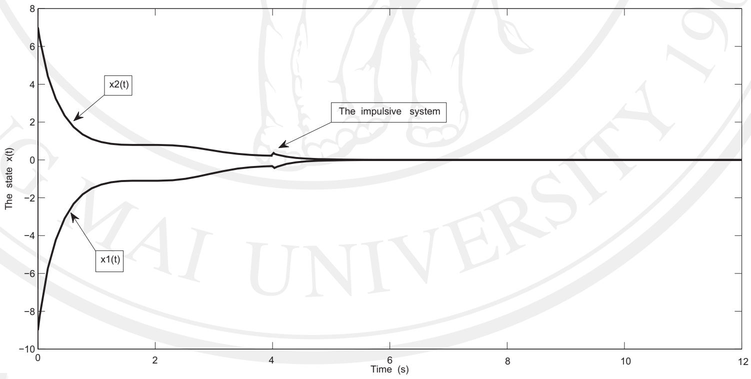

Figure 5.3: The simulation of solutions the uncertain impulsive switched LPD system with time-varying delays and nonlinear perturbations (5.13)