Chapter 1

Introduction

Stock Exchange started trading officially for about 400 years ago. Many investors are trying to predict or forecast the price of the shares in many ways, such as a fundamental analysis, a technical analysis, and so on. Mathematicians have created discrete mathematical models of stock price movement such as Binomial model, continuous mathematical models of stock price, such as geometric Brownian motion:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)$$

where $\{W(t)\}_{t\geq 0}$ is a Brownian motion under actual probability, $\mu(t)$ is the drift, $\sigma(t)$ is the volatility.

Investing in stocks are the risk which are different from savings in banks, so investors are looking for tools to measure a risk of investment. There are many popular and widely used risk measures, such as a Modern Portfolio Theory (MPT), in 1952, Harry Markowitz used mean return and variance of stock to compare the risk of investment in stocks. If you invested in portfolio(many kind of assets), then you can minimize risk of investment by minimizing variance. A popular risk measure is Value-at-Risk(VaR), which is the maximum possible loss at level α, defined by

$$VaR_\alpha(X) = F^{-1}(\alpha) = \inf\{x \in \mathbb{R} : F(x) \geq \alpha\}.$$

(From the publication of JP Morgans "Risk Metrics Technical Document" in 1995.) After that, the problem of modeling of the risk measures had been developed very fast. In 1999 Artnerz et al proposed axioms of coherent risk measures (details in the next chapter). The Tail Value-at-Risk(TVaR), is a coherent risk measure, but Value-at-Risk is not, where

$$TVaR_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 F^{-1}(t)dt.$$

Which is the average of $VaR_\alpha(.)$ over the α-upper tail.

In 1996, Wang introduced Wang's distortion function(or Wang transform) to define a price of risk in insurance to charge customers. This distortion function is:

$$h_\lambda(x) = \Phi(\Phi^{-1} + \lambda)$$

where $\lambda \in \mathbb{R}$ and Φ is the distribution function of the standard normal.

Wang's distortion can be used to finding the risk measures and pricing vanilla options, and it is a special case of distortion risk measure in general form.

$$\rho_h(X) = \int_0^\infty h(1 - F_X(x))dx + \int_{-\infty}^0 [h(1 - F_X(x)) - 1]dx.$$

By the nature of risk, the risk should increase as time increases.

Using stock price model under geometric Brownian motion, Treussard(2006) found that the VaR and TVaR are not consistent under actual probability. But, they are consistent under risk neutral probability.

Under actual probability,

$$VaR_\alpha(T) = S(0)\Big(e^{rT} - \exp\Big\{(\mu - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)\Big\}\Big)$$

then the present value of $VaR_\alpha(.)$ under actual probabilities is

$$\begin{aligned} PV(VaR_\alpha(T)) &= \exp\{-rT\}VaR_\alpha(T) \\ &= S(0)\Big[1 - \exp\Big\{(\mu - r - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)\Big\}\Big]. \end{aligned}$$

If $\mu - r - \frac{1}{2}\sigma^2 > 0$, so that $VaR_\alpha(T)$ will becoming negative for large value of T. Also,

$$TVaR_\alpha(T) = S(0)\Big(e^{rT} - \frac{1}{\alpha}\exp\{\mu T\}\Phi(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha))\Big)$$

then the present value of $TVaR_\alpha(.)$ under actual probabilities is

$$\begin{aligned} PV(TVaR_\alpha(T)) &= \exp\{-rT\}TVaR_\alpha(T) \\ &= S(0)\Big[1 - \frac{1}{\alpha}\exp\{(\mu - r)T\}\Phi(-\sigma\sqrt{T} - \Phi^{-1}(1-\alpha))\Big]. \end{aligned}$$

If $\mu - r > 0$, then $TVaR_\alpha(T)$ becomes negative for large value of T.

Under risk neutral probability, $VaR_\alpha(.)$ is

$$VaR_\alpha(T) = S(0)\Big(e^{rT} - \exp\Big\{(r - \frac{1}{2}\sigma^2)T - \sigma\sqrt{T}\Phi^{-1}(1-\alpha)\Big\}\Big).$$

So that, the present value of $VaR_\alpha(.)$ is

$$\begin{aligned} PV(VaR_\alpha(T)) &= \exp\{-rT\}VaR_\alpha(T) \\ &= S(0)\Big[1-\exp\Big\{(-\frac{1}{2}\sigma^2)T-\sigma\sqrt{T}\Phi^{-1}(1-\alpha)\Big\}\Big]. \end{aligned}$$

Then, $VaR_\alpha(T)$ is always increasing as T is increasing.
Under risk neutral probability, $TVaR_\alpha(.)$ is

$$TVaR_\alpha(T) = S(0)\Big(e^{rT}-\frac{1}{\alpha}\exp\{rT\}\Phi(-\sigma\sqrt{T}-\Phi^{-1}(1-\alpha))\Big).$$

So that, the present value of $TVaR_\alpha(.)$ under risk neutral probabilities is

$$\begin{aligned} PV(TVaR_\alpha(T)) &= \exp\{-rT\}TVaR_\alpha(T) \\ &= S(0)\Big[1-\frac{1}{\alpha}\Phi(-\sigma\sqrt{T}-\Phi^{-1}(1-\alpha))\Big]. \end{aligned}$$

Then, $TVaR_\alpha(T)$ is always increasing as T is increasing.

In 2010, Nguyen et al. [2] investigated also Treussard's work, for the case of one stock, but in a more realistic model than Black-Scholes model, namely Levy process models. Let $S(t)$ be a stock price position at a time t with the initial value $S(0)$, an investor either puts $S(0)$ into the bank with a risk-free rate r or put it on stock market. By time t, the investor either has $S(0)e^{rt}$ or $S(t)$. If the investor decides to invest money to the stock market, then

if $S(t) \geq S(0)e^{rt}$, he gains money,

if $S(t) < S(0)e^{rt}$, he loses money.

Considered the risk of random variable

$$X(t) = S(0)e^{rt} - S(t).$$

They used distortion risk measures to compute $VaR_\alpha(X(t))$, $TVaR_\alpha(X(t))$, Wang's distortion risk measure $\rho_w(X(t))$ and strictly concave distortion $\rho_h(X(t))$ on Itô process. They also showed results in the shifted poison process and random walks.

In real situations, investors do not invest in one stock but invest in many kinds of assets, for example golds, bonds and commodities, according to diversification principle. In this thesis, we are interested in the investment of many stocks. Let the portfolio $V(t)$ be denoted as

$$V(t) = n_1 S_1(t) + n_2 S_2(t) + \ldots + n_m S_m(t),$$

$S_i(t)$ = a stock price position at a time t of stock i.

n_i = number of stock invested in $S_i(t)$, $i = 1, 2, \ldots, m$.

Since $V(t)$ is a financial portfolio position at a time t, then with the initial value $V(0)$, an investor either puts $V(0)$ into the bank with a risk-free rate r or put it on stock market. By time t, the investor either has $V(0)e^{rt}$ or $V(t)$. If the investor decides to invest money to the finance market, then

case 1 if $V(t) \geq V(0)e^{rt}$, he gains money, or

case 2 if $V(t) < V(0)e^{rt}$, he loses money.

We will consider the risk of random variable

$$Y(t) = V(0)e^{rt} - V(t)$$

where we have a portfolio, $V(t) = \sum_{i=1}^{m} n_i S_i(t)$.

$S_i(t)$ = a stock price position at a time t of stock i.

n_i = number of stock invested in $S_i(t)$, $i = 1, 2, \ldots, m$.

Assume that $S_i(t),\ i = 1, 2, \ldots, m$, are comonotonic.

Let the risk of random variable of stock $S_i(t)$ be

$$Y_i(t) = S_i(0)e^{rt} - S_i(t).$$

So that, the risk measure of portfolio $V(t)$ is

$$\rho_h(Y(t)) = \sum_{i=1}^{m} n_i \rho_h(Y_i(t)).$$

In this research, we use a multiple asset model to find a price of stock $S_i(t)$. Defined by

$$dB(t) = rB(t)dt$$

$$dS_i(t) = S_i(t)\{\mu_i dt + \sum_{j=1}^{m} \sigma_{ij} dW_j(t)\}$$

where $B(t)$ is a cash bond with interest rate r, $\{W_j(t)\}_{t\geq 0}$, $j = 1,\ldots,m$, are independent Brownian motions, $\mu_i(t)$ are the drift, $\sigma_{ij}(t)$ are volatility. We assume that the matrix $\sigma = (\sigma_{ij})$ is invertible. By Itô formula with initial value $S_i(0)$ we have

$$S_i(t) = S_i(0)\exp\Big\{\int_0^t \big(\mu_i(s) - \frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2(s)\big)ds + \int_0^t \sum_{j=1}^{m}\sigma_{ij}(s)dW_j(s)\Big\}$$

Let μ_i, σ_{ij} be known constants, and we assuming that all of stocks are comonotonic.

We will consider $VaR_\alpha(Y(t))$, $TVaR_\alpha(Y(t))$, Wang's distortion risk measure $\rho_w(Y(t))$ and general strictly concave distortion $\rho_h(Y(t))$ in Black-Scholes model under actual and risk neutral probabilities.