Chapter 3

Main results

From a multiple asset model

$$dS_i(t) = S_i(t)\Big(\sum_{j=1}^{m}\sigma_{ij}(t)dW_j(t) + \mu_i(t)dt\Big)$$

where $\{W_j(t)\}_{t\geq 0}$, $j = 1, \ldots, m$, are independent Brownian motions, $\mu_i(t)$ are the drift, $\sigma_{ij}(t)$ are volatility. We assume that the matrix $\sigma = (\sigma_{ij})$ is invertible. By Itô formula with initial value $S_i(0)$ we have

$$S_i(t) = S_i(0)\exp\Big\{\int_0^t \big(\mu_i(s) - \frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2(s)\big)ds + \int_0^t\sum_{j=1}^{m}\sigma_{ij}(s)dW_j(s)\Big\}.$$

Let μ_i, σ_{ij} be constants and assume known.

We have

$$S_i(t) = S_i(0)\exp\Big\{\big(r - \frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2\big)t + \sum_{j=1}^{m}\sigma_{ij}W_j(t)\Big\}$$

$i = 1, 2, \ldots, m$ and $W_j(t) \backsim N(0,t) = \sqrt{t}z_j$. We have

$$S_i(t) = S_i(0)\exp\Big\{\big(\mu_i - \frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2\big)t + \sum_{j=1}^{m}\sigma_{ij}\sqrt{t}z_j\Big\}.$$

Then, we find cumulative distribution function(CDF) to compute distortion risk measures.

$$\begin{aligned} P(S_i(t)\leq x) &= P(S_i(0)\exp\Big\{\big(\mu_i - \frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2\big)t + \sum_{j=1}^{m}\sigma_{ij}\sqrt{t}z_j\Big\}\leq x) \\ &= P\Big((\mu_i - \frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2)t + \sum_{j=1}^{m}\sigma_{ij}\sqrt{t}z_j \leq ln(\frac{x}{S_i(0)})\Big) \\ &= P(\sqrt{t}\sum_{j=1}^{m}\sigma_{ij}z_j \leq ln(\frac{x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sum_{j=1}^{m}\sigma_{ij}^2)t). \end{aligned}$$

Since

$$\sum_{j=1}^{m} \sigma_{ij} z_j \backsim N(0, \sum_{j=1}^{m} \sigma_{ij}^2) \backsim \sqrt{\sum_{j=1}^{m} \sigma_{ij}^2} z$$

then, from here let denote $\sigma_i^2 = \sum_{j=1}^{m} \sigma_{ij}^2$ and $\sigma_i = \sqrt{\sum_{j=1}^{m} \sigma_{ij}^2}$.

Therefore the actual probability distribution $F_t(x)$ of $S_i(t)$, a geometric Brownian motion, is

$$\begin{aligned} F_t(x) = P(S_i(t) \leq x) &= P\left(z \leq \frac{ln(\frac{x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i \sqrt{t}}\right) \\ &= \Phi\left(\frac{ln(\frac{x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i \sqrt{t}}\right) \end{aligned} \tag{3.1}$$

and by using choquet integral the risk measure of $Y_i(t)$ is of the form

$$\rho_h(Y_i(t)) = \int_0^{\infty} h(P(Y_i(t) > x))dx + \int_{-\infty}^{0} [h(P(Y_i(t) > x)) - 1]dx$$

where $h(.)$ ia an arbitrary distortion function.

Under risk neutral probabilities when μ_i are replaced by r, then the multiple asset model is

$$dS_i(t) = S_i(t)\Big(\sum_{j=1}^{m} \sigma_{ij}(t) d\widetilde{W}_j(t) + rdt\Big).$$

Where $\{\widetilde{W}_j(t)\}_{t \geq 0},\ j = 1, 2, \ldots, m$, are independent Brownian motion, from Girsanov's Theorem, if a process $\theta(t), t \in [0, T]$ satisfies Girsanov's condition then the risk neutral probability measure Q exists and under Q,

$$\widetilde{W}_j(t) = W_j(t) + \int_0^{\infty} \theta(s)ds$$

therefore the risk neutral probability distribution $F_t(x)$ of $S_i(t)$ in Black-Scholes model, is

$$\begin{aligned} F_t(x) = Q(S_i(t) \leq x) &= Q\left(z \leq \frac{ln(\frac{x}{S_i(0)}) - (r - \frac{1}{2}\sigma_i^2)t}{\sigma_i \sqrt{t}}\right) \\ &= \Phi\left(\frac{ln(\frac{x}{S_i(0)}) - (r - \frac{1}{2}\sigma_i^2)t}{\sigma_i \sqrt{t}}\right). \end{aligned} \tag{3.2}$$

3.1 General case for risk in Black-Scholes model

Under actual probabilities, we will find a distortion risk measure. Consider a risk measure of portfolio $V(t)$.

By Choquet integral, we have

$$\begin{aligned}\rho(Y(t)) &= \sum_{i=1}^{m} n_i \rho(Y_i(t)) \\ &= \sum_{i=1}^{m} n_i \Big[\int_0^{\infty} h(P(Y_i(t) > x))dx + \int_{-\infty}^{0} [h(P(Y_i(t) > x)) - 1]dx\Big]\end{aligned}$$

where

$$h : [0,1] \to [0,1]$$

$$h(0) = 0, h(1) = 1$$

and h is increasing and strictly concave.

Recall that

$$\begin{aligned}P(Y_i(t) > x) &= P(S_i(0)e^{rt} - S_i(t) > x) \\ &= P(S_i(t) < S_i(0)e^{rt} - x) \\ &= \begin{cases} \Phi\left(\dfrac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right) & \text{if } x < S_i(0)e^{rt} \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore

$$\begin{aligned}\rho(Y_i(t)) = &\int_0^{S_i(0)e^{rt}} h\left(\Phi\left[\frac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right]\right) dx \\ &+ \int_{-\infty}^{0} \left[h\left(\Phi\left[\frac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right]\right) - 1\right] dx.\end{aligned}$$

Now, if we let

$$y_i = \frac{S_i(0)e^{rt} - x}{S_i(0)}$$

we have

$$\begin{aligned}\rho(Y_i(t)) = &\int_0^{S_i(0)e^{rt}} h\left(\Phi\left[\frac{ln(y_i) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right]\right) dx \\ &+ \int_{-\infty}^{0} \left[h\left(\Phi\left[\frac{ln(y_i) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right]\right) - 1\right] dx.\end{aligned}$$

Let

$$z_i = \frac{ln(y_i) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}$$

then

$$y_i = \exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}(z_i)\Big\}.$$

Let

$$C_i = \frac{rt - \Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t}{\sigma_i\sqrt{t}}$$

we obtain

$$\begin{aligned}\rho(Y_i(t)) = S_i(0)\int_{-\infty}^{C_i} h(\Phi[z_i])\sigma_i\sqrt{t}\,\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}dz_i \\ + S_i(0)\int_{C_i}^{\infty}[h(\Phi[z_i]) - 1]\sigma_i\sqrt{t}\,\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}dz_i.\end{aligned}$$

Note that since h is increasing. h has countable discontinuity points. Moreover, for cohorent risk measures, the distortion h is strictly concave which implies that h' is continuous almost everywhere.

By integration by parts, if we let

$$u_i = h(\Phi[z_i])\ , du_i = h'(\Phi[z_i])d\Phi[z_i] \text{ and}$$
$$dv_i = \sigma_i\sqrt{t}\,\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}dz_i\ , v_i = \exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}.$$

The first part of $\rho(Y_i(t))$ is

$$\begin{aligned}\rho^1(Y_i(t)) &= S_i(0)\Big(u_i v_i\Big|_{-\infty}^{C_i} - \int_{-\infty}^{C_i} v_i du_i\Big) \\ &= S_i(0)\Big(h(\Phi[z_i])\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}\Big|_{-\infty}^{C_i} \\ &\quad - \int_{-\infty}^{C_i}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}h'(\Phi[z_i])d\Phi[z_i]\Big) \\ &= S_i(0)\Big(h(\Phi[C_i])\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}C_i\Big\} \\ &\quad - \int_{-\infty}^{C_i}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}h'(\Phi[z_i])d\Phi[z_i]\Big).\end{aligned}$$

And by integration by parts, if we let

$$u_i = h(\Phi[z_i]) - 1\ , du_i = h'(\Phi[z_i])d\Phi[z_i] \text{ and}$$
$$dv_i = \sigma_i\sqrt{t}\,\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}dz_i\ , v_i = \exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}.$$

The second part of $\rho(Y_i(t))$ is

$$\begin{aligned}\rho^2(Y_i(t)) &= S_i(0)\Big(u_iv_i\Big|_{C_i}^{\infty} - \int_{C_i}^{\infty} v_i du_i\Big)\\ &= S_i(0)\Big([h(\Phi[z_i]) - 1]\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}\Big|_{-\infty}^{C_i}\\ &\quad - \int_{C_i}^{\infty}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}h'(\Phi[z_i])d\Phi[z_i]\Big)\\ &= S_i(0)\Big(-[h(\Phi[C_i]) - 1]\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}C_i\Big\}\\ &\quad - \int_{C_i}^{\infty}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}h'(\Phi[z_i])d\Phi[z_i]\Big).\end{aligned}$$

Then

$$\begin{aligned}\rho(Y_i(t)) &= \rho^1(Y_i(t)) + \rho^2(Y_i(t))\\ &= S_i(0)\Big(h(\Phi[C_i])\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}C_i\Big\}\\ &\quad - \int_{-\infty}^{C_i}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}h'(\Phi[z_i])d\Phi[z_i]\Big)\\ &\quad + S_i(0)\Big(-[h(\Phi[C_i]) - 1]\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}C_i\Big\}\\ &\quad - \int_{C_i}^{\infty}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}h'(\Phi[z_i])d\Phi[z_i]\Big)\end{aligned}$$

$$\begin{aligned}&= S_i(0)\Big(\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}C_i\Big\}\\ &\quad - \int_{-\infty}^{\infty}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}h'(\Phi[z_i])d\Phi[z_i]\Big)\\ &= S_i(0)\Big(\exp\{rt\} - \int_{-\infty}^{\infty}h'(\Phi[z_i])\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}d\Phi[z_i]\Big).\end{aligned}$$

Since

$$d\Phi[z_i] = \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}(z_i)^2\Big\}dz_i$$

then

$$\begin{aligned}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}d\Phi[z_i] &= \exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}(z_i)^2\Big\}dz_i\\ &= \frac{1}{\sqrt{2\pi}}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i - \frac{1}{2}(z_i)^2\Big\}dz_i\\ &= \frac{1}{\sqrt{2\pi}}\exp\{\mu_it\}\exp\Big\{-\frac{1}{2}(z_i)^2 + \sigma_i\sqrt{t}z_i - \frac{1}{2}\sigma_i^2t\Big\}dz_i\\ &= \frac{1}{\sqrt{2\pi}}\exp\{\mu_it\}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}z_i\Big)^2\Big\}dz_i.\end{aligned}$$

So that

$$\rho(Y_i(t)) = S_i(0)\Big(\exp\{rt\} - \int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\{\mu_i t\}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}z_i\Big)^2\Big\}dz_i\Big)$$
$$= S_i(0)e^{rt}\left[1 - e^{(\mu_i - r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}\Big)^2\Big\}dz_i\right].$$

The present value of $\rho(Y_i(t))$ are

$$PV(\rho(Y_i(t))) = e^{-rt}\rho(Y_i(t))$$
$$= S_i(0)\left[1 - e^{(\mu_i - r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}\Big)^2\Big\}dz_i\right].$$

Observe that if

$$\mu_i - r \leq 0$$

then $\rho(Y_i(t))$ or $PV(\rho(Y_i(t)))$ are always increasing as t goes to ∞.

So we only consider the case when

$$\mu_i - r > 0 \ \ \forall i, \ i = 1, 2, \ldots, m.$$

We have risk measure of portfolio is

$$\rho(Y(t)) = \sum_{i=1}^{m} n_i \rho(Y_i(t))$$
$$= \sum_{i=1}^{m} n_i S_i(0)e^{rt}\Big[1 - e^{(\mu_i - r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}\Big)^2\Big\}dz_i\Big]$$

if $\mu_i - r > 0$ and $\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}\Big)^2\Big\}dz_i$ are never equal to 0 then $\rho(Y(t))$ is eventually negative at long horizons. The following Theorems illustrate the situation under some conditions.

Theorem 3.1.1. *For each $i = 1, 2, \ldots, m$ if $h'(\Phi(z_i))$ is bounded below by some constants $C \neq 0$ on $[0, 1]$, $i = 1, 2, \ldots, m$ then $\rho(Y(t))$ is negative as t goes to ∞.*

Proof. We have

$$\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}\Big)^2\Big\}dz_i \geq C\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}\Big)^2\Big\}dz_i$$
$$\geq C$$
$$> 0.$$

Therefore

$$e^{(\mu_i - r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\ ,\ i=1,2,\ldots,m$$

tend to ∞ as $t \to \infty$. So, $\rho(Y_i(t))$ are negative for $t > \dfrac{\ln C_i}{\mu_i - r}$, $i = 1, 2, \ldots, m$. Hence $\rho(Y(t))$ is negative as t goes to ∞. ■

It implies that under "actual probability", the risk measure of portfolio in Theorem 3.1.1 is not consistent with time.

Theorem 3.1.2. *If* $\mu_i - r - \frac{1}{2}\sigma_i^2 > 0$, $i = 1, 2, \ldots, m$ *then* $\rho(Y(t))$ *is negative as* t *goes to* ∞.

Proof. Note that since h is increasing then $h'(\Phi[z_i])$ is nonnegative. Moreover, since h' is continuous and h' cannot be 0 (due to the definition of $\rho(Y_i(t))$), there exists a closed subset A_i of $[0, 1]$ with non-zero measure and a constant a_0^i such that

$$h'(u) \geq a_0^i \text{ for all } u \in A_i$$

then

$$\begin{aligned}
&\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\\
&=\int_{\Phi^{-1}(A_i)} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\\
&\quad+\int_{R\setminus\Phi^{-1}(A_i)} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\\
&\geq\int_{\Phi^{-1}(A_i)} a_0^i\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\\
&\quad+\int_{R\setminus\Phi^{-1}(A_i)} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i.
\end{aligned}$$

Since for each $i = 1, 2, \ldots, m$, A_i is close subset of $[0, 1]$, we can assume that A_i is an interval in $[0, 1]$. Therefore for each A_i, $\Phi^{-1}(A_i)$ is also an interval and assume that

$$\Phi^{-1}(A_i) = [a_i, b_i] \quad , \ i = 1, 2, \ldots, m$$

where $a_i \neq b_i$, Hence,

$$\begin{aligned}
&\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\\
\geq&\int_{a_i}^{b_i} a_0^i\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\\
&+\int_{R\setminus A_i} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\\
\geq& a_0^i\Big[\Phi\big(b_i-\sigma_i\sqrt{t}\big)-\Phi\big(a_i-\sigma_i\sqrt{t}\big)\Big]\\
&+\int_{R\setminus A_i} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i.
\end{aligned}$$

By Mean Value Theorem, we have

$$[\Phi\big(b_i-\sigma_i\sqrt{t}\big)-\Phi\big(a_i-\sigma_i\sqrt{t}\big)\Big] \geq (b_i-a_i)\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}(b_i-\sigma_i\sqrt{t})^2\Big\}.$$

Therefore

$$\begin{aligned}
&e^{(\mu_i-r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i\\
\geq& e^{(\mu_i-r)t}a_0^i(b_i-a_i)\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(b_i-\sigma_i\sqrt{t}\Big)^2\Big\}\\
\geq& a_0^i(b_i-a_i)\frac{1}{\sqrt{2\pi}}\exp\Big\{\Big(\mu_i-r-\frac{1}{2}\sigma_i^2\Big)t+b_i\sigma_i\sqrt{t}-\frac{1}{2}b_i^2\Big\}\ ,\ i=1,2,\ldots,m
\end{aligned}$$

implies

$$e^{(\mu_i-r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i$$

are getting larger than 1 for t large and therefore $\rho(Y_i(t))$ are becoming negative for t large $\forall i,\ i=1,2,\ldots,m$.

Hence $\rho(Y(t))$ is negative as t goes to ∞. ■

Therefore under "actual probability", the risk measure of portfolio in Theorem 3.1.2 is not monotone increasing with time.

Theorem 3.1.3. *If $h'(1)>0$ then $\rho(Y(t))$ is negative as t goes to ∞.*

Proof. Since h is the concave distortion function, so h' is deceasing.

So, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{ -\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i \\
\geq &\int_{-\infty}^{\sigma_i\sqrt{t}} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{ -\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i \\
\geq &\frac{1}{2}h'\big(\Phi[\sigma_i\sqrt{t}]\big) \\
\geq &\frac{1}{2}h'(1) > 0 \quad \forall i,\ i=1,2,\dots,m.
\end{aligned}$$

Therefore

$$e^{(\mu_i-r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{ -\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i$$

are getting larger than 1 for t large and hence $\rho(Y_i(t))$ are becoming negative for $t > \dfrac{\ln 2 - \ln h'(1)}{\mu_i - r}\ \forall i,\ i=1,2,\dots,m.$
Hence $\rho(Y(t))$ is also negative as t goes to ∞. ■

Therefore under "actual probability", the risk measure of portfolio in Theorem 3.1.3 is not consistent with time.

Theorem 3.1.4. *If*

$$\lim_{t\to\infty}\big(e^{(\mu_i-r)t}h'\Big(\Phi\Big(\sigma_i\sqrt{t}\Big)\Big)\big) = +\infty\ \forall i,\ i=1,2,\dots,m$$

then $\rho(Y(t))$ is negative as t goes to ∞.

Proof. Indeed, similar to the above theorem, we have

$$\begin{aligned}
&e^{(\mu_i-r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{ -\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i \\
&\qquad\geq e^{(\mu_i-r)t}h'\Big(\Phi\Big(\sigma_i\sqrt{t}\Big)\Big).
\end{aligned}$$

Therefore,

$$e^{(\mu_i-r)t}\int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{ -\frac{1}{2}\Big(z_i-\sigma_i\sqrt{t}\Big)^2\Big\}dz_i$$

are getting larger than 1 for t large and therefore $\rho(Y_i(t))$ are becoming negative for t large $\forall i,\ i=1,2,\dots,m$

Hence $\rho(Y(t))$ is negative as t goes to ∞. ■

Therefore under "actual probability", the risk measure of portfolio in Theorem 3.1.4 is not consistent with time.

Under risk neutral probabilities, multiple asset model is

$$dS_i(t) = S_i(t)\Big(\sum_{j=1}^{m}\sigma_{ij}(t)d\widetilde{W}_j(t) + rdt\Big)$$

Then we use (3.2) to find a distortion risk measure. When $\mu_i = r\ \forall i,\ i = 1, 2, \ldots, m$. We have a risk measure of portfolio is

$$\begin{aligned}\rho(Y(t)) &= \sum_{i=1}^{m} n_i\rho(Y_i(t))\\ &= \sum_{i=1}^{m} n_iS_i(0)e^{rt}\Big[1 - \int_{-\infty}^{\infty} h'(\Phi[z_i])\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\Big(z_i - \sigma_i\sqrt{t}\Big)^2\Big\}dz_i\Big]\end{aligned}$$

which result in that $\rho(Y(t))$ is increasing with respect to t (since $h'(\phi[z_i]) \geq 0$ and $\exp\{-\frac{1}{2}(z_i - \sigma_i\sqrt{t})^2\}$ is decreasing with respect to t). In conclusion, the risk measure of portfolio under "risk neutral probability" is consistent with time.

3.2 Value-at-Risk in Black-Scholes model

For the case of $VaR_\alpha(.)$, the distortion function is

$$h_\alpha(u) = 1_{(\alpha,1]}(u).$$

Finding $VaR_\alpha(.)$ by using Choquet integral, we have

$$\begin{aligned}VaR_\alpha(Y(t)) &= \sum_{i=1}^{m} n_iVaR_\alpha(Y_i(t))\\ &= \sum_{i=1}^{m} n_i\Big[\int_0^{\infty} h_\alpha(P(Y_i(t) > x))dx + \int_{-\infty}^{0}[h_\alpha(P(Y_i(t) > x)) - 1]dx\Big].\end{aligned}$$

Since

$$\begin{aligned}P(Y_i(t) > x) &= P(S_i(0)e^{rt} - S_i(t) > x)\\ &= P(S_i(t) < S_i(0)e^{rt} - x)\\ &= \begin{cases}\Phi\left(\dfrac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right) & \text{if } x < S_i(0)e^{rt}\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

then

$$h_\alpha(P(Y_i(t) > x)) = 1$$

if and only if

$$\Phi\left(\frac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right) > \alpha$$

if and only if

$$\frac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}} > \Phi^{-1}(\alpha)$$

if and only if

$$x < S_i(0)e^{rt} - S_i(0)\exp\Big\{(\mu_i - \frac{1}{2}\sigma_i^2)t + \sigma_i\sqrt{t}\Phi^{-1}(\alpha)\Big\}$$

if and only if

$$x < S_i(0)e^{rt} - S_i(0)\exp\Big\{(\mu_i - \frac{1}{2}\sigma_i^2)t - \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}.$$

Let

$$\begin{aligned} x^* &= S_i(0)e^{rt} - S_i(0)\exp\Big\{(\mu_i - \frac{1}{2}\sigma_i^2)t - \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\} \\ &= S_i(0)e^{rt}\Big[1 - \exp\Big\{(\mu_i - r - \frac{1}{2}\sigma_i^2)t - \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}\Big]. \end{aligned}$$

Case 1: If

$$\mu_i - r - \frac{1}{2}\sigma_i^2 \le 0$$

or

$$\mu_i - r - \frac{1}{2}\sigma_i^2 > 0 \text{ and } t \le \left(\frac{\sigma_i\Phi^{-1}(1-\alpha)}{(\mu_i - r - \frac{1}{2}\sigma_i^2)}\right)^2$$

then

$$x^* > 0.$$

So

$$\begin{aligned} VaR_\alpha(Y_i(t)) &= \int_0^{x^*} dx = x^* \\ &= S_i(0)e^{rt}\Big[1 - \exp\Big\{(\mu_i - r - \frac{1}{2}\sigma_i^2)t - \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}\Big]. \end{aligned}$$

Case 2:

If

$$\mu_i - r - \frac{1}{2}\sigma_i^2 > 0 \text{ and } t > \left(\frac{\sigma_i\Phi^{-1}(1-\alpha)}{(\mu_i - r - \frac{1}{2}\sigma_i^2)}\right)^2$$

then

$$x^* < 0.$$

So

$$VaR_\alpha(Y_i(t)) = \int_{x^*}^{0} (-1)dx = x^*.$$

Therefore, in both cases, we have

$$VaR_\alpha(Y_i(t)) = x^* = S_i(0)e^{rt}\Big[1 - \exp\Big\{(\mu_i - r - \frac{1}{2}\sigma_i^2)t - \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}\Big].$$

Thus, the present value of $VaR_\alpha(Y_i(t))$ are

$$\begin{aligned} PV(VaR_\alpha(Y_i(t))) &= e^{-rt}VaR_\alpha(Y_i(t)) \\ &= S_i(0)\Big[1 - \exp\Big\{(\mu_i - r - \frac{1}{2}\sigma_i^2)t - \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}\Big]. \end{aligned}$$

Thus, if

$$\mu_i - r - \frac{1}{2}\sigma_i^2 > 0 \ \forall i, \ i = 1, 2, \ldots, m$$

then for each $i, \ i = 1, 2, \ldots, m$ the value for $VaR_\alpha(Y_i(t))$ or $PV(VaR_\alpha(Y_i(t)))$ will become negative for $t > \left(\dfrac{\sigma_i\Phi^{-1}(1-\alpha)}{\mu_i - r - \frac{1}{2}\sigma_i^2}\right)^2$ and keep decreasing afterwards. Since, portfolio $V(t) = n_1S_1(t) + n_2S_2(t) + \ldots + n_mS_m(t)$, n_i = number of stock in $S_i(t)$, $i = 1, 2, \ldots, m$.

We have

$$\begin{aligned} VaR_\alpha(Y(t)) &= VaR_\alpha\big(V(0)e^{rt} - V(t)\big) \\ &= VaR_\alpha\big((\sum_{i=1}^{m} n_iS_i(0))e^{rt} - \sum_{i=1}^{m} n_iS_i(t)\big) \\ &= VaR_\alpha\big(\sum_{i=1}^{m}(n_iS_i(0)e^{rt} - n_iS_i(t))\big) \end{aligned}$$

by comonotonic property, we have

$$\begin{aligned} VaR_\alpha(Y(t)) &= \sum_{i=1}^{m} VaR_\alpha\big(n_i(S_i(0)e^{rt} - S_i(t))\big) \\ &= \sum_{i=1}^{m} n_iVaR_\alpha\big(S_i(0)e^{rt} - S_i(t)\big) \\ &= \sum_{i=1}^{m} n_iVaR_\alpha\big(Y_i(t)\big). \end{aligned}$$

If

$$\mu_i - r - \frac{1}{2}\sigma_i^2 > 0 \qquad \forall i,\ i = 1, 2, \ldots, m$$

then

$$VaR_\alpha(Y(t)) = \sum_{i=1}^{m} n_i \Big(S_i(0)e^{rt}\Big[1 - \exp\Big\{(\mu_i - r - \frac{1}{2}\sigma_i^2)t - \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}\Big]\Big).$$

The value for $VaR_\alpha(Y(t))$ will become negative for t large and keeps decreasing afterwards. Thus, under actual probability, $VaR_\alpha(Y(t))$ is not monotone with time horizons. So that, The Value-at-Risk of portfolio is not consistent with time.

In risk neutral probability measure when μ_i are replace by r for $i = 1, 2, \ldots, m$ the value of $VaR_\alpha(Y(t))$ reduce to

$$VaR_\alpha(Y(t)) = \sum_{i=1}^{m} n_i \Big(S_i(0)e^{rt}\Big[1 - \exp\Big\{(-\frac{1}{2}\sigma_i^2)t - \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}\Big]\Big).$$

It implies that $VaR_\alpha(Y(t))$ is always increasing as t is increasing in risk neutral probability. Thus, the Value-at-Risk of portfolio is consistent with time.

3.3 Tail Value-at-Risk in Black-Scholes model

The distortion function in $TVaR_\alpha(.)$ is

$$h_\alpha(u) = min\Big\{1, \frac{u}{\alpha}\Big\}$$

Finding $TVaR_\alpha(.)$ by using Choquet integral, we have

$$\begin{aligned} TVaR_\alpha(Y(t)) &= \sum_{i=1}^{m} n_i TVaR_\alpha(Y_i(t)) \\ &= \sum_{i=1}^{m} n_i \Big[\int_0^\infty h_\alpha(P(Y_i(t) > x))dx + \int_{-\infty}^{0} [h_\alpha(P(Y_i(t) > x)) - 1]dx\Big]. \end{aligned}$$

Observe that

$$h_\alpha[P(Y_i(t) > x)] = 1$$

if and only if

$$\frac{P(Y(t) > x)}{\alpha} > 1$$

if and only if

$$P(Y(t) > x) > \alpha$$

if and only if

$$P(Y(t) \le x) \le 1 - \alpha$$

if and only if

$$x \le VaR_\alpha(Y_i(t)) = x^*.$$

Note that

$$\begin{aligned} P(Y_i(t) > x) &= P(S_i(0)e^{rt} - S_i(t) > x) \\ &= P(S_i(t) < S_i(0)e^{rt} - x) \\ &= \begin{cases} \Phi\left(\dfrac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right) & \text{if } x < S_i(0)e^{rt} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Case 1:

If

$$x^* > 0.$$

Then

$$\begin{aligned} TVaR_\alpha(Y_i(t)) &= \int_0^{x^*} dx + \int_{x^*}^{S_i(0)e^{rt}} \Big[\frac{P(Y_i(t) > x)}{\alpha}\Big] dx \\ &= x^* + \int_{x^*}^{S_i(0)e^{rt}} \Bigg[\frac{1}{\alpha}\Phi\left(\frac{ln\left(\frac{S_i(0)e^{rt} - x}{S_i(0)}\right) - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}\right)\Bigg] dx. \end{aligned}$$

Now, if we let

$$y_i = \frac{S_i(0)e^{rt} - x}{S_i(0)}$$

we have

$$TVaR_\alpha(Y_i(t) = x^* + \frac{S_i(0)}{\alpha}\int_0^{\frac{S_i(0)e^{rt}-x^*}{S_i(0)}} \Phi\left(\frac{ln(y_i) - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}\right) dy_i.$$

Now, let

$$z_i = \frac{ln(y_i) - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}$$

then

$$TVaR_\alpha(Y_i(t)) = x^* + \frac{S_i(0)}{\alpha}\sigma_i\sqrt{t}\int_{-\infty}^{-\Phi^{-1}(1-\alpha)} \Phi(z_i)\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}dz_i.$$

By integration by parts, if we let

$$u_i = \Phi(z_i) \text{ and } dv_i = \sigma_i\sqrt{t}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}dz_i$$

then we have

$$\begin{aligned}
&TVaR_\alpha(Y(t))\\
&= x^* + \frac{S_i(0)}{\alpha}(1-\alpha)\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}\\
&\quad - \tfrac{S_i(0)}{\alpha}\int_{-\infty}^{\Phi^{-1}(1-\alpha)} \exp\Big\{\Big(\mu_i - \tfrac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}\frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}z_i^2\Big\}dz_i\\
&= x^* + S_i(0)\exp\Big\{\Big(\mu_i - \tfrac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}\\
&\quad - \tfrac{S_i(0)}{\alpha}\exp\Big\{\Big(\mu_i - \tfrac{1}{2}\sigma_i^2\Big)t\Big\}\exp\Big\{\frac{1}{2}t\sigma_i^2\Big\}\int_{-\infty}^{-\Phi^{-1}(1-\alpha)} \frac{1}{\sqrt{2\pi}}\exp\Big\{-\frac{1}{2}\big(z_i - \sigma_i\sqrt{t}\big)^2\Big\}dz_i\\
&= S_i(0)e^{rt} - \tfrac{S_i(0)}{\alpha}\exp\{\mu_i t\}\Phi\Big[-\Phi^{-1}(1-\alpha) - \sigma_i\sqrt{t}\Big]\\
&= S_i(0)e^{rt}\Big[1 - \tfrac{1}{\alpha}\exp\{(\mu_i - r)t\}\Phi\big[-\Phi^{-1}(1-\alpha) - \sigma_i\sqrt{t}\big]\Big].
\end{aligned}$$

Case 2 : if

$$x^* \leq 0$$

then

$$\begin{aligned}
TVaR_\alpha(Y_i(t)) &= \int_0^\infty \frac{P(Y_i(t) > x)}{\alpha}dx + \int_{x^*}^0 \Big[\frac{P(Y_i(t) > x)}{\alpha} - 1\Big]dx\\
&= \frac{1}{\alpha}\int_0^{S_i(0)e^{rt}} \Phi\left(\frac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right)dx\\
&\quad + \int_{x^*}^0 \left[\frac{1}{\alpha}\Phi\left(\frac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)}) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right) - 1\right]dx.
\end{aligned}$$

Now, if we let

$$y_i = \frac{S_i(0)e^{rt} - x}{S_i(0)},\ d_i = \exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}\Phi^{-1}(1-\alpha)\Big\}$$

we have

$$\begin{aligned}
TVaR_\alpha(Y_i(t)) &= \frac{S_i(0)}{\alpha}\int_0^{e^{rt}} \Phi\left(\frac{ln(y_i) - \big(\mu_i - \frac{1}{2}\sigma_i^2\big)t}{\sigma_i\sqrt{t}}\right)dy_i\\
&\quad + S_i(0)\int_{e^{rt}}^{d_i} \left[\frac{1}{\alpha}\Phi\left(\frac{ln(y_i) - \big(\mu_i - \frac{1}{2}\sigma_i^2\big)t}{\sigma_i\sqrt{t}}\right) - 1\right]dy_i.
\end{aligned}$$

Now, let

$$z_i = \frac{ln(y_i) - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}} \text{ and } q_i = \frac{rt - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}$$

then

$$\begin{aligned} &TYaR_\alpha(Y_i(t)) \\ &= \frac{S_i(0)}{\alpha}\int_{-\infty}^{q_i}\Phi(z)\sigma_i\sqrt{t}\text{exp}\Big\{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + z_i\sigma_i\sqrt{t}\Big\}dz_i \\ &\quad + S_i(0)\int_{q_i}^{-\Phi^{-1}(1-\alpha)}\left[\frac{\Phi(z_i)}{\alpha} - 1\right]\sigma_i\sqrt{t}\text{exp}\Big\{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + z_i\sigma_i\sqrt{t}\Big\}dz_i. \end{aligned}$$

By integration by parts, if we let

$$u_i = \Phi(z_i) \text{ and } dv_i = \sigma_i\sqrt{t}\text{exp}\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}dz_i$$

then the first part of $TVaR_\alpha(Y_i(t))$ is

$$\begin{aligned} TVaR_\alpha^1(Y_i(t)) = &\frac{S_i(0)}{\alpha}\Phi\left[\frac{rt - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}\right]e^{rt} \\ &- \frac{S_i(0)}{\alpha}e^{\mu_i t}\Phi\left[\frac{rt - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}} - \sigma_i\sqrt{t}\right]. \end{aligned}$$

And by integration by parts, if we let

$$u_i = \frac{1}{\alpha}\Phi(z_i) - 1 \text{ and } dv_i = \sigma_i\sqrt{t}\text{exp}\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}z_i\Big\}dz_i$$

then the second part of $TVaR_\alpha(Y_i(t))$ is

$$\begin{aligned} TVaR_\alpha^2(Y_i(t)) = &-S_i(0)\left(\frac{1}{\alpha}\Phi\left[\frac{rt - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}\right] - 1\right)e^{rt} \\ &- \frac{S_i(0)}{\alpha}e^{\mu_i t}\Phi\Big[-\Phi^{-1}(1-\alpha) - \sigma_i\sqrt{t}\Big] \\ &+ \frac{S_i(0)}{\alpha}e^{\mu_i t}\Phi\left[\frac{rt - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}}\right]. \end{aligned}$$

Then

$$TVaR_\alpha(Y_i(t)) = S_i(0)e^{rt} - \frac{1}{\alpha}S_i(0)e^{\mu_i t}\Phi\Big[-\Phi^{-1}(1-\alpha) - \sigma_i\sqrt{t}\Big]$$

Hence, in both cases, $TVaR_\alpha(Y_i(t))$ has the same form and is

$$\begin{aligned} TVaR_\alpha(Y_i(t)) &= S_i(0)e^{rt} - \frac{1}{\alpha}S_i(0)\text{exp}\{\mu_i t\}\Phi\big[-\Phi^{-1}(1-\alpha) - \sigma_i\sqrt{t}\big] \\ &= S_i(0)e^{rt}\Big[1 - \frac{1}{\alpha}\text{exp}\{(\mu_i - r)t\}\Phi\big[-\Phi^{-1}(1-\alpha) - \sigma_i\sqrt{t}\big]\Big]. \end{aligned}$$

The present value of $TVaR_\alpha(Y_i(t))$ are

$$\begin{aligned} PV(TVaR_\alpha(Y_i(t))) &= e^{-rt}TVaR_\alpha(Y_i(t)) \\ &= S_i(0)\Big[1-\frac{1}{\alpha}\exp\{(\mu_i-r)t\}\Phi\big[-\Phi^{-1}(1-\alpha)-\sigma_i\sqrt{t}\big]\Big]. \end{aligned}$$

Since

$$\begin{aligned} \Phi\Big[-\Phi^{-1}(1-\alpha)-\sigma_i\sqrt{t}\Big] &< \Phi\big[-\Phi^{-1}(1-\alpha)\big] \\ &= \Phi\Big[\Phi^{-1}(\alpha)\Big] = \alpha. \end{aligned}$$

Therefore

if

$$\mu_i - r > 0, \forall i,\ i = 1,2,\ldots,m$$

then $TVaR_\alpha(Y_i(t))$ or $PV(TVaR_\alpha(Y_i(t)))$, $\forall i,\ i = 1,2,\ldots,m$ will be decreasing and becoming negative for large value of t.

If

$$\mu_i - r \leq 0$$

then $TVaR_\alpha(Y_i(t))$ or $PV(TVaR_\alpha(Y_i(t)))$ are increasing as t gets to ∞.

Since, portfolio $V(t) = n_1S_1 + n_2S_2(t) + \ldots + n_mS_m(t)$, n_i = number of stock in $S_i(t)$, $i = 1,2,\ldots,m$.

So that

$$\begin{aligned} TVaR_\alpha(Y(t)) &= TVaR_\alpha\big(V(0)e^{rt} - V(t)\big) \\ &= TVaR_\alpha\big((\sum_{i=1}^{m} n_iS_i(0))e^{rt} - (\sum_{i=1}^{m} n_iS_i(t))\big) \\ &= TVaR_\alpha\big(\sum_{i=1}^{m}(n_iS_i(0)e^{rt} - n_iS_i(t))\big) \end{aligned}$$

by comonotonic property, we have

$$\begin{aligned} TVaR_\alpha(Y(t)) &= \sum_{i=1}^{m} TVaR_\alpha\big(n_i(S_i(0)e^{rt} - S_i(t))\big) \\ &= \sum_{i=1}^{m} n_iTVaR_\alpha\big(S_i(0)e^{rt} - S_i(t)\big) \\ &= \sum_{i=1}^{m} n_iTVaR_\alpha\big(Y_i(t)\big) \end{aligned}$$

$$TVaR_\alpha(Y(t)) = \sum_{i=1}^{m} n_i\Big(S_i(0)e^{rt}\Big[1-\frac{1}{\alpha}\exp\{(\mu_i-r)t\}\Phi\big[-\Phi^{-1}(1-\alpha)-\sigma_i\sqrt{t}\big]\Big]\Big)$$

then $TVaR_\alpha(Y(t))$ will becoming negative for large value of t if $\mu_i - r > 0$, $\forall i,\ i = 1, 2, \ldots, m$ and at $t = 0$, we have $TVAR_\alpha(Y(0)) = 0$. Thus, under actual probability, $TVaR_\alpha(Y(t))$ is not increasing as t increasing. So that the Tail Value-at-Risk of portfolio is not consistent with time.

In risk neutral probability measure when μ_i are replace by r for $i = 1, 2, \ldots, m$, the value of $TVaR_\alpha(Y(t))$ reduce to

$$TVaR_\alpha(Y(t)) = \sum_{i=1}^{m} n_i\Big(S_i(0)e^{rt}\Big[1-\frac{1}{\alpha}\Phi\big[-\Phi^{-1}(1-\alpha)-\sigma_i\sqrt{t}\big]\Big]\Big)$$

so that it is always increasing as t is increasing in risk neutral probability. Hence, the Tail Value-at-Risk of portfolio is consistent with time under risk neutral probability.

3.4 Risks based on Wang's distortion function in Black-Scholes model

The Wang's distortion function is

$$h_\lambda(u) = \Phi\big[\Phi^{-1}(u)+\lambda\big], \qquad \lambda > 0.$$

The risk measure under Wang's distortion function is

$$\begin{aligned}\rho_w(Y(t)) &= \sum_{i=1}^{m} n_i\rho_w(Y_i(t)) \\ &= \sum_{i=1}^{m} n_i\Big[\int_0^\infty h_\lambda(P(Y_i(t)>x))dx + \int_{-\infty}^{0}[h_\lambda(P(Y_i(t)>x))-1]dx\Big]\end{aligned}$$

and

$$\begin{aligned}P(Y_i(t)>x) &= P(S_i(0)e^{rt}-S_i(t)>x) \\ &= P(S_i(t)<S_i(0)e^{rt}-x) \\ &= \begin{cases}\Phi\left(\dfrac{ln(\frac{S_i(0)e^{rt}-x}{S_i(0)})-(\mu_i-\frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}}\right) & \text{if } x < S_i(0)e^{rt} \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Therefore

$$\rho_W(Y_i(t)) = \int_0^{S_i(0)e^{rt}} \Phi\left[\frac{ln\left(\frac{S_i(0)e^{rt}-x}{S_i(0)}\right) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}} + \lambda\right] dx$$
$$+ \int_{-\infty}^{0}\left[\Phi\left[\frac{ln\left(\frac{S_i(0)e^{rt}-x}{S_i(0)}\right) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}} + \lambda\right] - 1\right] dx.$$

Now if we let

$$y_i = \frac{S_i(0)e^{rt} - x}{S_i(0)}.$$

we have

$$\rho_W(Y_i(t)) = \int_0^{S_i(0)e^{rt}} \Phi\left[\frac{ln(y_i) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}} + \lambda\right] dx$$
$$+ \int_{-\infty}^{0}\left[\Phi\left[\frac{ln(y_i) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}} + \lambda\right] - 1\right] dx.$$

Let

$$z_i = \frac{ln(y_i) - (\mu_i - \frac{1}{2}\sigma_i^2)t}{\sigma_i\sqrt{t}} + \lambda$$

then

$$y_i = \exp\left\{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i\sqrt{t}(z_i - \lambda)\right\}.$$

Let

$$C_i = \frac{rt - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}} + \lambda$$

we obtain

$$\rho_W(Y_i(t)) = S_i(0)\int_{-\infty}^{C_i} \Phi[z_i]\sigma_i\sqrt{t}\exp\left\{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i\sqrt{t}(z_i - \lambda)\right\}dz_i$$
$$= S_i(0)\int_{C_i}^{\infty} [\Phi[z_i] - 1]\sigma_i\sqrt{t}\exp\left\{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i\sqrt{t}(z_i - \lambda)\right\}dz_i.$$

By integration by parts, if we let

$$u_i = \Phi(z_i) \text{ and } dv_i = \sigma_i\sqrt{t}\exp\left\{\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i\sqrt{t}(z_i - \lambda)\right\}dz_i$$

then the first part of $\rho_W(Y_i(t))$ is

$$\rho_W^1(Y_i(t)) = S_i(0)\Phi\left[\frac{rt - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}} + \lambda\right] e^{rt}$$
$$- S_i(0)\exp\left\{\mu_i t - \lambda\sigma_i\sqrt{t}\right\}\Phi\left[\frac{rt - \left(\mu_i - \frac{1}{2}\sigma_i^2\right)t}{\sigma_i\sqrt{t}} + \lambda - \sigma_i\sqrt{t}\right].$$

And also by integration by parts, if we let

$$u_i = \Phi(z_i) - 1 \text{ and } dv_i = \sigma_i\sqrt{t}\exp\Big\{\Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t + \sigma_i\sqrt{t}(z_i - \lambda)\Big\}dz_i$$

then the second part of $\rho_W(Y_i(t))$ is

$$\begin{aligned}\rho_W^2(Y_i(t)) = &-S_i(0)\left(\Phi\left[\frac{rt - \Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t}{\sigma_i\sqrt{t}} + \lambda\right] - 1\right)e^{rt}\\ &- S_i(0)\exp\Big\{\mu_i t - \lambda\sigma_i\sqrt{t}\Big\}\left(1 - \Phi\left[\frac{rt - \Big(\mu_i - \frac{1}{2}\sigma_i^2\Big)t}{\sigma_i\sqrt{t}} + \lambda - \sigma_i\sqrt{t}\right]\right).\end{aligned}$$

Then

$$\begin{aligned}\rho_W(Y_i(t)) = \rho_W(Y_i^1(t)) + \rho_W^2(Y_i(t)) &= S_i(0)e^{rt} - S_i(0)\exp\Big\{\mu_i t - \sigma_i\sqrt{t}\lambda\Big\}\\ &= S_i(0)e^{rt}\Big[1 - \exp\Big\{(\mu_i - r)t - \sigma_i\sqrt{t}\lambda\Big\}\Big].\end{aligned}$$

The present value of $\rho_W(Y_i(t))$ are

$$\begin{aligned}PV(\rho_W(Y_i(t))) &= e^{-rt}\rho_W(Y_i(t))\\ &= S_i(0)\Big[1 - \exp\Big\{(\mu_i - r)t - \sigma_i\sqrt{t}\lambda\Big\}\Big]\end{aligned}$$

if

$$\mu_i - r > 0 \ \ \forall i, \ i = 1, 2, \ldots, m$$

then $\rho_W(Y_i(t))$ are become negative for $t > \left(\dfrac{\sigma_i\lambda}{\mu_i - r}\right)^2 \ \forall i, \ i = 1, 2, \ldots, m.$
As same as $VaR_\alpha(Y(t))$ and $TVaR_\alpha(Y(t))$. We have risk measure of portfolio under Wang's distortion function is

$$\rho_W(Y(t)) = \sum_{i=1}^{m} n_i\Big(S_i(0)e^{rt}\Big[1 - \exp\Big\{(\mu_i - r)t - \sigma_i\sqrt{t}\lambda\Big\}\Big]\Big).$$

If $\mu_i - r > 0$, $\forall i, \ i = 1, 2, \ldots, m$ then $\rho_W(Y(t))$ is becoming negative for t large.

In conclusion, "under actual probability", the risk measure of portfolio under Wang's distortion function is not consistent with time.

Under risk neutral probability, when $\mu_i = r \ \forall i, \ i = 1, 2, \ldots, m$, we have a risk measure of portfolio is

$$\rho_W(Y(t)) = \sum_{i=1}^{m} n_i\Big(S_i(0)e^{rt}\Big[1 - \exp\Big\{-\sigma_i\sqrt{t}\lambda\Big\}\Big]\Big).$$

So that, $\rho_W(Y(t))$ is increasing as t is increasing.

In conclusion, "under risk neutral probability", the risk measure of portfolio under Wang's distortion function is consistent with time.