CHAPTER 2
THEORY

2.1 Fiber-reinforced composite materials

A Composite material is a structural material consisted of two or more different materials which are mechanically or metallurgically bonded together at a macroscopic level. One constituent is called the reinforcing phase and the one which is embedded is called the matrix (Figure 2.1). The reinforcing phase material could be in form of particles, flakes, or fibers. The matrix phase materials are generally continuous. Composite materials offer several other advantages over conventional materials such as steel, aluminum, and other types of metal. These advantages include improved strength, stiffness, fatigue and impact resistance, thermal conductivity, and corrosion resistance.

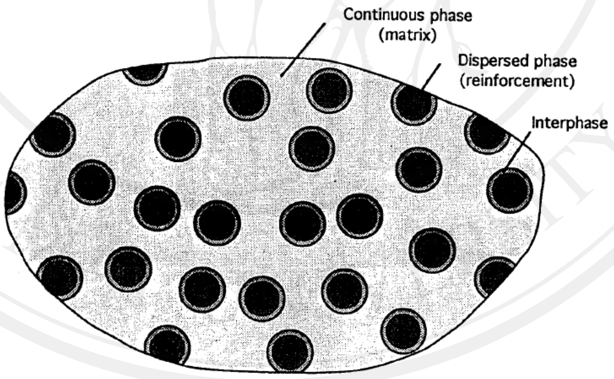


Figure 2.1 Phases of a composite materials.

(Source: Isaac, 1994)

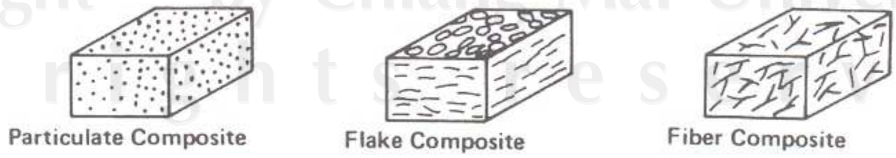


Figure 2.2 Classification of composite materials.

(Source: Laszlo, 2003)

There are many types of composite materials and several methods of classifying them. One method is based on geometry of the reinforcement (Figure 2.2).

(1) Particulate composite materials consist of particles immersed in matrices. They are usually isotropic because the particles are added randomly. Particulate composites have advantages such as improved strength, increased operating temperature, oxidation resistance, etc.
(2) Flake composite materials consist of flat reinforcements of matrices. They provide advantages such as high out-of-plane flexural modulus, higher strength, and low cost. However, flakes cannot be oriented easily and only a limited number of materials are available for use.
(3) Fiber-reinforced composite materials consist of matrices reinforced by short (discontinuous) or long (continuous) fiber; fibers are generally anisotropic.

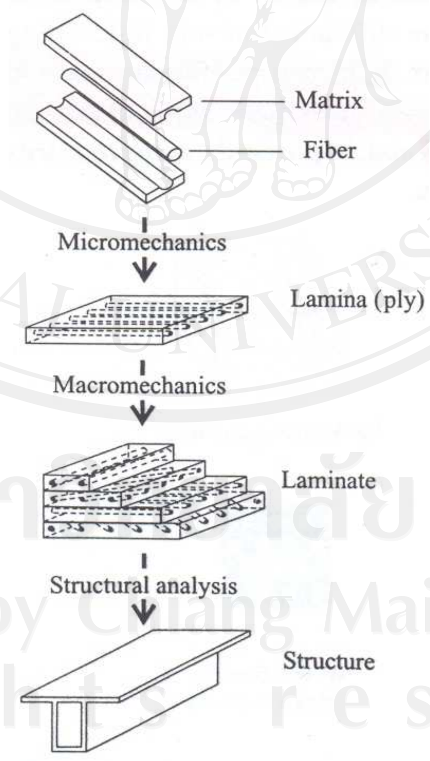


Figure 2.3 The levels of analysis for a structure made of laminated composite.
(Source: Laszlo, 2003)

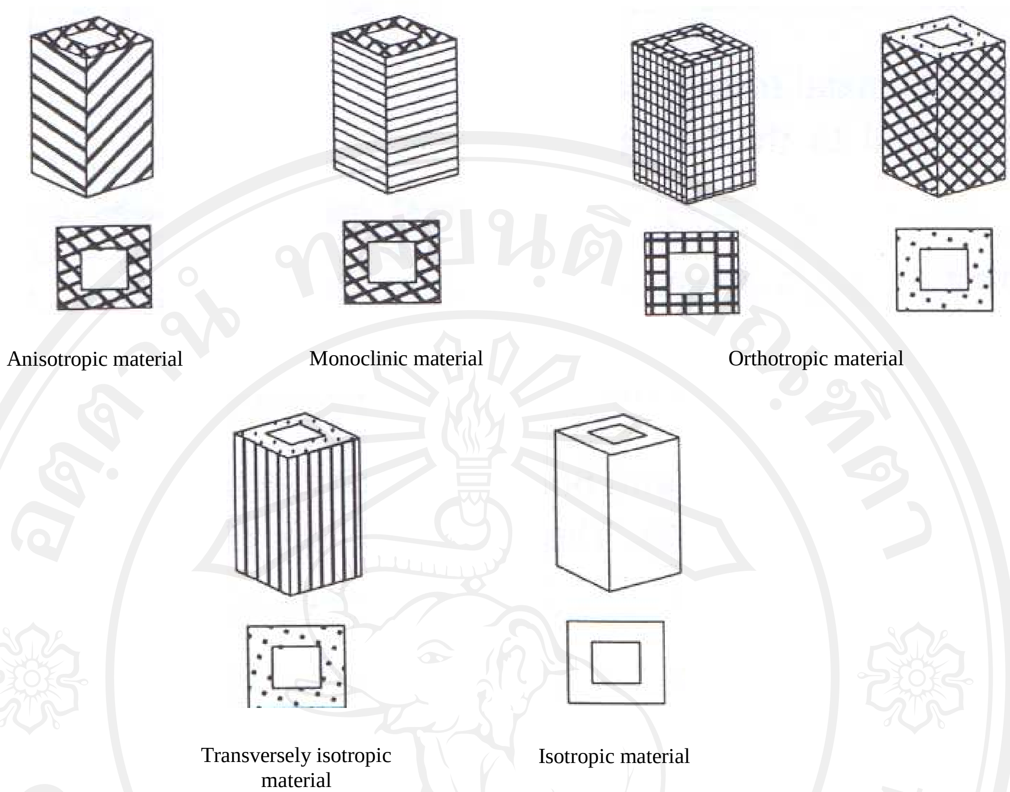


Figure 2.4 Illustrations of possible fiber orientations.
(Source: Laszlo, 2003)

Fiber-reinforced composite materials consist of two materials, a reinforcement material called fiber and a base material called matrix material. Fibers are the principal load-carrying members, they are kept together by the matrix material which acts as a load-transfer medium between fibers, and protects fibers from being exposed to the environment. Fiber reinforced composite materials are often made in the form of a thin layer, called lamina or ply. Within each lamina, the fibers can be continuous or discontinuous, woven, unidirectional, bidirectional, or randomly distributed. Stacking a number of such lamina in the direction of lamina thickness called laminate, which can form the desired structure such as bars, beams or plates (Figure 2.3). The sequence of various orientations of a fiber reinforced composite layer in a laminate is termed the lamination scheme or stacking sequence. The lamination scheme and material properties of individual lamina provide an added flexibility to designs to tailor the stiffness and strength of the laminate in order to match the structural stiffness and strength requirements. Depending on the arrangements of the fibers, the material may behave differently in different directions.

According to their behavior, fiber-reinforced composite materials may be characterized as anisotropic, monoclinic, orthotropic, transversely isotropic, or isotropic materials (Figure 2.4).

2.1.1 Stress-strain relationships

In macromechanical analyses of fiber-reinforced composite materials that are large with respect to the fiber diameter, the fiber and matrix properties may be averaged, and the material may be treated as linear and elastic behavior. Thus, the stress-strain relationships follow Hook's law for a three-dimensional body in x_1-x_2-x_3 coordinate system in matrix form is

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&C_{14}&C_{15}&C_{16}\\ C_{21}&C_{22}&C_{23}&C_{24}&C_{25}&C_{26}\\ C_{31}&C_{32}&C_{33}&C_{34}&C_{35}&C_{36}\\ C_{41}&C_{42}&C_{43}&C_{44}&C_{45}&C_{46}\\ C_{51}&C_{52}&C_{53}&C_{54}&C_{55}&C_{56}\\ C_{61}&C_{62}&C_{63}&C_{64}&C_{65}&C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix} \tag{2.1}$$

where $\sigma_1, \sigma_2, \sigma_3$ Normal components of stress parallel to x_1, x_2, and x_3 direction, respectively

$\tau_{23}, \tau_{13}, \tau_{12}$ Shearing stress components in x_1-x_2-x_3 coordinate system

$\varepsilon_1, \varepsilon_2, \varepsilon_3$ Normal strains in x_1, x_2, and x_3 direction, respectively

$\gamma_{23}, \gamma_{13}, \gamma_{12}$ Shearing strain components in x_1-x_2-x_3 coordinate system

$[C_{ij}]$ The element of stiffness in x_1-x_2-x_3 coordinate system

inverting equation (2.1), the general strain-stress relationship for a three-dimensional body in x_1-x_2-x_3 coordinate system is

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \gamma_{23}\\ \gamma_{13}\\ \gamma_{12}\end{bmatrix}=\begin{bmatrix}S_{11}&S_{12}&S_{13}&S_{14}&S_{15}&S_{16}\\ S_{21}&S_{22}&S_{23}&S_{24}&S_{25}&S_{26}\\ S_{31}&S_{32}&S_{33}&S_{34}&S_{35}&S_{36}\\ S_{41}&S_{42}&S_{43}&S_{44}&S_{45}&S_{46}\\ S_{51}&S_{52}&S_{53}&S_{54}&S_{55}&S_{56}\\ S_{61}&S_{62}&S_{63}&S_{64}&S_{65}&S_{66}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \tau_{23}\\ \tau_{13}\\ \tau_{12}\end{bmatrix} \tag{2.2}$$

where $[S_{ij}]$ The element of compliance in x_1-x_2-x_3 coordinate system

Due to the symmetry of the stiffness matrix $[C]$, the 36 constants in equation (2.1) actually reduce to 21 independent constants. This also implies that only 21 independent constants are in the general compliance matrix $[S]$ of equation (2.2). Further reduction in the number of independent stiffness (or compliance) parameters comes from the so-called material symmetry. When elastic material parameters at a point have the same values for every pair of coordinate systems that are mirror images of each other in a certain plane, that plane is called a material plane of symmetry. The difference in material plane of symmetry may be characterized material as anisotropic, monoclinic, orthotropic, transversely isotropic, or isotropic materials.

(1) Anisotropic materials: when there are no symmetry planes with respect to the alignment of the fibers the material, the stiffness matrix can be written as equation (2.3) and the compliance matrix can be written as equation (2.4)

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \tag{2.3}$$

$$[S]=\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \tag{2.4}$$

(2) Monoclinic materials: when there is a symmetry plane with respect to the alignment of the fibers, the stiffness matrix can be written as equation (2.5) and the compliance matrix can be written as equation (2.6)

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \tag{2.5}$$

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & \frac{\nu_{61}}{G_{12}} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & \frac{\nu_{62}}{G_{12}} \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & \frac{\nu_{63}}{G_{12}} \\ 0 & 0 & 0 & \frac{1}{G_{23}} & \frac{\nu_{54}}{G_{13}} & 0 \\ 0 & 0 & 0 & \frac{\nu_{45}}{G_{23}} & \frac{1}{G_{13}} & 0 \\ \frac{\nu_{16}}{E_1} & \frac{\nu_{26}}{E_2} & \frac{\nu_{36}}{E_3} & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \tag{2.6}$$

where E_1, E_2, E_3 Modulus of elasticity in tension and compression in x_1, x_2, and x_3 direction, respectively

G_{12}, G_{13}, G_{23} Modulus of elasticity in shear in the x_1-x_2, x_1-x_3, and x_2-x_3, respectively

ν_{ij} Poisson's ratio, defined as the ratio of transverse strain in the j^{th} direction to the axial strain in the i^{th} direction when stressed in the i^{th} direction, and

$$\frac{\nu_{ij}}{E_i}=\frac{\nu_{ji}}{E_j} \tag{2.7}$$

(3) Orthotropic materials: when there are three mutually perpendicular symmetry planes with respect to the alignment of the fibers, the stiffness matrix can be written as equation (2.8) and the compliance matrix be written as equation (2.9)

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{2.8}$$

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \tag{2.9}$$

(4) Transversely isotropic materials: when there are three mutually perpendicular symmetry planes with respect to the alignment of the fibers and in one of the planes of symmetry the material is treated as isotropic, the stiffness matrix can be written as equation (2.10) and the compliance matrix can be written as equation (2.11)

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{2.10}$$

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{21}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_{23})}{E_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{13}} \end{bmatrix} \qquad (2.11)$$

(5) Isotropic materials: when there are no preferred directions and every plane is a plane of symmetry, the stiffness matrix can be written as equation (2.12) and the compliance matrix can be written as equation (2.13)

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{C_{11}-C_{12}}{2} \end{bmatrix} \qquad (2.12)$$

$$[S]=\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \qquad (2.13)$$

where E Modulus of elasticity in tension and compression of isotropic material

ν Poisson's ratio of isotropic material

The stiffness and compliance matrix from equation (2.3)-(2.13) are referred to a coordinate system that coincides with the principal material coordinate system. The coordinate system used in the problem formulation, in general, does not coincide with the principal material coordinate system. Further, composite laminates have several layers; each with different orientation of their material coordinates with respect to the laminate coordinates. Thus, there is a need to establish transformation one coordinate system to the corresponding quantities in another coordinate system.

2.1.2 Transformation of material coefficients

Due to the laminates have several layers, each with different orientation of their material coordinates system; the problem coordinate system does not coincide with the material coordinate system (Figure 2.5). Thus there is a need to establish transformation relations along stresses and strains in the material coordinate system of each layer to the problem coordinate system. In general, two coordinate systems are used to describe a laminate: (x, y, z) denotes the problem coordinate system which used to write the governing equations of a laminate and (x_1, x_2, x_3) is the material coordinate system of a typical layer in the laminate. The x_1x_2-plane and the xy-plane are parallel; the x_3-axis is parallel to the z-axis. The x_1-axis is oriented at an angle of $+\theta$ counterclockwise from the x-axis (Figure 2.6).

In x-y-z coordinate system, the stress-strain relationship can be written as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & \overline{C}_{14} & \overline{C}_{15} & \overline{C}_{16} \\ \overline{C}_{21} & \overline{C}_{22} & \overline{C}_{23} & \overline{C}_{24} & \overline{C}_{25} & \overline{C}_{26} \\ \overline{C}_{31} & \overline{C}_{32} & \overline{C}_{33} & \overline{C}_{34} & \overline{C}_{35} & \overline{C}_{36} \\ \overline{C}_{41} & \overline{C}_{42} & \overline{C}_{43} & \overline{C}_{44} & \overline{C}_{45} & \overline{C}_{46} \\ \overline{C}_{51} & \overline{C}_{52} & \overline{C}_{53} & \overline{C}_{54} & \overline{C}_{55} & \overline{C}_{56} \\ \overline{C}_{61} & \overline{C}_{62} & \overline{C}_{63} & \overline{C}_{64} & \overline{C}_{65} & \overline{C}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} \tag{2.14}$$

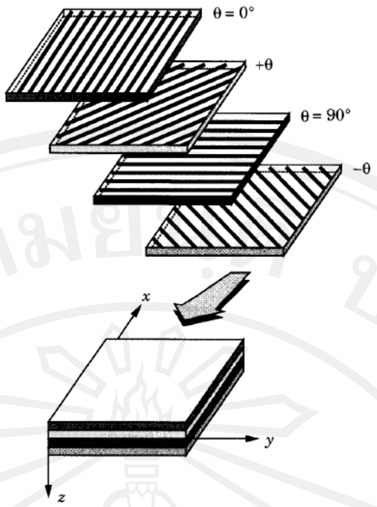


Figure 2.5 A laminate made up of lamina with different fiber orientations.
(Source: Reddy, 2004)

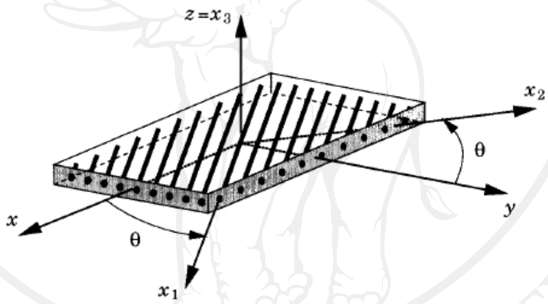


Figure 2.6 A lamina with material and problem coordinate systems.
(Source: Reddy, 2004)

where

$\sigma_x, \sigma_y, \sigma_z$	Normal components of stress parallel to *x*, *y*, and *z* directions, respectively
$\tau_{yz}, \tau_{xz}, \tau_{xy}$	Shearing stress components in *x-y-z* coordinate system
$\varepsilon_x, \varepsilon_y, \varepsilon_z$	Normal strains in *x*, *y*, and *z* directions, respectively
$\gamma_{yz}, \gamma_{xz}, \gamma_{xy}$	Shearing strain components in *x-y-z* coordinate system
$[\overline{C}_{ij}]$	The element of stiffness matrix in *x-y-z* coordinate system,

$$\overline{C}_{11} = C_{11}c^4 - 4C_{16}c^3s + 2(C_{12} + 2C_{66})c^2s^2 - 4C_{26}cs^3 + C_{22}s^4$$
$$\overline{C}_{12} = C_{12}c^4 + 2(C_{16} - C_{26})c^3s + (C_{11} + C_{22} - 4C_{66})c^2s^2 + 2(C_{26} - C_{16})cs^3 + C_{12}s^4$$
$$\overline{C}_{13} = C_{13}c^2 - 2C_{36}cs + C_{23}s^2$$

$$
\begin{aligned}
\overline{C}_{14} &= C_{14}c^3 + (C_{15} - 2C_{46})c^2s + (C_{24} - 2C_{56})cs^2 + C_{25}s^3 \\
\overline{C}_{15} &= C_{15}c^3 - (C_{14} + 2C_{56})c^2s + (C_{25} + 2C_{46})cs^2 - C_{24}s^3 \\
\overline{C}_{16} &= C_{16}c^4 + (C_{11} - C_{12} - 2C_{66})c^3s + 3(C_{26} - C_{16})c^2s^2 \\
&\qquad + (2C_{66} + C_{12} - C_{22})cs^3 - C_{26}s^4 \\
\overline{C}_{22} &= C_{22}c^4 + 4C_{26}c^3s + 2(C_{12} + 2C_{66})c^2s^2 + 4C_{16}cs^3 + C_{11}s^4 \\
\overline{C}_{23} &= C_{23}c^2 + 2C_{36}cs + C_{13}s^2 \\
\overline{C}_{24} &= C_{24}c^3 + (C_{25} + 2C_{46})c^2s + (C_{14} + 2C_{56})cs^2 + C_{15}s^3 \\
\overline{C}_{25} &= C_{25}c^3 + (2C_{56} - C_{24})c^2s + (C_{15} - 2C_{46})cs^2 - C_{14}s^3 \\
\overline{C}_{26} &= C_{26}c^4 + (C_{12} - C_{22} + 2C_{66})c^3s + 3(C_{16} - C_{26})c^2s^2 \\
&\qquad + (C_{11} - C_{12} - 2C_{66})cs^3 - C_{16}s^4 \\
\overline{C}_{33} &= C_{33} \\
\overline{C}_{34} &= C_{34}c + C_{35}s \\
\overline{C}_{35} &= C_{35}c - C_{34}s \\
\overline{C}_{36} &= (C_{13} - C_{23})cs + C_{36}(c^2 - s^2) \\
\overline{C}_{44} &= C_{44}c^2 + C_{55}s^2 + 2C_{45}cs \\
\overline{C}_{45} &= C_{45}(c^2 - s^2) + (C_{55} - C_{44})cs \\
\overline{C}_{46} &= C_{46}c^3 + (C_{56} + C_{14} - C_{24})c^2s + (C_{15} - C_{25} - C_{46})cs^2 - C_{56}s^3 \\
\overline{C}_{55} &= C_{55}c^2 + C_{44}s^2 - 2C_{45}cs \\
\overline{C}_{56} &= C_{56}c^3 + (C_{15} - C_{25} - C_{46})c^2s + (C_{24} - C_{14} - C_{56})cs^2 + C_{46}s^3 \\
\overline{C}_{66} &= 2(C_{16} - C_{26})c^3s + (C_{11} + C_{22} - 2C_{12} - 2C_{66})c^2s^2 \\
&\qquad + 2(C_{26} - C_{16})cs^3 + C_{66}(c^4 + s^4)
\end{aligned}
\tag{2.15}
$$

where c, s $\cos\theta$ and $\sin\theta$, respectively

inverting equation (2.14), can be written strain-stress relationship in *x*-*y*-*z* coordinate as

$$
\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}
=
\begin{bmatrix}
\overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & \overline{S}_{14} & \overline{S}_{15} & \overline{S}_{16} \\
\overline{S}_{21} & \overline{S}_{22} & \overline{S}_{23} & \overline{S}_{24} & \overline{S}_{25} & \overline{S}_{26} \\
\overline{S}_{31} & \overline{S}_{32} & \overline{S}_{33} & \overline{S}_{34} & \overline{S}_{35} & \overline{S}_{36} \\
\overline{S}_{41} & \overline{S}_{42} & \overline{S}_{43} & \overline{S}_{44} & \overline{S}_{45} & \overline{S}_{46} \\
\overline{S}_{51} & \overline{S}_{52} & \overline{S}_{53} & \overline{S}_{54} & \overline{S}_{55} & \overline{S}_{56} \\
\overline{S}_{61} & \overline{S}_{62} & \overline{S}_{63} & \overline{S}_{64} & \overline{S}_{65} & \overline{S}_{66}
\end{bmatrix}
\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}
\tag{2.16}
$$

where $[\bar{S}_{ij}]$ The element of compliance in *x-y-z* coordinate system,

$$\left[\bar{S}_{ij}\right]^{-1} = \left[\bar{C}_{ij}\right] \tag{2.17}$$

In most structural applications fiber-reinforced composite materials are used in the form of thin laminates loaded in the plane of the laminate. Thus, fiber-reinforced composite materials can be considered to be under a condition of plane stress with all stress components in the out-of-plane direction being zero. This assumption then reduces the three-dimensional stress-strain relationship, equation (2.14), to two-dimensional stress-strain relationship in *x-y-z* coordinate system as

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{2.18}$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{2.19}$$

where $[\bar{Q}_{ij}]$ The element of plane stress-reduce stiffness,

$$\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \end{bmatrix} \tag{2.20}$$

$$\begin{bmatrix} \bar{Q}_{44} & \bar{Q}_{45} \\ \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} = \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix} \tag{2.21}$$

2.2 Fiber-reinforced composite plates

Plates that are straight, plane, two-dimensional structural components of which one dimension, referred to the thickness, is much smaller than the other dimensions. Plates have free, simply supported and clamped boundary conditions, including elastic supports and elastic restraints, or, in some case, even point supports. The static and dynamic loads carried by plates are predominantly perpendicular to the plate surface. These external loads are carried by internal bending and twisting moments and by transverse shear forces.

A rigorous elastic analysis would require, for instance, that the plate should be considered as a three-dimensional continuum. Needless to say, such an approach is highly impractical since it would create almost insurmountable mathematical difficulties. Even if a solution could be found, the resulting costs would be, in most cases, prohibitively high. Consequently, in order to rationalize the plate analyses, plate is distinguished into three categories with inherently different structural behavior and, hence, different governing differential equations. The three plate-types might be categorized, to some extent, using their ratio of thickness to governing length (h/L).

(1) Stiff plates (h/L = < 1/10) are thin plates with flexural rigidity, carrying loads two dimensionally, mostly by internal bending and twisting moments and by transverse shear, generally in a manner similar to beams.

(2) Moderately thick plates (h/L = 1/10 – 1/5) are in many respects similar to stiff plates, with the notable exception that the effects of transverse shear forces on the normal stress components are also taken into account.

(3) Thick plates (h/L > 1/5) have an internal stress condition that resembles that of three-dimensional continua.

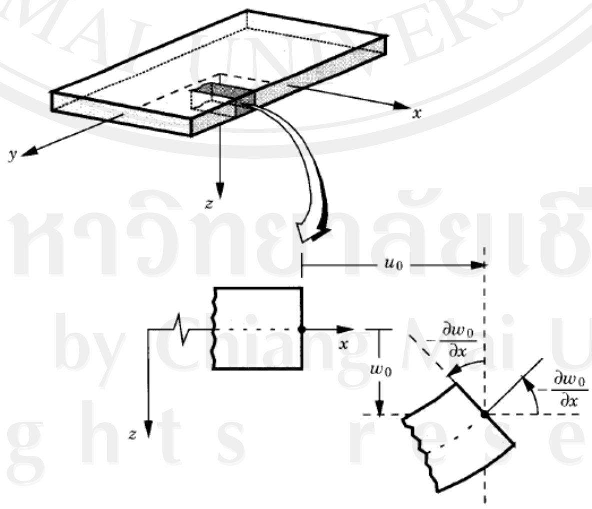


Figure 2.7 Undeformed and deformed geometries of an edge of a plate under Kirchhoff hypothesis.
(Source: Reddy, 2004)

Fiber-reinforced composite laminates are formed by stacking layers of different composite materials and/or fiber orientation. By construction, fiber-reinforced composite laminates have their planar dimensions one to two orders of magnitude larger than their thickness. Often fiber-reinforced composite laminates are used in applications that require membrane and bending strengths. Therefore, fiber-reinforced composite laminates are treated as plate type structure.

2.2.1 The classical laminated plate theory

The classical laminated plate theory (CLPT) is an extension of the Kirchhoff plate theory to laminated composite plates. In the classical laminated plate theory it is assumed that the Kirchhoff hypothesis holds:

(1) There is no deformation in the middle plane of the plate. This plane remains neutral during bending.

(2) Points of the plate lying initially on a normal-to the middle plane of the plate remain on the normal-to-the-middle surface of the plate after bending.

(3) The normal stresses in the direction transverse to the plate can be disregarded in comparison to the stress in other directions.

According to Kirchhoff hypothesis, the displacement field in *x*-*y*-*z* coordinate system (Figure 2.7) is assumed to be

$$u(x,y,z,t) = u_o(x,y,t) - z\frac{\partial w_o}{\partial x}$$
$$v(x,y,z,t) = v_o(x,y,t) - z\frac{\partial w_o}{\partial y}$$
$$w(x,y,z,t) = w_o(x,y,t) \tag{2.22}$$

where

u, v, w	Components of displacements in *x*, *y*, and *z* directions, respectively
u_0, v_0, w_0	Components of displacements of middle surface in *x*, *y*, and *z* directions, respectively
z	Distance from middle surface

from equation (2.22), the strains are therefore given by

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix} - z \begin{Bmatrix} \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial^2 w_0}{\partial y^2} \\ 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{2.23}$$

where $\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0$ Normal strains of middle surface in *x*, *y*, and *z* directions, respectively

$\kappa_x, \kappa_y, \kappa_{xy}$ Curvatures of middle surface of plate

Considering all stress over the thickness of the plate (Figure 2.8), the normal stress and the shearing stress give the in-plane forces, moments (per unit length), and the transverse shear forces (per unit length) acting on a small element as (Figure 2.9)

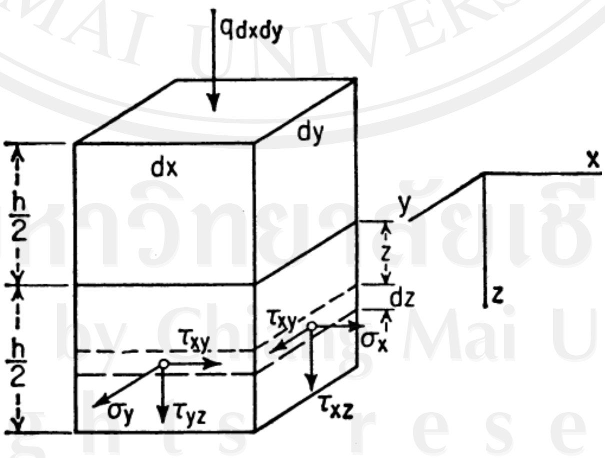


Figure 2.8 Stresses on plate element.
(Source: Timoshenko, 1959)

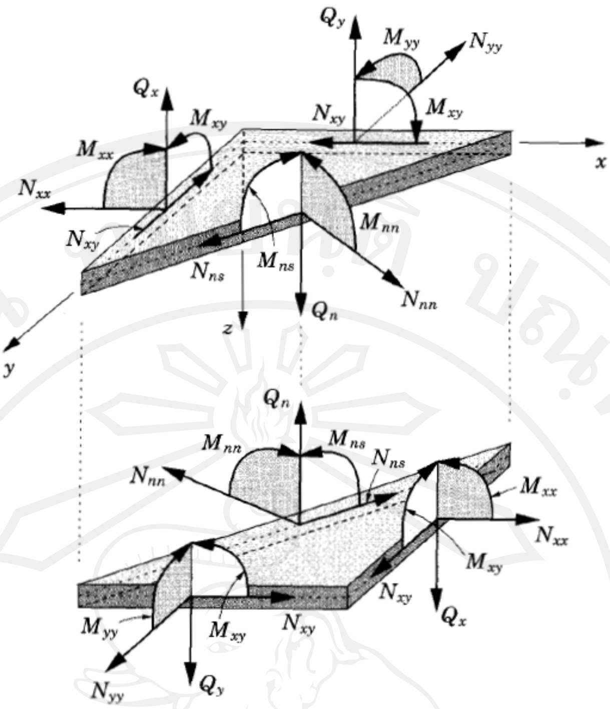


Figure 2.9 Force and moment resultants on a plate element.
(Source: Reddy, 2004)

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z\, dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z\, dz$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} dz$$

where N_x, N_y Normal forces per unit length of sections of a plate perpendicular to x and y directions, respectively

N_{xy} Shearing force per unit length of section of a plate perpendicular to x direction

M_x, M_y	Bending moments per unit length of sections of a plate perpendicular to x and y directions, respectively
M_{xy}	Twisting moment per unit length of section of a plate perpendicular to x direction
Q_x, Q_y	Shearing forces parallel to z direction per unit length of sections of a plate perpendicular to x and y directions, respectively
h	Thickness of a plate

combining equation (2.18), (2.19) and (2.23), the in-plane forces, moments (per unit length) and the transverse shear forces (per unit length) can be written as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{2.24}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{2.25}$$

$$\begin{bmatrix} Q_y \\ Q_x \end{bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{2.26}$$

where	$[A_{ij}]$	The element of extensional stiffness
	$[B_{ij}]$	The element of coupling stiffness
	$[D_{ij}]$	The element of bending stiffness,

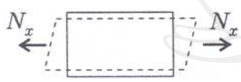
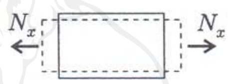
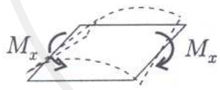
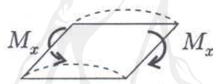
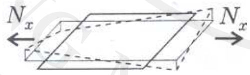
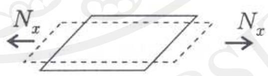
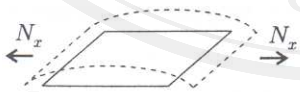
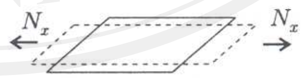
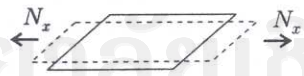
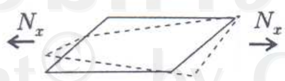
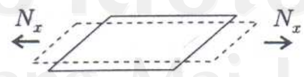
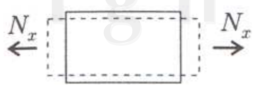
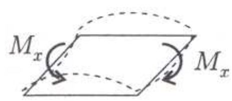
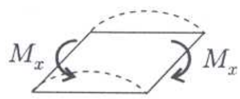
$$[A_{ij}] = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k - z_{k-1})$$

$$[B_{ij}] = \frac{1}{2} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^2 - z_{k-1}^2)$$

$$[D_{ij}] = \frac{1}{3} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (z_k^3 - z_{k-1}^3) \tag{2.27}$$

The [*A*], [*B*], and [*D*] matrices represent the stiffness of a laminate and describe the response of the laminate to in-plane forces and moments. Examination of [*A*], [*B*], and [*D*] matrices shows that the different types of couplings may occur as illustrated in Table 2.1. When the coupling terms shown in the last column are zero, there is no coupling. The coupling terms $A_{26}, D_{26}, B_{26}, B_{22}$ can be illustrated in a similar manner by applying a force N_y and a moment M_y in the *y*-*z* plane.

Table 2.1 Illustration of the coupling term.
(Source: Laszlo, 2003)

Coupling	No coupling	Element
Extension-shear		
		A_{16}
Bending-twist		
		D_{16}
Extension-twist		
		B_{16}
In-plane-out-of-plane		
		B_{11}
		B_{12}
		B_{66}
Extension-extension		
		A_{12}
Bending-extension		
		D_{12}

(1) Extension-shear coupling: when the elements A_{16}, A_{26} are not zero, in-plane normal forces N_x, N_y cause shear deformation γ_{xy}^0, and a twist force N_{xy} cause elongations in the *x* and *y* directions.

(2) Bending-twist coupling: when the elements D_{16}, D_{26} are not zero, bending moments M_x, M_y cause twist of the laminate κ_{xy}, and a twist moment M_{xy} causes curvatures in the *x-z* and *y-z* planes.

(3) Extension-twist and bending-shear coupling: when the elements B_{16}, B_{26} are not zero, in-plane normal forces N_x, N_y cause twist κ_{xy}, and bending moments M_x, M_y results in shear deformation γ_{xy}^0.

(4) In-plane-out-of-plane coupling: when the elements B_{ij} are not zero, in-plane forces N_x, N_y, N_{xy} cause out-of-plane deformations (curvatures) of the laminate, and the moments M_x, M_y, M_{xy} cause in-plane deformations in the *x-y* plane.

(5) Extension-extension coupling: when the elements A_{12} is not zero, a normal force N_x causes elongation in the *y* direction ε_y^0, and a normal force N_y causes elongation in the *x* direction ε_x^0.

(6) Bending-bending coupling: when the elements D_{12} is not zero, a bending moment M_x causes curvature of the laminate in the *y-z* plane κ_y, and a bending moment M_y causes curvature of the laminate in the *x-z* plane κ_x.

2.2.2 Typical fiber-reinforced composite lamination scheme

Based on angle, material, and thickness of lamina, the symmetry or antisymmetry of a laminate may zero out some elements of the three stiffness matrices [*A*], [*B*], and [*D*]. These are important because they may result in reducing or zeroing out the coupling of forces and bending moments, normal and shear forces, or bending and twisting moments. Most commonly fiber-reinforced composite laminates used forms (Figure 2.10) are:

(1) Unidirectional laminates: where the fiber orientation angle is same in all lamina, it can be implied that $[B]=0$.

(2) Cross-ply laminates: where the orientation angles in alternate layer are $.../0/90/0/90/...$. It can be implied that $A_{16}=A_{26}=0$, $B_{16}=B_{26}=0$, and $D_{16}=D_{26}=0$.

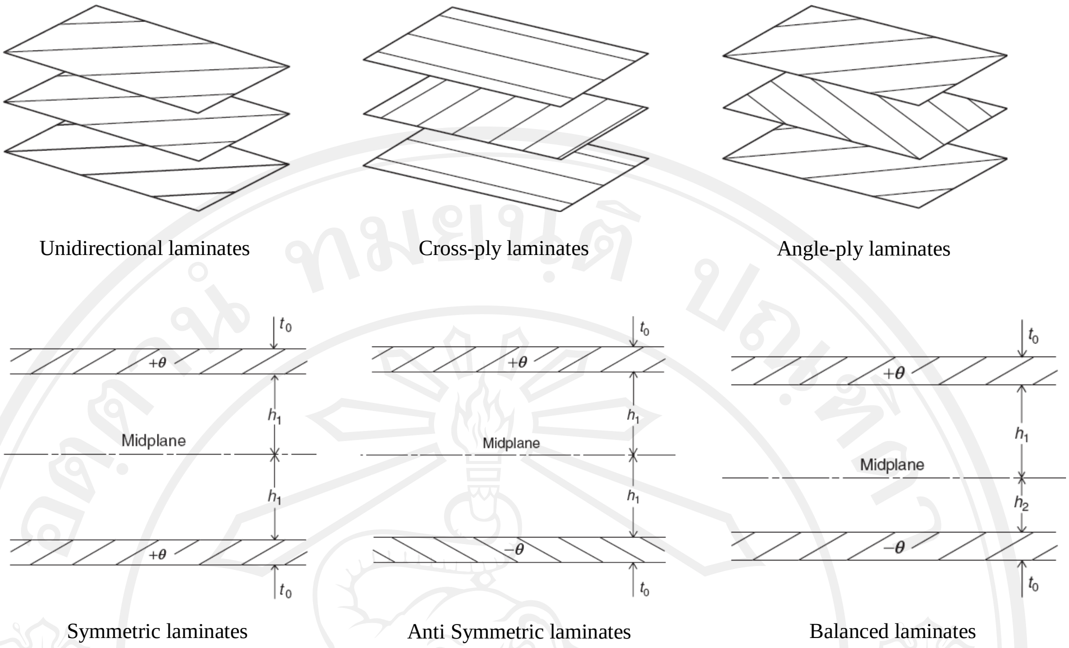


Figure 2.10 Type of the lamination schemes.
(Source: Autar, 2006)

(3) Angle-ply laminates: where the orientation angles in alternate layer are $.../\theta/-\theta/\theta/-\theta/...$ when $\theta \neq 0°, 90°$. It can be implied that $A_{16} = A_{26} = 0$.

(4) Symmetric laminates: where the lamina orientation is symmetrical about the center line of the laminated plate; that is for each lamina above the mid plane, there is identical lamina in all respects (material, thickness, fiber orientation) at an equal distance below the mid plane. It can be implied that $[B] = 0$.

(5) Anti symmetric laminates: where for every lamina of $+\theta°$ orientation at distance *z*, there is an identical $-\theta°$ orientation lamina at distance –*z*. It can be implied that $A_{16} = A_{26} = 0$, and $D_{16} = D_{26} = 0$.

(6) Balance laminates: where for every lamina of $+\theta°$ orientation, there is an identical $-\theta°$ orientation lamina, somewhere within the laminate. It can be implied that $A_{16} = A_{26} = 0$.

2.2.3 Boundary conditions

The conditions along each edge of the plate must be specified. Boundary conditions of plates in bending can be generally classified as one of the following.

(1) Clamped edge (C): If the edge of a plate is clamped, the deflection along this edge is zero, and the tangent plane to the deflected middle surface along this edge coincides with the initial position of the middle plane of the plate. Assuming the clamped edge to be given by $x = a$, the boundary conditions are

$$(w)_{x=a} = 0$$
$$\left(\frac{\partial w}{\partial x}\right)_{x=a} = 0 \tag{2.28}$$

(2) Simply supported edge (S): If the edge of a plate is simply supported, the deflection along this edge must be zero. At the same time this edge can rotate freely with respect to the edge line; there are no bending moments along this edge. Assuming the simply supported edge to be given by $x = a$, the boundary conditions are

$$(w)_{x=a} = 0$$
$$(M_x)_{x=a} = 0 \tag{2.29}$$

(3) Free edge (F): If the edge of a plate is entirely free, it is natural to assume that along this edge there are no bending moments and supplement shear forces. Assuming the free edge to be given by $x = a$, the boundary conditions are

$$(M_x)_{x=a} = 0$$
$$(V_x)_{x=a} = 0 \tag{2.30}$$

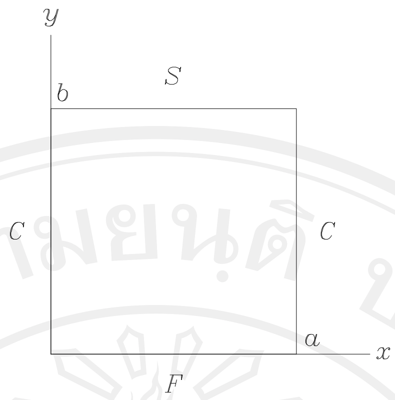


Figure 2.11 The CFCS boundary conditions.

The notation for boundary conditions is, for example, CFCS, in which the first and third letters mean the boundary conditions along $x=0$ and $x=a$ respectively, and the second and fourth letters mean the boundary conditions along $y=0$ and $y=b$ respectively (Figure 2.11).

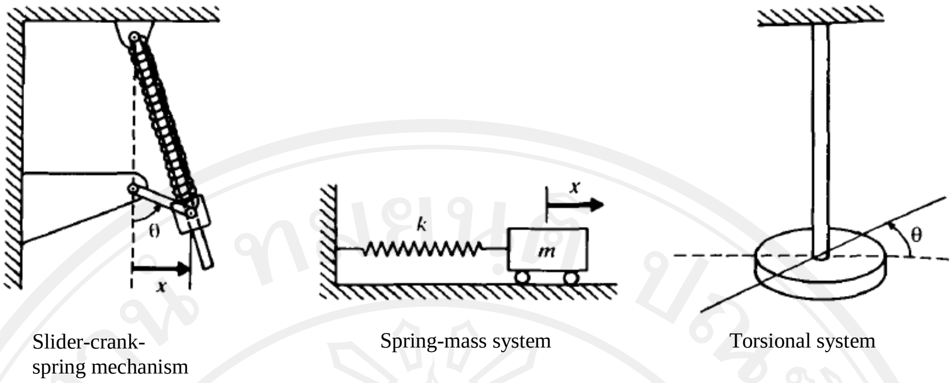
2.3 Vibration of fiber-reinforced composite plates

Vibration is any motion that repeats itself after an interval of time. A vibratory system, in general, includes a means for storing potential energy (spring or elasticity), a means for storing kinetic energy (mass or inertia), and a means by which energy is gradually lost (damper). It involves the transfer of its potential energy to kinetic energy and kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle of vibration and must be replaced by an external source if a state of steady vibration is to be maintained.

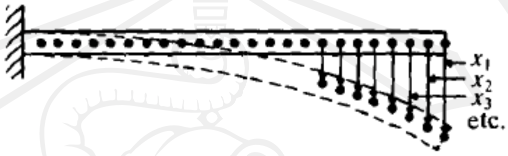
Vibration system models can be divided into two classes, discrete and continuous (or distributed). The systems do depend on system parameters such as mass, damping, and elasticity (Figure 2.12).

(1) Discrete systems where mass, damping, and elasticity are assumed to be present only at certain discrete points in the system.

(2) Continuous systems are systems with varying distributions of mass, damping, and elasticity.



(a) Discrete systems



(b) Continuous systems: Cantilever beam

Figure 2.12 Discrete and continuous systems.
(Source: Rao, 1999)

2.3.1 Continuous systems

The continuous systems are designated by infinite number of degrees of freedom (DOFs). Displacement of continuous systems is described by a continuous function of position and time, and consequently will be governed by partial differential equations (PDEs). The behavior of discrete systems, by contrast, is defined by finite number of DOFs and the corresponding equation is ordinary differential equations (ODEs). In general, the equations of motion for continuous system are a transcendental equation that yields an infinite number of natural frequencies and mode shapes. The vibration frequencies and their modes are conventionally ordered as a sequence ω_1, ω_2, ω_3… with $\omega_{n+1} > \omega_n$. The lowest frequency of vibration is denoted ω_1. A mode shape is a specific pattern of vibration executed by a mechanical system at a specific frequency and the lowest natural frequencies tends to have a long wavelength; the wavelength decreases for higher frequency modes. As illustrated in Figure 2.13, the first and second mode shapes of simply supported plate has a single and double sine waves in *x* coordinate direction, respectively.

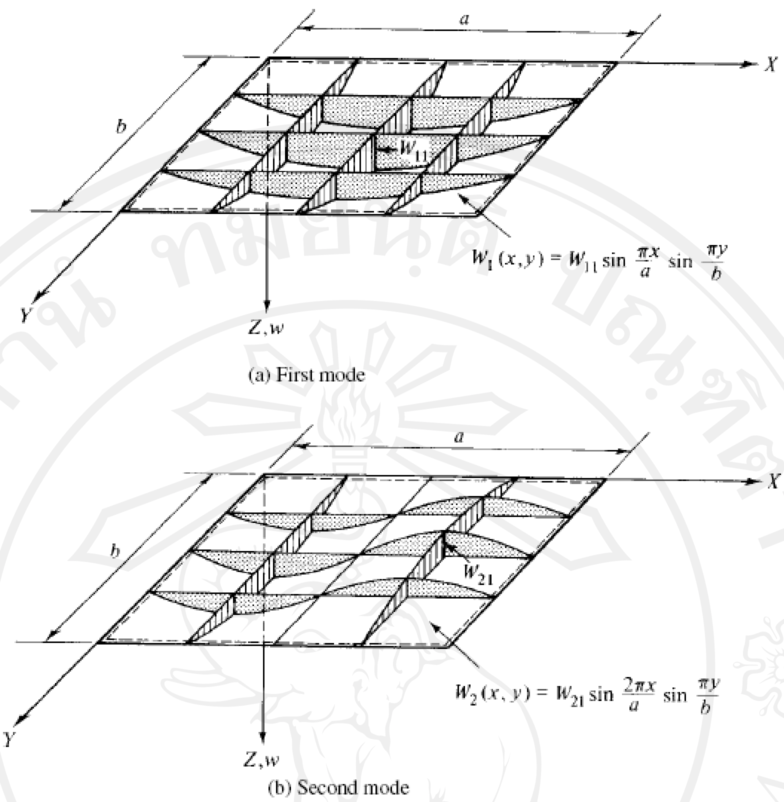


Figure 2.13 First and second modes of simply supported plate.
(Source: Rudolph, 2004)

Most of time, continuous systems are approximated as discrete systems due to the fact that the ordinary differential equation is easier to be solved than the partial differential equation. Although the treatment of a system as continuous gives exact results, the exact methods available for dealing with continuous systems are limited to a narrow selection of problems. Hence most of the continuous systems are studied by treating them as an approximate method or a numerical method. In this regard, many methods exist. But here, only three methods are addressed for solving the free vibration of symmetrically laminated composite rectangular plates with various boundary conditions.

(1) The extended Kantorovich method
(2) The Rayleigh-Ritz method
(3) The finite element method

2.3.2 The extended Kantorovich method

The extended Kantorovich method is an approximate method which used a separable function in the form of function *X*(*x*) and function *Y*(*y*). A separable function is applied to the dynamic system energy equation which yields the partial differential equation. The variation method is used to reduce the partial differential equations to ordinary differential equations in the *x* and *y* coordinates direction, with a constant coefficient. The iterative calculation is used to evaluate the natural frequency from the ordinary differential equation, and to force the final solution required to satisfy the boundary conditions. These iterations are repeated until the result converges to a desired degree.

Hamiton's principle is a generalization of the principle of virtual displacement within the dynamics of a system. The principle assumes that the system under consideration is characterized by two energy functions: the potential energy and the kinetic energy.

$$\delta\left[\Pi - K\right] = 0 \tag{2.31}$$

where Π Potential energy

K Kinetic energy

The potential energy of the symmetrically laminated composite rectangular plate (Figure 2.14) can be written as

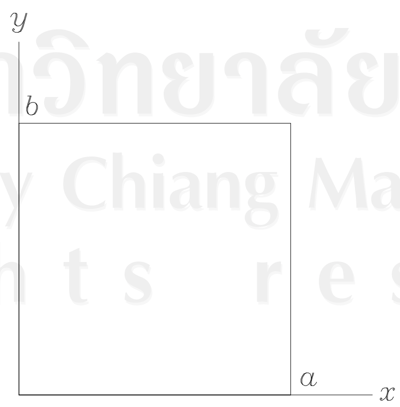


Figure 2.14 The rectangular plate.

$$\Pi = \frac{1}{2}\int_0^a\int_0^b\int_{-h/2}^{h/2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\right]dx\,dy\,dz \tag{2.32}$$

substitute equation (2.23) into equation (2.32)

$$\Pi = \frac{1}{2}\int_0^a\int_0^b\int_{-h/2}^{h/2}\left[\sigma_x(\varepsilon_x^0 - z\kappa_x) + \sigma_y(\varepsilon_y^0 - z\kappa_y) + \tau_{xy}(\gamma_{xy}^0 - z\kappa_{xy})\right]dx\,dy\,dz$$

By the in-plane forces-stress relationship and the moment-stress relationship, the potential energy can be written as

$$\Pi = \frac{1}{2}\int_0^a\int_0^b\left[N_x\varepsilon_x^0 + N_y\varepsilon_y^0 + N_{xy}\gamma_{xy}^0 + M_x\kappa_x + M_y\kappa_y + M_{xy}\kappa_{xy}\right]dx\,dy \tag{2.33}$$

Based upon the first assumption of the Kirchhoff hypothesis, the ε_x^0, ε_y^0, and γ_{xy}^0 are zero. Thus, combining equation (2.25) with equation (2.33)

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^a\int_0^b\Bigg[& D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + 2D_{12}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{\partial^2 w_0}{\partial y^2}\right) + 4D_{16}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{\partial^2 w_0}{\partial x\partial y}\right) \\ & + D_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 4D_{26}\left(\frac{\partial^2 w_0}{\partial y^2}\right)\left(\frac{\partial^2 w_0}{\partial x\partial y}\right) + 4D_{66}\left(\frac{\partial^2 w_0}{\partial x\partial y}\right)^2\Bigg]dxdy\end{aligned} \tag{2.34}$$

The kinetic energy of the symmetrically laminated composite rectangular plate (Figure 2.15) can be written as

$$K = \frac{1}{2}\int_0^a\int_0^b m(\omega\, w_0)^2\,dxdy \tag{2.35}$$

where m Mass per unit area of plate

ω Natural circular frequency

substitute equation (2.34) and (2.35) in the application of Hamiton's principle as equation (2.31)

$$\delta\left\{\frac{1}{2}\int_0^a\int_0^b\left[D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2+2D_{12}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{\partial^2 w_0}{\partial y^2}\right)+4D_{16}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{\partial^2 w_0}{\partial x\partial y}\right)\right.\right.$$
$$\left.+D_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2+4D_{26}\left(\frac{\partial^2 w_0}{\partial y^2}\right)\left(\frac{\partial^2 w_0}{\partial x\partial y}\right)+4D_{66}\left(\frac{\partial^2 w_0}{\partial x\partial y}\right)^2\right]dxdy$$
$$\left.-\frac{1}{2}\int_0^a\int_0^b m(\omega\ w_0)^2\,dxdy\right\}=0 \qquad (2.36)$$

Assume the solution as

$$w_0(x,y)=X(x)Y(y) \qquad (2.37)$$

substitute equation (2.37) into equation (2.36)

$$\delta\left\{\frac{1}{2}\int_0^a\int_0^b\left[D_{11}\left(\frac{\partial^2 X}{\partial x^2}Y\right)^2+2D_{12}\left(\frac{\partial^2 X}{\partial x^2}Y\right)\left(X\frac{\partial^2 Y}{\partial y^2}\right)+D_{22}\left(X\frac{\partial^2 Y}{\partial y^2}\right)^2\right.\right.$$
$$\left.+4D_{66}\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\right)^2+4\left(D_{16}\left(\frac{\partial^2 X}{\partial x^2}Y\right)+D_{26}\left(X\frac{\partial^2 Y}{\partial y^2}\right)\right)\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y}\right)\right]dxdy$$
$$\left.-\frac{1}{2}\int_0^a\int_0^b\left[mX^2Y^2\omega^2\right]dx\,dy\right\}=0 \qquad (2.38)$$

If $X(x)$ is defined as priori, equation (2.38) can be rewritten as

$$\delta\left\{\frac{1}{2}\int_0^b\left[S_{1x}D_{11}Y^2+2S_{2x}D_{12}Y\left(\frac{\partial^2 Y}{\partial y^2}\right)+S_{3x}D_{22}\left(\frac{\partial^2 Y}{\partial y^2}\right)^2+4S_{4x}D_{66}\left(\frac{\partial Y}{\partial y}\right)^2\right.\right.$$
$$\left.+4S_{5x}D_{16}Y\left(\frac{\partial Y}{\partial y}\right)+4S_{6x}D_{26}\left(\frac{\partial Y}{\partial y}\right)\left(\frac{\partial^2 Y}{\partial y^2}\right)\right]dy$$
$$\left.-\frac{1}{2}\int_0^b S_{3x}mY^2\omega^2 dy\right\}=0 \qquad (2.39)$$

where $S_{1x}=\int_0^a\left(\frac{\partial^2 X}{\partial x^2}\right)^2 dx$ $\qquad S_{2x}=\int_0^a\left(X\frac{\partial^2 X}{\partial x^2}\right)dx$

$$S_{3x} = \int_0^a X^2 dx \qquad S_{4x} = \int_0^a \left(\frac{\partial X}{\partial x}\right)^2 dx$$

$$S_{5x} = \int_0^a \left(\frac{\partial X}{\partial x}\right)\left(\frac{\partial^2 X}{\partial x^2}\right) dx \qquad S_{6x} = \int_0^a X\left(\frac{\partial X}{\partial x}\right) dx \tag{2.40}$$

The variational method and integration by parts equation (2.39) yields

$$\int_0^b \Bigg[\left(2S_{3x}D_{22}\left(\frac{\partial^4 Y}{\partial y^4}\right) + 4S_{2x}D_{12}\left(\frac{\partial^2 Y}{\partial y^2}\right) - 8S_{4x}D_{66}\left(\frac{\partial^2 Y}{\partial y^2}\right) + 2S_{1x}D_{11}Y - 2S_{3x}m\omega^2 Y\right)\delta Y + \frac{\partial}{\partial y}\left(-2S_{3x}D_{22}\left(\frac{\partial^3 Y}{\partial y^3}\right) - 2S_{2x}D_{12}\left(\frac{\partial Y}{\partial y}\right) + 8S_{4x}D_{66}\left(\frac{\partial Y}{\partial y}\right) + 4S_{5x}D_{16}Y\right)\delta Y + \frac{\partial Y}{\partial y}\left(2S_{3x}D_{22}\left(\frac{\partial^2 Y}{\partial y^2}\right) + 4S_{6x}D_{26}\left(\frac{\partial Y}{\partial y}\right) + 2S_{2x}D_{12}Y\right)\frac{\partial \delta Y}{\partial y}\Bigg] dy = 0$$

The fourth order ordinary differential equations and the boundary conditions along y sides, as shown in equations (2.41)-(2.43), are obtained by setting the coefficients of δY and $(\partial \delta Y / \partial y)$ to zero separately.

$$S_{3x}D_{22}\frac{d^4 Y}{dy^4} + (2S_{2x}D_{12} - 4S_{4x}D_{66})\frac{d^2 Y}{dy^2} + (S_{1x}D_{11} - S_{3x}m\omega^2)Y = 0 \tag{2.41}$$

$$V_y = S_{3x}D_{22}\frac{d^3 Y}{dy^3} + (S_{2x}D_{12} - 4S_{4x}D_{66})\frac{dY}{dy} - 2S_{5x}D_{16}Y \tag{2.42}$$

$$M_y = S_{3x}D_{22}\frac{d^2 Y}{dy^2} + 2S_{6x}D_{26}\frac{dY}{dy} + S_{2x}D_{12}Y \tag{2.43}$$

Similarly when $Y(y)$ is defined as priori, the fourth order ordinary differential equations can be written as equation (2.44) and the boundary conditions along $x = 0$ and $x = a$ as equations (2.45) and (2.46)

$$S_{3y}D_{11}\frac{d^4 X}{dx^4} + (2S_{2y}D_{12} - 4S_{4y}D_{66})\frac{d^2 X}{dx^2} + (S_{1y}D_{22} - S_{3y}m\omega^2)X = 0 \tag{2.44}$$

$$V_x = S_{3y} D_{11} \frac{d^3 X}{dx^3} + (S_{2y} D_{12} - 4S_{4y} D_{66}) \frac{dX}{dx} - 2S_{5y} D_{26} X \qquad (2.45)$$

$$M_x = S_{3y} D_{11} \frac{d^2 X}{dx^2} + 2S_{6y} D_{16} \frac{dX}{dx} + S_{2y} D_{12} X \qquad (2.46)$$

where

$$S_{1y} = \int_0^b \left(\frac{\partial^2 Y}{\partial y^2} \right)^2 dy \qquad S_{2y} = \int_0^b \left(Y \frac{\partial^2 Y}{\partial y^2} \right) dy$$

$$S_{3y} = \int_0^b Y^2 dy \qquad S_{4y} = \int_0^b \left(\frac{\partial Y}{\partial y} \right)^2 dy$$

$$S_{5y} = \int_0^b \left(\frac{\partial Y}{\partial y} \right) \left(\frac{\partial^2 Y}{\partial y^2} \right) dy \qquad S_{6y} = \int_0^b Y \left(\frac{\partial Y}{\partial y} \right) dy \qquad (2.47)$$

The fourth order ordinary differential equations in equation (2.41) can be rewritten in a simple form as

$$\frac{d^4 Y}{dy^4} + 2n_1 \frac{d^2 Y}{dy^2} + n_2 Y = 0 \qquad (2.48)$$

where

$$n_1 = \frac{S_{2x} D_{12} - 2S_{4x} D_{66}}{S_{3x} D_{22}}$$

$$n_2 = \frac{S_{1x} D_{11} - S_{3x} m\omega^2}{S_{3x} D_{22}}$$

The characteristic equation of equation (2.48) is

$$q^4 + 2n_1 q^2 + n_2 = 0 \qquad (2.49)$$

whose four roots are

$$q_{1,2,3,4} = \pm\sqrt{-n_1 \pm \sqrt{n_1^2 - n_2}} \qquad (2.50)$$

From equation (2.50) it follows that the nature of the solution depends on whether the expression under the inner square root is positive, zero, or negative. Thus, there are four distinct cases.

(1) If $n_1 > 0$ and $(n_1^2 - n_2) > 0$, all four roots are imaginary:

$$q_{1,2} = \pm i\sqrt{n_1 + \sqrt{n_1^2 - n_2}}$$

$$q_{3,4} = \pm i\sqrt{n_1 - \sqrt{n_1^2 - n_2}}$$

$$Y(y) = C_{1y}\sin(q_1 y) + C_{2y}\cos(q_1 y) + C_{3y}\sin(q_3 y) + C_{4y}\cos(q_3 y)$$

(2) If $n_1 < 0$ and $n_2 < 0$, two roots are imaginary and the other are real:

$$q_{1,2} = \pm i\sqrt{\sqrt{n_1^2 - n_2} + n_1}$$

$$q_{3,4} = \pm\sqrt{\sqrt{n_1^2 - n_2} - n_1}$$

$$Y(y) = C_{1y}\sin(q_1 y) + C_{2y}\cos(q_1 y) + C_{3y}\sinh(q_3 y) + C_{4y}\cosh(q_3 y)$$

(3) If $n_1 < 0$ and $(n_1^2 - n_2) < 0$, roots are in complex conjugate pairs:

$$q_{1,2} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{n_2} - n_1} \pm \frac{i}{\sqrt{2}}\sqrt{\sqrt{n_2} + n_1}$$

$$q_{3,4} = -\frac{1}{\sqrt{2}}\sqrt{\sqrt{n_2} - n_1} \pm \frac{i}{\sqrt{2}}\sqrt{\sqrt{n_2} + n_1}$$

$$\begin{aligned} Y(y) = {} & [C_{1y}\cos(q_3 y) + C_{2y}\sin(q_3 y)]\cosh(q_1 y) \\ & + [C_{3y}\cos(q_3 y) + C_{4y}\sin(q_3 y)]\sinh(q_1 y) \end{aligned}$$

(4) If $n_1 < 0$ and $0 < n_2 < n_1^2$, all four roots are real:

$$q_{1,2} = \pm\sqrt{-n_1 + \sqrt{n_1^2 - n_2}}$$

$$q_{3,4} = \pm\sqrt{-n_1 - \sqrt{n_1^2 - n_2}}$$

$$Y(y) = C_{1y}\sinh(q_1 y) + C_{2y}\cosh(q_1 y) + C_{3y}\sinh(q_3 y) + C_{4y}\cosh(q_3 y)$$

In this study considering a case $S_{1x} D_{11} < S_{3x}\, m\omega^2$, the solution can be written as follows:

$$Y(y) = C_{1y}\sin(q_1 y) + C_{2y}\cos(q_1 y) + C_{3y}\sinh(q_3 y) + C_{4y}\cosh(q_3 y) \tag{2.51}$$

where q_1 and q_3 Modal parameters in y coordinate direction, and

$$q_1^2 - q_3^2 = \frac{2S_{2x}D_{12} - 4S_{4x}D_{66}}{S_{3x}D_{22}} \tag{2.52}$$

$$q_1^2\, q_3^2 = \frac{S_{3x}m\omega^2 - S_{1x}D_{11}}{S_{3x}D_{22}} \tag{2.53}$$

Similarly, the fourth order ordinary differential equations in equation (2.44) can be rewritten in a simple form as

$$\frac{d^4X}{dx^4} + m_1\frac{d^2X}{dx^2} + m_2X = 0 \tag{2.54}$$

where $m_1 = \dfrac{S_{2y}D_{12} - 2S_{4y}D_{66}}{S_{3y}D_{11}}$

$m_2 = \dfrac{S_{1y}D_{22} - S_{3y}m\omega^2}{S_{3y}D_{11}}$

The characteristic equation of equation (2.54) is

$$p^4 + 2m_1p^2 + m_2 = 0 \tag{2.55}$$

whose four roots are

$$p_{1,2,3,4} = \pm\sqrt{-m_1 \pm \sqrt{m_1^2 - m_2}} \tag{2.56}$$

From equation (2.56) it follows that the nature of the solution depends on whether the expression under the inner square root is positive, zero, or negative. Thus, there are four distinct cases.

(1) If $m_1 > 0$ and $(m_1^2 - m_2) > 0$, all four roots are imaginary:

$$p_{1,2} = \pm i\sqrt{m_1 + \sqrt{m_1^2 - m_2}}$$

$$p_{3,4} = \pm i\sqrt{m_1 - \sqrt{m_1^2 - m_2}}$$

$$X(x) = C_{1x}\sin(p_1 x) + C_{2x}\cos(p_1 x) + C_{3x}\sin(p_3 x) + C_{4x}\cos(p_3 x)$$

(2) If $m_1 < 0$ and $m_2 < 0$, two roots are imaginary and the other are real:

$$p_{1,2} = \pm i\sqrt{\sqrt{m_1^2 - m_2} + m_1}$$

$$p_{3,4} = \pm\sqrt{\sqrt{m_1^2 - m_2} - m_1}$$

$$X(x) = C_{1x}\sin(p_1 x) + C_{2x}\cos(p_1 x) + C_{3x}\sinh(p_3 x) + C_{4x}\cosh(p_3 x)$$

(3) If $m_1 < 0$ and $(m_1^2 - m_2) < 0$, roots are in complex conjugate pairs:

$$p_{1,2} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{m_2} - m_1} \pm \frac{i}{\sqrt{2}}\sqrt{\sqrt{m_2} + m_1}$$

$$p_{3,4} = -\frac{1}{\sqrt{2}}\sqrt{\sqrt{m_2} - m_1} \pm \frac{i}{\sqrt{2}}\sqrt{\sqrt{m_2} + m_1}$$

$$X(x) = [C_{1x}\cos(p_3 x) + C_{2x}\sin(p_3 x)]\cosh(p_1 x) + [C_{3x}\cos(p_3 x) + C_{4x}\sin(p_3 x)]\sinh(p_1 x)$$

(4) If $m_1 < 0$ and $0 < m_2 < m_1^2$, all four roots are real:

$$p_{1,2} = \pm\sqrt{-m_1 + \sqrt{m_1^2 - m_2}}$$

$$p_{3,4} = \pm\sqrt{-m_1 - \sqrt{m_1^2 - m_2}}$$

$$X(x) = C_{1x}\sinh(p_1 x) + C_{2x}\cosh(p_1 x) + C_{3x}\sinh(p_3 x) + C_{4x}\cosh(p_3 x)$$

In this study considering a case $S_{1y} D_{22} < S_{3y}\, m\omega^2$, the solution can be written as follows:

$$X(x) = C_{1x}\sin(p_1 x) + C_{2x}\cos(p_1 x) + C_{3x}\sinh(p_3 x) + C_{4x}\cosh(p_3 x) \qquad (2.57)$$

where $\quad p_1$ and p_2 $\qquad$ Modal parameters in the x coordinate direction, and

$$p_1^2 - p_3^2 = \frac{2S_{2y}D_{12} - 4S_{4y}D_{66}}{S_{3y}D_{11}} \qquad (2.58)$$

$$p_1^2\, p_3^2 = \frac{S_{3y}m\omega^2 - S_{1y}D_{22}}{S_{3y}D_{11}} \qquad (2.59)$$

The iterative calculation is used to evaluate the natural frequency and to develop a final solution to satisfy the boundary conditions.

(1) The iterative calculation begins by choosing a basis function in the x or y coordinate direction, using the procedures shown in Figure 2.15, and choosing the $X_0(x)$ as a basis function. S_{1x} through S_{6x} is calculated from $X_0(x)$.

(2) In the first iteration, substitute the solution equation (2.51) in the boundary conditions and use q_3 as a function of q_1, or q_1 as a function of q_3 from the relationship equation (2.52). Then find the eigenvalue q_1 or q_3, the eigenvector $Y_1(y)$, and the natural circular frequency from equation (2.53).

X_0 : Choose basis function

$\Downarrow$

$W_{01} = \boxed{X_0}\, Y_1$: Iteration no.1

$\Downarrow$

$W_{11} = X_1 \boxed{Y_1}$: Iteration no.2

$\Downarrow$

$W_{12} = \boxed{X_1}\, Y_2$: Iteration no.3

$\Downarrow$

$W_{22} = X_2 \boxed{Y_2}$: Iteration no.4

⋮

Converged solution

Figure 2.15 Iteration procedures.

(3) In the second iteration, substitute the solution equation (2.57) in the boundary conditions and use p_3 as a function of p_1, or p_1 as a function of p_3 from the relationship equation (2.58); S_{1y} through S_{6y} is calculated from the eigenvector $Y_1(y)$ obtained from step (2). Then find the eigenvalue p_1 or p_3, the eigenvector $X_1(x)$, and the natural circular frequency from equation (2.59).

(4) Compare the natural frequency from step (3) and (2). If the difference satisfies the specified tolerance level, the last natural circular frequency can be taken as the final solution. Otherwise continue the iterative calculation by repeating steps (2) to (4).

2.3.3 The Rayleigh-Ritz method

The Rayleigh-Ritz method is an approximate method which can be considered an extension of Rayleigh's method. It is based on the premise that a closer approximation to the exact natural mode can be obtained by superposing a number of assumed functions than by using a single assumed function, as in Rayleigh's method. If the assumed functions are suitably chosen, this method provides not only the approximate value of the fundamental frequency but also the approximate values of the higher natural frequencies and the mode shapes. An arbitrary number of functions can be used, and the number of frequencies that can be obtained is equal to the number of functions used. A large number of functions, although it involves more computational work, leads to more accurate results.

By equation (2.36), the Lagrangian equation can be written as

$$L = \frac{1}{2}\int_0^a\int_0^b \left[D_{11}\left(\frac{\partial^2 w_0}{\partial x^2}\right)^2 + 2D_{12}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{\partial^2 w_0}{\partial y^2}\right) + 4D_{16}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\left(\frac{\partial^2 w_0}{\partial x \partial y}\right) + D_{22}\left(\frac{\partial^2 w_0}{\partial y^2}\right)^2 + 4D_{26}\left(\frac{\partial^2 w_0}{\partial y^2}\right)\left(\frac{\partial^2 w_0}{\partial x \partial y}\right) + 4D_{66}\left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 \right] dxdy - \frac{1}{2}\int_0^a\int_0^b m(\omega w)^2\, dxdy \tag{2.60}$$

Assume the solution as

$$w_0(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij} f_i(x) g_j(y) \tag{2.61}$$

where A_{ij} Arbitrary coefficients

$f_i(x)$ Functions which satisfy the boundary conditions along the x coordinate direction

$g_j(y)$ Functions which satisfy the boundary conditions along the y coordinate direction

The arbitrary coefficients are to be determined so that the solution provides the best possible approximation to the natural frequency. Substituting equation (2.61) into equation (2.60), the resulting expression is partially differentiated with respect to each of the arbitrary coefficients. To make the natural frequency stationary, each of partial derivatives is equal to zero and obtain

$$\frac{\partial L}{\partial A_{ij}} = 0 \tag{2.62}$$

Equation (2.62) yields a set of linear algebraic equations in the arbitrary coefficients and also contains the undetermined natural circular frequency. This defines an algebraic eigenvalue problem. The solution of this eigenvalue problem generally gives *n* natural circular frequencies and *n* eigenvectors. When this eigenvector is substituted into equation (2.61), the best possible approximation solution of the plate is obtained.

$$[K]\{A\} - \omega^2 [M]\{A\} = 0 \tag{2.63}$$

where $[K]$ Stiffness matrix

$[M]$ Mass matrix

$\{A\}$ Arbitrary coefficients vector,

$$K_{kl} = \sum_{k=1}^{m}\sum_{l=1}^{n}\Bigg[D_{11}\int_0^a f_{k,xx} f_{i,xx} dx \int_0^b g_l g_j dy + D_{12}\int_0^a f_{k,xx} f_i dx \int_0^b g_l g_{j,yy} dy$$
$$+ D_{12}\int_0^a f_k f_{i,xx} dx \int_0^b g_{l,yy} g_j dy + 2D_{16}\int_0^a f_{k,xx} f_{i,x} dx \int_0^b g_l g_{j,y} dy$$
$$+ 2D_{16}\int_0^a f_k f_{i,xx} dx \int_0^b g_{l,y} g_j dy + D_{22}\int_0^a f_k f_i dx \int_0^b g_{l,yy} g_{j,yy} dy$$
$$+ 2D_{26}\int_0^a f_{k,x} f_i dx \int_0^b g_{l,y} g_{j,yy} dy + 2D_{26}\int_0^a f_k f_{i,x} dx \int_0^b g_{l,yy} g_{j,y} dy$$
$$+ 4D_{66}\int_0^a f_{k,x} f_{i,x} dx \int_0^b g_{l,y} g_{j,y} dy \Bigg] \tag{2.64}$$

$$M_{kl} = \sum_{k=1}^{m}\sum_{l=1}^{n}\left[m\omega^2 \int_0^a f_i f_k dx \int_0^b g_l g_j dy \right] \tag{2.65}$$

where $f_{i,x}, g_{j,y}$ The first differentiation with respect to the subscripted variable

$f_{i,xx}, g_{j,yy}$ The second differentiation with respect to the subscripted variable

2.3.4 The finite element method

The finite element method is a numerical method used for analyzing structure and continuum. Usually the problem addressed is too complicated to be solved satisfactorily by analytical methods. The basic idea of the finite element method is to view a given domain as an assemblage of simple geometric shapes, called elements, (Figure 2.16). These elements assemble through interconnection at a finite number of points on each element called nodes. Within the domain of each element we assume a simple general solution to the governing equations. The specific solution for each element becomes a function of unknown solution values at nodes. The application of the general solution form to all elements results in a finite set of algebraic equations is solved for unknown nodal values by using numerical procedures.

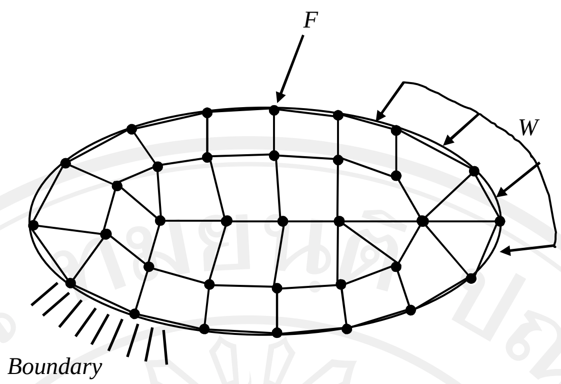


Figure 2.16 Two-dimensional continuum domain.
(Source: Knight, 1993)

The principle of virtual work states that a virtual change of the internal strain energy must be offset by an identical change in external work due to the applied loads,

$$\delta U = \delta V \tag{2.66}$$

where U Strain energy
V External work
δ Virtual operator

The virtual strain energy is

$$\delta U_1 = \int_{vol} \{\delta\varepsilon\}^T \{\sigma\}\, d(vol) \tag{2.67}$$

where $\{\varepsilon\}$ Strain vector
$\{\sigma\}$ Stress vector
vol Volume of element

For linear elastic material, the stresses is related to the strains by

$$\{\sigma\} = [C]\{\varepsilon\} \tag{2.68}$$

where $[C]$ Stiffness matrix

For small deformations, the strains is related to the displacements by

$$\{\varepsilon\} = [\partial]\{u\} \tag{2.69}$$

where $[\partial]$ Derivative operator matrix
$\{u\}$ Displacement vector

By using an assumed displacement function to define the displacement of every material point in the element

$$\{u\} = [N]\{d\} \tag{2.70}$$

where $[N]$ Element shape functions or interpolation functions
$\{d\}$ Nodal displacement

substituting equation (2.70) into (2.69), the strains may be related to the nodal displacement by

$$\{\varepsilon\} = [\partial][N]\{d\}$$
$$\{\varepsilon\} = [B]\{d\} \tag{2.71}$$

where $[B]$ Strain-displacement matrix, based on the element shape functions

substituting equation (2.71) into (2.68)

$$\{\sigma\} = [C][B]\{d\} \tag{2.72}$$

combining equation (2.72) with equation (2.71), and nothing that $\{d\}$ does not vary over the volume, equation (2.67) can be written as

$$\delta U_1 = \{\delta d\}^T \int_{vol} [B]^T [C][B]\{d\} d(vol) \tag{2.73}$$

Another form of virtual strain energy is when a surface moves against a distributed resistance, as in a foundation stiffness. This may be written as

$$\delta U_2 = \int_{area_f} \{\delta w_n\}^T \{\sigma\} d(area_f) \tag{2.74}$$

where	$\{w_n\}$	Motion normal to the surface
	$\{\sigma\}$	Stress carried by the surface
	$area_f$	Area of the distributed resistance

Both $\{w_n\}$ and $\{\sigma\}$ will usually have only one non-zero component. The point-wise normal displacement is related to the nodal displacements by

$$\{w_n\} = [N_n]\{d\} \tag{2.75}$$

where $[N_n]$ Matrix of shape functions for normal motions at the surface

The stress, $\{\sigma\}$ is

$$\{\sigma\} = k\{w_n\} \tag{2.76}$$

where k Foundation stiffness in units of force per length per unit area

substituting equation (2.75) into (2.76) into equation (2.74)

$$\delta U_2 = \{\delta d\}^T k \int_{area_f} [N_n]^T [N_n]\{d\} d(area_f) \tag{2.77}$$

Next, the external virtual work will be considered. The inertial effects will be studied first

$$\delta V_1 = -\int_{vol} \{\delta w\}^T \frac{\{F^a\}}{vol} d(vol) \tag{2.78}$$

where $\{F^a\}$ Acceleration (D'Alembert) force vector

According to Newton's second law

$$\frac{\{F^a\}}{vol} = \rho \frac{\partial^2}{\partial t^2}\{w\} \tag{2.79}$$

where ρ Density

t Time

The displacements with the element are related to the nodal displacements by

$$\{w\} = [N]\{d\} \tag{2.80}$$

where $[N]$ Matrix of shape functions

combining equations (2.78), (2.79), and (2.80) and assuming that ρ is constant over the volume

$$\delta V_1 = -\{\delta d\}^T \rho \int_{vol} [N]^T [N] \frac{\partial^2}{\partial t^2}\{d\} d(vol) \tag{2.81}$$

The pressure force vector formulation starts with

$$\delta V_2 = \int_{area_p} \{\delta w_n\}^T \{P\} d(area_p) \tag{2.82}$$

where $\{P\}$ Applied pressure vector (normally contains only one non-zero component)

$area_p$ Area over which pressure acts

combining equation (2.80) and (2.82)

$$\delta V_2 = \{\delta d\}^T \int_{area_p} [N_n]^T \{P\} d(area_p) \tag{2.83}$$

Unless otherwise noted, pressures are applied to the outside surface of each element and are normal to curved surfaces, if applicable. Nodal forces applied to the element can be accounted for by

$$\delta V_3 = \{\delta d\}^T \{F_e^{nd}\} \tag{2.84}$$

where $\{F_e^{nd}\}$ Nodal forces applied to the element

All material properties for stress analysis elements are evaluated at the average temperature of each element. Finally, equations (2.66), (2.73), (2.77), (2.81), (2.83) and (2.84) may be combined to give

$$\begin{aligned}
&\{\delta d\}^T \int_{vol} [B]^T [C][B]\{d\} d(vol) - \{\delta d\}^T \int_{vol} [B]^T [C]\{e^{th}\} d(vol) \\
&+ \{\delta d\}^T k \int_{area_f} [N_n]^T [N_n]\{d\} d(area_f) = -\{\delta d\}^T \rho \int_{vol} [N]^T [N] \frac{\partial^2}{\partial^2 t}\{d\} d(vol) \\
&\quad + \{\delta d\}^T \int_{area_p} [N_n]^T \{P\} d(area_p) + \{\delta d\}^T \{F_e^{nd}\}
\end{aligned} \tag{2.85}$$

Noting that the $\{d\}^T$ vector is a set of arbitrary virtual displacements common in all of the above terms, the condition required to satisfy equation (2.85) reduces to

$$([K_e] + [K_e^f])\{d\} - \{F_e^{th}\} = [M_e]\{\ddot{d}\} + \{F_e^{pr}\} + \{F_e^{nd}\} \tag{2.86}$$

where $[K_e] = \int_{vol} [B]^T [C][B] d(vol)$ Element stiffness

$[K_e^f] = k \int_{area_f} [N_n]^T [N_n] d(area_f)$ Element foundation stiffness matrix

$\{F_e^{th}\} = \int_{vol} [B]^T [C]\{\varepsilon^{th}\} d(vol)$ Element thermal load vector

$$[M_e] = \rho \int_{vol} [N]^T [N] d(vol) \qquad \text{Element mass matrix}$$

$$\{\ddot{d}\} = \frac{\partial^2}{\partial t^2}\{d\} \qquad \text{Acceleration vector}$$

$$\{F_e^{pr}\} = \int_{area_p} [N_n]^T \{P\} d(area_p) \qquad \text{Element pressure vector}$$

Equation (2.86) represents the equilibrium equation on a one element basis. The equation of motion for an undamped system, expressed in matrix notation is

$$[M]\{\ddot{d}\} + [K]\{d\} = \{0\} \tag{2.87}$$

where $[K]$ The structure stiffness matrix

For a linear system, free vibrations will be harmonic of the form:

$$\{d\} = \{\phi\}_i \cos(\omega_i t) \tag{2.88}$$

where $\{\phi\}_i$ Eigenvector representing the mode shape of the i^{th} natural frequency

ω_i The i^{th} natural circular frequency

Thus, equation (2.88) becomes:

$$\left(-\omega_i^2[M] + [K]\right)\{\phi\}_i = \{0\} \tag{2.89}$$

This equality is satisfied if either $\{\phi\}_i = \{0\}$ or if the determinant of $\left([K] - \omega^2[M]\right)$ is zero. The first option is the trivial one and, therefore, is not of interest. Thus, the second one gives the solution:

$$\left|[K] - \omega^2[M]\right| = 0 \tag{2.90}$$

This is an eigenvalue problem which may be solved for up to n values of ω^2 and n eigenvectors $\{\phi\}_i$ which satisfy equation (2.89) where n is the number of DOFs.

Rather than outputting the natural circular frequencies (ω), the natural frequencies (f) are output; where:

$$f_i = \frac{\omega_i}{2\pi} \tag{2.91}$$

where f_i The i[th] natural frequency