CHAPTER 3
MATERIALS AND METHODS

3.1 Research method

To achieve the research objective, the research method is divided into four parts

(1) Study the extended Kantorovich method to evaluate the mode shapes and the natural frequencies of symmetrically laminated composite rectangular plates with various boundary conditions.

(2) Study the Rayleigh-Ritz method to evaluate the mode shapes and the natural frequencies of symmetrically laminated composite rectangular plates with various boundary conditions.

(3) Study the finite element method to evaluate the mode shapes and the natural frequencies of symmetrically laminated composite rectangular plates with various boundary conditions.

(4) Comparison the mode shapes and the natural frequencies obtained by the extended Kantorovich method and the Rayleigh-Ritz method with the finite element method.

The numerical examples of some lamination schemes (Table 3.1) are explained in each study to gain understanding.

3.2 Study the extended Kantorovich method

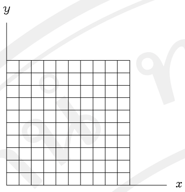
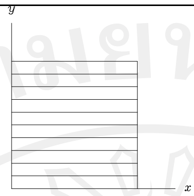
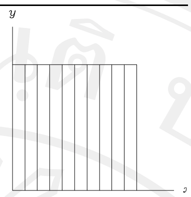
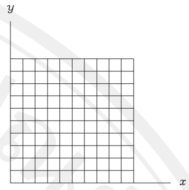
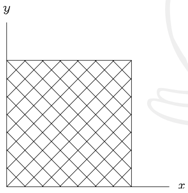
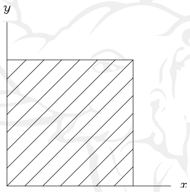
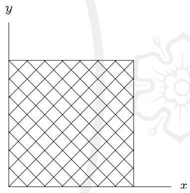
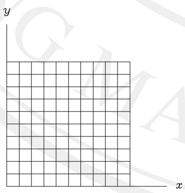
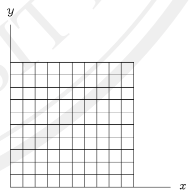
The extended Kantorovich method is applied to the first natural frequency of [0/90/90/0] laminated composite square plates with CCCC boundary conditions (Figure 3.1). The extended Kantorovich method can be divided into three steps:

(1) Bending stiffness calculation for [0/90/90/0]

(2) Assume a basis function

(3) The extended Kantorovich calculation

Table 3.1 Schematic of numerical examples of each study

Study					Numerical method	
No.	Detail				Chapter 3	Method
1	Lamination schemes	 Cross-ply	 Unidirectional 0°	 Unidirectional 90°	 [0/90/90/0]	EKM
	Boundary conditions	15			CCCC	
	Number of frequency				1	
2	Lamination schemes	 Angle-ply 45°	 Unidirectional 45°		 [45/-45/-45/45]	EKM+RRM
	Boundary conditions	15			CCCC	
	Number of frequency				5	
3	Lamination schemes	 Cross-ply			 [0/90/90/0]	FEM
	Boundary conditions	CCCC			CCCC	
	Number of frequency	100			100	

3.2.1 Bending stiffness calculation for [0/90/90/0]

The bending stiffness calculation of plate concerns with the material properties and the fiber orientations. Combination equation (2.7) with equation (2.9), the compliance matrix can be written as

$$[S]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} = \begin{bmatrix} \frac{1}{138} & -\frac{0.3}{138} & 0 \\ -\frac{0.3}{138} & \frac{1}{8.96} & 0 \\ 0 & 0 & \frac{1}{7.10} \end{bmatrix} \quad (GPa)^{-1} \qquad (3.1)$$

inverting equation (3.1) the elements of the stiffness can be written as

$$\begin{aligned} C_{11} &= 138.800 \quad GPa \\ C_{12} &= 2.704 \quad GPa \\ C_{22} &= 9.013 \quad GPa \\ C_{66} &= 7.100 \quad GPa \end{aligned} \qquad (3.2)$$

combining equation (3.2), (2.15) and (2.20), the elements of the plane stress-reduce stiffness matrices of $\theta = 0°$ and $90°$ can be written as

$$[\overline{Q}]^0 = \begin{bmatrix} 138.800 & 2.704 & 0 \\ 2.704 & 9.013 & 0 \\ 0 & 0 & 7.100 \end{bmatrix} \quad GPa \qquad (3.3)$$

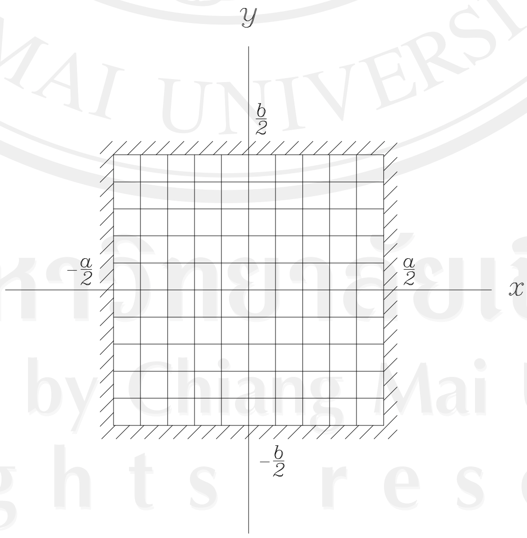


Figure 3.1 [0/90/90/0] square plates with CCCC boundary conditions.

$$[\overline{Q}]^{90} = \begin{bmatrix} 901.000 & 2.704 & 0 \\ 2.704 & 1.38 & 0 \\ 0 & 0 & 7.100 \end{bmatrix} \quad GPa \qquad (3.4)$$

substituting equation (3.3) and (3.4) into equation (2.27), the elements of the bending stiffness matrices of [0/90/90/0], can be written as

$$[D] = \frac{(h/2)^3 - (h/4)^3}{3} \begin{bmatrix} 138.800 & 2.704 & 0 \\ 2.704 & 9.013 & 0 \\ 0 & 0 & 7.100 \end{bmatrix}$$

$$+ \frac{(h/4)^3 - (0)^3}{3} \begin{bmatrix} 9.01\times10^{11} & 2.70\times10^{9} & 0 \\ 2.70\times10^{9} & 1.38\times10^{9} & 0 \\ 0 & 0 & 7.10\times10^{9} \end{bmatrix}$$

$$+ \frac{(0)^3 - (-h/4)^3}{3} \begin{bmatrix} 9.01\times10^{11} & 2.70\times10^{9} & 0 \\ 2.70\times10^{9} & 1.38\times10^{9} & 0 \\ 0 & 0 & 7.10\times10^{9} \end{bmatrix}$$

$$+ \frac{(-h/4)^3 - (-h/2)^3}{3} \begin{bmatrix} 1.38\times10^{11} & 2.70\times10^{9} & 0 \\ 2.70\times10^{9} & 9.01\times10^{9} & 0 \\ 0 & 0 & 7.10\times10^{9} \end{bmatrix}$$

$$[D] = \begin{bmatrix} 10.210 & 0.225 & 0 \\ 0.225 & 2.103 & 0 \\ 0 & 0 & 0.591 \end{bmatrix} h^3 \quad \frac{GPa}{m^3} \qquad (3.5)$$

3.2.2 Assume a basis function

A basis function in the extended Kantorovich method is neither required to satisfy the geometric nor the force boundary conditions because the iterative procedure will force the solution to satisfy all boundary conditions eventually. By Table 3.2, the transverse vibration of a beam is used as a basis function. The first iteration chooses the first natural frequency of the clamped-clamped beam as a basis function, equation (3.6).

Table 3.2 Common boundary conditions for the transverse vibration of a beam. (Source: Rao, 1999)

End conditions of beam	Frequency Equation	Mode shape (Normal function)	Value of $\beta_n l$
Simply-Simply	$\sin\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x]$	$\beta_1 l = \pi$ $\beta_2 l = 2\pi$ $\beta_3 l = 3\pi$ $\beta_4 l = 4\pi$
Free-Free	$\cos\beta_n l \cdot \cosh\beta_n l = 1$	$W_n(x) = C_n[\sin\beta_n x + \sinh\beta_n x + \alpha_n(\cos\beta_n x + \cosh\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l - \sinh\beta_n l}{\cosh\beta_n l - \cos\beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$
Clamped-Clamped	$\cos\beta_n l \cdot \cosh\beta_n l = 1$	$W_n(x) = C_n[\sinh\beta_n x - \sin\beta_n x + \alpha_n(\cosh\beta_n x - \cos\beta_n x)]$ where $\alpha_n = \left(\frac{\sinh\beta_n l - \sin\beta_n l}{\cos\beta_n l - \cosh\beta_n l}\right)$	$\beta_1 l = 4.730041$ $\beta_2 l = 7.853205$ $\beta_3 l = 10.995608$ $\beta_4 l = 14.137165$
Clamped-Free	$\cos\beta_n l \cdot \cosh\beta_n l = -1$	$W_n(x) = C_n[\sin\beta_n x - \sinh\beta_n x - \alpha_n(\cos\beta_n x - \cosh\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l + \sinh\beta_n l}{\cos\beta_n l + \cosh\beta_n l}\right)$	$\beta_1 l = 1.875104$ $\beta_2 l = 4.694091$ $\beta_3 l = 7.854757$ $\beta_4 l = 10.995541$
Clamped-Simply	$\tan\beta_n l - \tanh\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x - \sinh\beta_n x + \alpha_n(\cosh\beta_n x - \cos\beta_n x)]$ where $\alpha_n = \left(\frac{\sin\beta_n l - \sinh\beta_n l}{\cos\beta_n l - \cosh\beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$
Simply-Free	$\tan\beta_n l - \tanh\beta_n l = 0$	$W_n(x) = C_n[\sin\beta_n x + \alpha_n \sinh\beta_n x$ where $\alpha_n = \left(\frac{\sin\beta_n l}{\sinh\beta_n l}\right)$	$\beta_1 l = 3.926602$ $\beta_2 l = 7.068583$ $\beta_3 l = 10.210176$ $\beta_4 l = 13.351768$

$$X_0 = \cos(4.730) - \frac{\cos\left(\frac{4.730}{2}\right)}{\cosh\left(\frac{4.730}{2}\right)} \cosh(4.730x) \tag{3.6}$$

3.2.3 The extended Kantorovich calculation

In the extended Kantorovich calculation procedures (Figure 3.2), starting with substitute equation (3.6) into equation (2.40) to calculate S_{1x} through S_{4x}

$$S_{1x} = \int_{-a/2}^{a/2} \left(\frac{\partial^2 X}{\partial x^2}\right)^2 dx = 254.699 \qquad S_{2x} = \int_{-a/2}^{a/2} \left(X \frac{\partial^2 X}{\partial x^2}\right) dx = -6.260$$

$$S_{3x} = \int_{-a/2}^{a/2} X^2 dx = 0.509 \qquad S_{4x} = \int_{-a/2}^{a/2} \left(\frac{\partial X}{\partial x}\right)^2 dx = 6.260 \tag{3.7}$$

apply the clamped edges boundary conditions from equation (2.28), $(w)_{y=\pm b/2} = 0$ and $\left(\frac{\partial w}{\partial y}\right)_{y=\pm b/2} = 0$, into the solution equation (2.51), yields

$$Y_1 = C_{2y}\cos(q_1 y) + C_{4y}\cosh(q_3 y)$$

$$Y_1\big|_{b/2} = C_{2y}\cos\left(\frac{q_1}{2}\right) + C_{4y}\cosh\left(\frac{q_3}{2}\right) = 0 \tag{3.8}$$

$$\left.\frac{\partial Y_1}{\partial y}\right|_{b/2} = -C_{2y} q_1 \sin\left(\frac{q_1}{2}\right) + C_{4y} q_2 \sinh\left(\frac{q_3}{2}\right) = 0 \tag{3.9}$$

by equation (3.8) and (3.9), the solution equation can be written as equation (3.10) and the eigenvalue problem can be written as equation (3.11)

$$Y_1 = C_{2y}\cos(q_1 y) + C_{2y} \frac{q_1 \sin\left(\frac{q_1}{2}\right)}{q_2 \sinh\left(\frac{q_3}{2}\right)} \cosh(q_3 y) \tag{3.10}$$

$$q_3 \tanh\left(\frac{q_3}{2}\right) + q_1 \tan\left(\frac{q_1}{2}\right) = 0 \tag{3.11}$$

Substitute equation (3.7) and (3.5) into equation (2.52) to use q_3 as a function of q_1

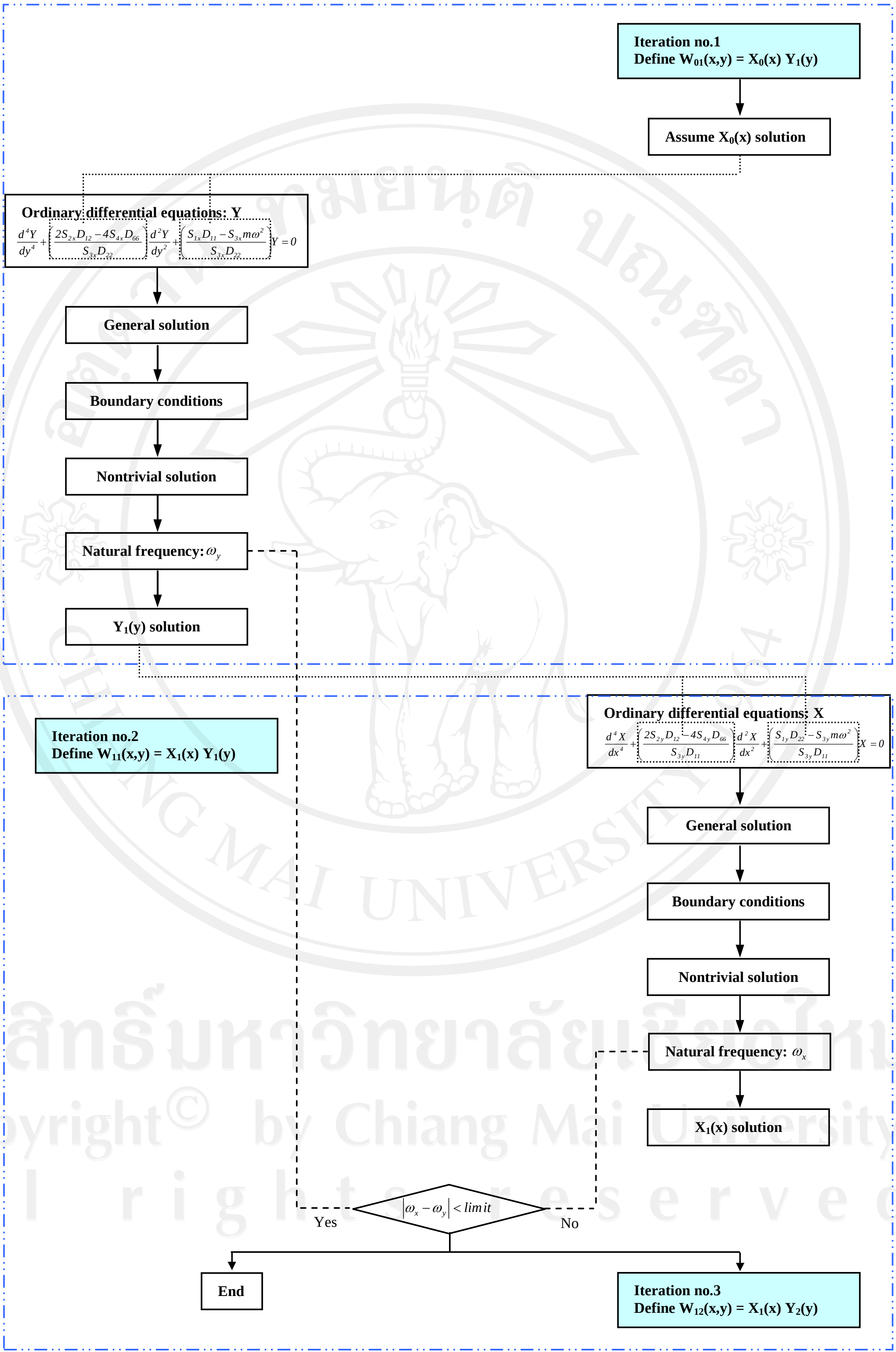


Figure 3.2 The extended Kantorovich calculation procedures.

$$q_3 = \sqrt{q_1^2 - \frac{2S_{2x}D_{12} - 4S_{4x}D_{66}}{S_{3x}D_{22}}} = \sqrt{16.4804 + q_1^2} \tag{3.12}$$

combining equation (3.12) with (3.11) to find the eigenvalue of modal parameter q_1. By Figure 3.3, the first eigenvalue of modal parameter $q_1 = 4.4155$, substituting q_1 into equation (3.12) obtained $q_3 = 5.9981$. Substitute q_1 and q_3 into equation (3.10) can be written the solution equation as

$$Y_1 = \cos(4.4155y) + 0.0591\cosh(5.9981y) \tag{3.13}$$

In the second iteration, the solution equation (3.13) from the first iteration is substituted into equation (2.47) to calculate S_{1y} through S_{4y}

$$S_{1y} = \int_{-b/2}^{b/2} \left(\frac{\partial^2 Y}{\partial y^2}\right)^2 dy = 227.116 \qquad S_{2y} = \int_{-b/2}^{b/2} \left(Y\frac{\partial^2 Y}{\partial y^2}\right) dy = -5.46309$$

$$S_{3y} = \int_{-b/2}^{b/2} Y^2 dy = 0.452145 \qquad S_{4y} = \int_{-b/2}^{b/2} \left(\frac{\partial Y}{\partial y}\right)^2 dy = 5.46309 \tag{3.14}$$

apply the clamped edges boundary conditions from equation (2.28), $(w)_{x=\pm a/2} = 0$ and $\left(\frac{\partial w}{\partial y}\right)_{x=\pm a/2} = 0$, into the solution equation (2.58), yields

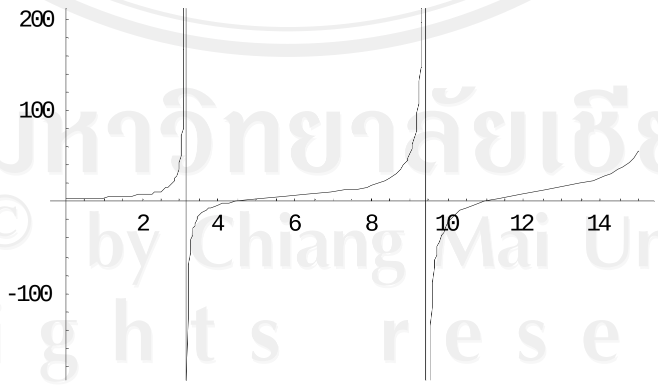


Figure 3.3 The eigenvalue of modal parameters q_1 on the first iteration.

$$X_1 = C_{2x}\cos(p_1 x) + C_{4x}\cosh(p_3 x)$$

$$X_1\big|_{a/2} = C_{2x}\cos\left(\frac{p_1}{2}\right) + C_{4x}\cosh\left(\frac{p_3}{2}\right) = 0 \tag{3.15}$$

$$\left.\frac{\partial X_1}{\partial x}\right|_{x/2} = -C_{2x} p_1 \sin\left(\frac{p_1}{2}\right) + C_{4x} p_2 \sinh\left(\frac{p_3}{2}\right) = 0 \tag{3.16}$$

by equation (3.15) and (3.16), the solution equation can be written as equation (3.17) and the eigenvalue problem can be written as equation (3.18)

$$X_1 = C_{2x}\cos(p_1 x) + C_{2x}\frac{p_1 \sin\left(\frac{p_1}{2}\right)}{p_2 \sinh\left(\frac{p_2}{2}\right)}\cosh(p_3 x) \tag{3.17}$$

$$p_3 \tanh\left(\frac{p_3}{2}\right) + p_1 \tan\left(\frac{p_1}{2}\right) = 0 \tag{3.18}$$

Substitute equation (3.14) and (3.5) into equation (2.49) to use p_3 as a function of p_1

$$p_3 = \sqrt{p_1^2 - \frac{2S_{2y}D_{12} - 4S_{4y}D_{66}}{S_{3y}D_{11}}} = \sqrt{3.33223 + p_1^2} \tag{3.19}$$

combining equation (3.19) with equation (3.18) to find the eigenvalue of modal parameter p_1. By Figure 3.4, the first eigenvalue of modal parameter $p_1 = 4.65436$, substituting p_1 into equation (3.19) obtained $p_3 = 4.99953$. Substitute p_1 and p_3 into equation (3.17) can be written the solution equation as

$$X_1 = \cos(4.65436x) + 0.111941\cosh(4.99953x) \tag{3.20}$$

In the third iteration, the solution equation (3.20) from the second iteration is substituted into equation (2.40) to calculate S_{1x} through S_{4x}

$$S_{1x} = \int_{-a/2}^{a/2}\left(\frac{\partial^2 X}{\partial x^2}\right)^2 dx = 246.299 \qquad S_{2x} = \int_{-a/2}^{a/2}\left(X\frac{\partial^2 X}{\partial x^2}\right)dx = -6.06841$$

$$S_{3x} = \int_{-a/2}^{a/2} X^2 dx = 0.491969 \qquad S_{4x} = \int_{-a/2}^{a/2} \left(\frac{\partial X}{\partial x}\right)^2 dx = 6.02841 \qquad (3.21)$$

apply the clamped edges boundary conditions from equation (2.28), $(w)_{y=\pm b/2} = 0$ and $\left(\frac{\partial w}{\partial y}\right)_{y=\pm b/2} = 0$, into the solution equation (2.51), yields

$$Y_2 = C_{2y}\cos(q_1 y) + C_{4y}\cosh(q_3 y)$$

$$Y_2\big|_{b/2} = C_{2y}\cos\left(\frac{q_1}{2}\right) + C_{4y}\cosh\left(\frac{q_3}{2}\right) = 0 \qquad (3.22)$$

$$\left.\frac{\partial Y_2}{\partial y}\right|_{b/2} = -C_{2y}q_1\sin\left(\frac{q_1}{2}\right) + C_{4y}q_2\sinh\left(\frac{q_3}{2}\right) = 0 \qquad (3.23)$$

by equation (3.22) and (3.23), the solution equation can be written as equation (3.24) and the eigenvalue problem can be written as equation (3.25)

$$Y_2 = C_{2y}\cos(q_1 y) + C_{2y}\frac{q_1\sin\left(\frac{q_1}{2}\right)}{q_2\sinh\left(\frac{q_3}{2}\right)}\cosh(q_3 y) \qquad (3.24)$$

$$q_3\tanh\left(\frac{q_3}{2}\right) + q_1\tan\left(\frac{q_1}{2}\right) = 0 \qquad (3.25)$$

Substitute equation (3.21) and (3.5) into equation (2.52) to use q_2 as a function of q_1

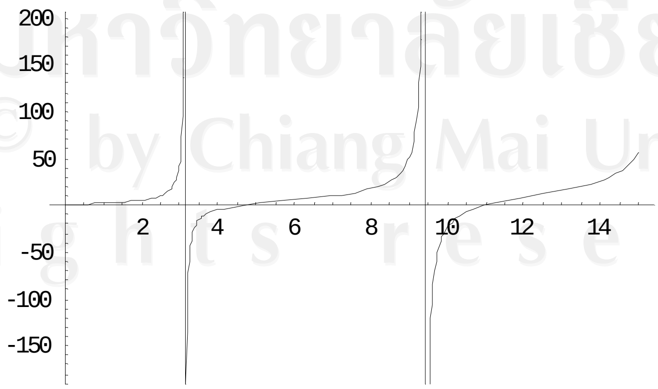


Figure 3.4 The eigenvalue of modal parameters p_1 on the second iteration.

$$q_3 = \sqrt{q_1^2 - \frac{2S_{2x}D_{12} - 4S_{4x}D_{66}}{S_{3x}D_{22}}} = \sqrt{16.4148 + q_1^2} \qquad (3.26)$$

combining equation (3.26) with (3.25) to find the eigenvalue of modal parameter q_1. By Figure 3.5, the first eigenvalue of modal parameter $q_1 = 4.4165$, substituting q_1 into equation (3.26) obtained and $q_3 = 5.9933$. Substitute q_1 and q_3 into equation (3.24) can be written the solution equation as

$$Y_2 = \cos(4.4165y) + 0.0593\cosh(5.9933y) \qquad (3.27)$$

In the fourth iteration, the solution equation (3.27) from the third iteration is substituted into equation (2.47) to calculate S_{1y} through S_{4y}

$$S_{1y} = \int_{-b/2}^{b/2} \left(\frac{\partial^2 Y}{\partial y^2}\right)^2 dy = 227.172 \qquad S_{2y} = \int_{-b/2}^{b/2} \left(Y\frac{\partial^2 Y}{\partial y^2}\right) dy = -5.46496$$

$$S_{3y} = \int_{-b/2}^{b/2} Y^2 dy = 0.452272 \qquad S_{4y} = \int_{-b/2}^{b/2} \left(\frac{\partial Y}{\partial y}\right)^2 dy = 5.46496 \qquad (3.28)$$

apply the clamped edges boundary conditions from equation (2.28), $(w)_{x=\pm a/2} = 0$ and $\left(\frac{\partial w}{\partial y}\right)_{x=\pm a/2} = 0$, into the solution equation (2.57), yields

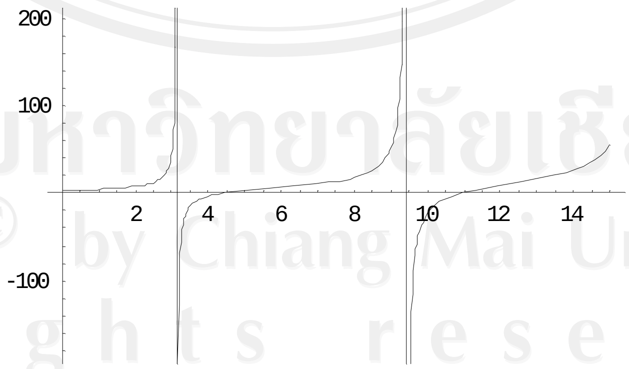


Figure 3.5 The eigenvalue of modal parameters q_1 on the third iteration.

$$X_2 = C_{2x}\cos(p_1 x) + C_{4x}\cosh(p_3 x)$$

$$X_2\big|_{a/2} = C_{2x}\cos\left(\frac{p_1}{2}\right) + C_{4x}\cosh\left(\frac{p_3}{2}\right) = 0 \tag{3.29}$$

$$\left.\frac{\partial X_2}{\partial x}\right|_{x/2} = -C_{2x}p_1\sin\left(\frac{p_1}{2}\right) + C_{4x}p_2\sinh\left(\frac{p_3}{2}\right) = 0 \tag{3.30}$$

by equation (3.29) and (3.30), the solution equation can be written as equation (3.31) and the eigenvalue problem can be written as equation (3.32)

$$X_2 = C_{2x}\cos(p_1 x) + C_{2x}\frac{p_1\sin\left(\frac{p_1}{2}\right)}{p_2\sinh\left(\frac{p_3}{2}\right)}\cosh(p_3 x) \tag{3.31}$$

$$p_3\tanh\left(\frac{p_3}{2}\right) + p_1\tan\left(\frac{p_1}{2}\right) = 0 \tag{3.32}$$

Substitute equation (3.28) and (3.5) into equation (2.58) to use p_3 as a function of p_1

$$p_3 = \sqrt{p_1^2 - \frac{2S_{2y}D_{12} - 4S_{4y}D_{66}}{S_{3y}D_{11}}} = \sqrt{3.33243 + p_1^2} \tag{3.33}$$

combining equation (3.33) with (3.32) to find the eigenvalue of modal parameter p_1. By Figure 3.6, the first eigenvalue of modal parameter $p_1 = 4.65436$, substituting p_1 into equation (3.33) obtained $p_3 = 4.99955$. Substitute p_1 and p_3 into equation (3.31) can be written the solution equation as

$$X_2 = \cos(4.65436x) + 0.11194\cosh(4.99955x) \tag{3.34}$$

The fourth iteration is the end iteration, due to the modal parameter in the *x* coordinate direction of the second and fourth iteration being identified, where $p_1 = 4.654$ and $p_3 = 4.999$. The plate mode shape is the product of the eigenvector in the *x* and *y* coordinate directions from the fourth and third iteration, respectively, $w(x,y) = X_2(x)\,Y_2(y)$.

The natural frequency is evaluated by substituting q_1, q_3, S_{1x} and S_{3x} obtained from the third iteration into equation (2.53)

$$f = \frac{1}{2\pi}\sqrt{\frac{(q_1^2 q_3^2)(S_{3x}D_{22}) + S_{1x}D_{11}}{S_{3x}m}} = 102.125 \qquad Hz$$

or by substituting p_1, p_3, S_{1y} and S_{3y} obtained from the fourth iteration, into equation (2.59)

$$f = \frac{1}{2\pi}\sqrt{\frac{(p_1^2 p_3^2)(S_{3y}D_{11}) + S_{1y}D_{22}}{S_{3y}m}} = 102.125 \qquad Hz$$

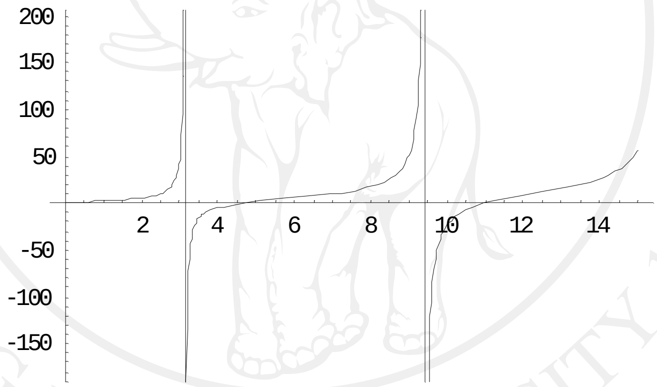


Figure 3.6 The eigenvalue of modal parameters p_1 on the fourth iteration.

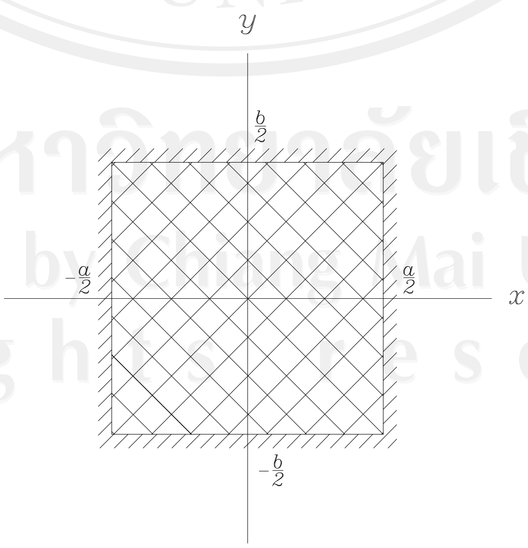


Figure 3.7 [45/-45/-45/45] square plates with CCCC boundary conditions.

3.3 Study the Rayleigh-Ritz method

For numerical calculation, the Rayleigh-Ritz method is applied to the first nine natural frequencies of [45/-45/-45/45] laminated composite square plates with CCCC boundary conditions (Figure 3.7). The Rayleigh-Ritz method can divide into three steps:

(1) Bending stiffness calculation for [45/-45/-45/45]

(2) Assume a basis function

(3) The Rayleigh-Ritz calculation

3.3.1 Bending stiffness calculation for [45/-45/-45/45]

The bending stiffness calculation for [45/-45/-45/45] is similar to [0/90/90/0], except substitution the fiber orientation. Combining equation (3.2), (2.15) and (2.20), the elements of the plane stress-reduce stiffness matrices of $\theta = 45°$ and -45° can be written as

$$[\overline{Q}]^{45} = \begin{bmatrix} 45.400 & 31.200 & 32.400 \\ 31.200 & 45.400 & 32.400 \\ 32.400 & 32.400 & 35.600 \end{bmatrix} \quad GPa \qquad (3.35)$$

$$[\overline{Q}]^{135} = \begin{bmatrix} 45.400 & 31.200 & -32.400 \\ 31.200 & 45.400 & -32.400 \\ -32.400 & -32.400 & 35.600 \end{bmatrix} \quad GPa \qquad (3.36)$$

substituting equation (3.35) and (3.36) into equation (2.27), the elements of the bending stiffness matrices of [45/-45/-45/45], can be written as

$$[D] = \frac{(h/2)^3 - (h/4)^3}{3} \begin{bmatrix} 4.54\times10^{10} & 3.12\times10^{10} & 3.24\times10^{10} \\ 3.12\times10^{10} & 4.54\times10^{10} & 3.24\times10^{10} \\ 3.24\times10^{10} & 3.24\times10^{10} & 3.56\times10^{10} \end{bmatrix}$$

$$+ \frac{(h/4)^3 - (0)^3}{3} \begin{bmatrix} 4.54\times10^{10} & 3.12\times10^{10} & -3.24\times10^{10} \\ 3.12\times10^{10} & 4.54\times10^{10} & -3.24\times10^{10} \\ -3.24\times10^{10} & -3.24\times10^{10} & 3.56\times10^{10} \end{bmatrix}$$

$$+\frac{(0)^3-(-h/4)^3}{3}\begin{bmatrix} 4.54\times10^{10} & 3.12\times10^{10} & -3.24\times10^{10} \\ 3.12\times10^{10} & 4.54\times10^{10} & -3.24\times10^{10} \\ -3.24\times10^{10} & -3.24\times10^{10} & 3.56\times10^{10} \end{bmatrix}$$

$$+\frac{(-h/4)^3-(-h/2)^3}{3}\begin{bmatrix} 4.54\times10^{10} & 3.12\times10^{10} & 3.24\times10^{10} \\ 3.12\times10^{10} & 4.54\times10^{10} & 3.24\times10^{10} \\ 3.24\times10^{10} & 3.24\times10^{10} & 3.56\times10^{10} \end{bmatrix}$$

$$[D]=\begin{bmatrix} 3.783 & 2.600 & 2.028 \\ 2.600 & 3.783 & 2.028 \\ 2.028 & 2.028 & 2.967 \end{bmatrix} h^3 \qquad \frac{GPa}{m^3} \tag{3.37}$$

3.3.2 Assume a basis function

The Rayleigh-Ritz method considers a linear combination of the several basis functions satisfied the boundary conditions. The basis functions are obtained from the first nine natural frequencies of [0/90/90/0] laminated composite square plates with CCCC boundary conditions, using the extended Kantorovich method. By equation (2.52) the solution is assume as

$$\begin{aligned} w(x,y) &= A_{11} f_1(x)g_1(y) + A_{22} f_2(x)g_2(y) + A_{33} f_3(x)g_3(y) + A_{44} f_4(x)g_4(y) \\ &+ A_{55} f_5(x)g_5(y) + A_{66} f_6(x)g_6(y) + A_{77} f_7(x)g_7(y) \\ &+ A_{88} f_8(x)g_8(y) + A_{99} f_9(x)g_9(y) \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} f_1(x) &= \cos(4.6543x) + 0.1119\cosh(4.9995x) \\ f_2(x) &= \cos(4.4770x) + 0.0708\cosh(5.7149x) \\ f_3(x) &= \cos(4.2731x) + 0.0367\cosh(6.7451x) \\ f_4(x) &= \sin(7.8277x) + 0.0251\sinh(8.0307x) \\ f_5(x) &= \sin(7.7599x) + 0.0189\sinh(8.5231x) \\ f_6(x) &= \sin(7.6652x) + 0.0123\sinh(9.2664x) \\ f_7(x) &= \cos(10.9828x) - 0.0053\cosh(11.1240x) \\ f_8(x) &= \cos(10.9471x) - 0.0044\cosh(11.4918x) \\ f_9(x) &= \cos(10.8935x) - 0.0032\cosh(12.0659x) \end{aligned} \tag{3.39}$$

$$g_1(y) = \cos(4.4164y) + 0.0593\cosh(5.9933y)$$
$$g_2(y) = \sin(7.7337y) + 0.0169\sinh(8.7212y)$$
$$g_3(y) = \cos(10.9327y) - 0.0040\cosh(11.6426y)$$
$$g_4(y) = \cos(3.9932y) + 0.0101\cosh(8.8061y)$$
$$g_5(y) = \sin(7.4927y) + 0.0050\sinh(10.8403y)$$
$$g_6(y) = \cos(10.7863y) - 0.0016\cosh(13.3183y)$$
$$g_7(y) = \cos(3.7412y) + 0.0013\cosh(12.1018y)$$
$$g_8(y) = \sin(7.2639y) + 0.0010\sinh(13.6058y)$$
$$g_9(y) = \cos(10.617y) - 0.0004\cosh(15.6487y) \quad (3.40)$$

3.3.3 The Rayleigh-Ritz calculation

In the Rayleigh-Ritz calculation, starting with substitute equation (3.38) and (3.37) into equation (2.55), the elements of the stiffness matrix can be obtained as Table 3.3. Similarly, substituting equation (3.38) into equation (2.56), the elements of the mass matrix can be obtained as Table 3.4.

Combining the elements of the stiffness matrix and mass matrix into equation (2.54)

$$[K]\{A\} - \omega^2[M]\{A\} = 0 \quad (3.41)$$

where $$\{A\} = \begin{Bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{44} \\ A_{55} \\ A_{66} \\ A_{77} \\ A_{88} \\ A_{99} \end{Bmatrix} \quad (3.42)$$

Equation (3.41) leads to an algebraic eigenvalue problem, solving this eigenvalue problem obtains nine circular natural frequencies as the following

Table 3.3 The elements of the stiffness matrix

K_{ij}		*i*								
		1	2	3	4	5	6	7	8	9
j	1	1.40677x10^6	0	282614	0	-1.45804x10^6	0	388268	0	368227
	2	0	5.77011x10^6	0	1.27742x10^6	0	4.14230x10^6	0	1.47569x10^6	0
	3	282614	0	1.67394x10^7	0	-3.72410x10^6	0	171154	0	2.99419x10^6
	4	0	1.27742x10^6	0	5.38678x10^6	0	889441	0	3.80643x10^6	0
	5	-1.45804x10^6	0	-3.72410x10^6	0	1.46541x10^7	0	-3.54664x10^6	0	-9.72897x10^6
	6	0	4.14230x10^6	0	889441	0	3.39847x10^7	0	8.90015x10^6	0
	7	388268	0	171154	0	-3.54664x10^6	0	1.64780x10^7	0	1.55105x10^6
	8	0	1.47569x10^6	0	3.80643x10^6	0	8.90015x10^6	0	3.29645x10^7	0
	9	368227	0	2.99419x10^6	0	-9.72897x10^6	0	1.55105x10^6	0	6.44739x10^7

Table 3.4 The elements of the mass matrix

M_{ij}		*i*								
		1	2	3	4	5	6	7	8	9
	1	0.22250	0	-0.00004	0	0	0	-0.00003	0	0.00013
	2	0	0.22389	0	0	0	0	0	-0.00005	0
	3	-0.00004	0	0.21628	0	0	0	0.00012	0	-0.00003
	4	0	0	0	0.21111	0	-0.00004	0	0	0
j	5	0	0	0	0	0.22848	0	0	0	0
	6	0	0	0	-0.00004	0	0.23236	0	0	0
	7	-0.00003	0	0.00012	0	0	0	0.21540	0	-0.00001
	8	0	-0.00005	0	0	0	0	0	0.227619	0
	9	0.00013	0	-0.00003	0	0	0	-0.00001	0	0.23417

$$\omega_1^2 = 5.5656\times10^6 \qquad \omega_6^2 = 8.8665\times10^7$$
$$\omega_2^2 = 1.7999\times10^7 \qquad \omega_7^2 = 1.0859\times10^8$$
$$\omega_3^2 = 2.8105\times10^7 \qquad \omega_8^2 = 1.8767\times10^8$$
$$\omega_4^2 = 4.4172\times10^7 \qquad \omega_9^2 = 2.8517\times10^8$$
$$\omega_5^2 = 7.6102\times10^7 \tag{3.43}$$

substituting each circular natural frequency into equation (3.41) to obtain eigenvector. For example, substituting $\omega_1^2 = 5.5656\times10^6$ into equation (3.41), the eigenvector can be obtained as the following

$$\{A\} = \begin{Bmatrix} 1.00000 \\ 1.24174\times10^{-16} \\ 0.00829 \\ -3.37652\times10^{-17} \\ 0.12054 \\ -2.38982\times10^{-17} \\ 0.00121 \\ 5.01901\times10^{-18} \\ 0.01232 \end{Bmatrix} \tag{3.44}$$

substituting equation (3.44) into equation (3.38), the approximation solution of each plate mode shape is obtained.

To obtain the natural frequency, the circular natural frequency is substituted into the following equation

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{5.56560\times10^6}{m}} = 93.867 \qquad Hz$$

3.4 Study the finite element method

For numerical calculation, the finite element method is applied to the natural frequencies of [0/90/90/0] laminated composite square plates with CCCC boundary conditions (Figure 3.1). The finite element method typically involves three steps (Table 3.5):

(1) Pre processing steps

(2) Processing steps

(3) Post processing steps

3.4.1 Pre processing step

The model generation is conducted in this processor, involving material definition, creation of a model, and, finally, meshing. The important tasks within this processor are:

(1) Specify the type of element

(2) Define the real constants

(3) Define Material properties

(4) Create the model geometry

(5) Generate the mesh

3.4.2 Processing step

This processor is used for obtaining the solution for the finite element model that is generated within the Pre processing step. The important tasks within this processor are:

(1) Define analysis type and analysis options

(2) Specify boundary conditions

(3) Obtain solution

3.4.3 Post processing step

In this step, the results over the entire or a portion of the model are reviewed. This includes the plotting of contours, vector displays, deformed shapes, and listings of the results.

Table 3.5 The finite element steps.

Step	Detail	Description
Pre processing	Type of element:	
	Element	Shell with 100 layers
	Geometry	Quadrilateral and Triangle
	Node	8
	Degree of freedom	6
	Real constants:	
	Number of layers	4
	Lamination scheme	[0/90/90/0]
	Layer thickness (m)	0.0025
	Material properties:	
	Young's modulus (GPa)	E_x = 138, E_y = 8.96, E_z = 8.96
	Shear modulus (GPa)	G_{xy} = 7.10, G_{yz} = 2.82, G_{xz} = 7.10
	Poisson's ratio	ν_{xy} = 0.30, ν_{yz} = 0.59, ν_{xz} = 0.30
	Density (kg/m^3)	1600
	Model geometry:	
	Width (m)	1
	Length (m)	1
	Number of element (Figure 3.8):	64x64
Processing	Analysis type:	Modal
	Option for analysis:	
	Method	Subspace
	Number of modes to extract	100
	Number of modes to expand	100
	Boundary conditions:	CCCC
	Solve the system:	Current LS
Post processing	View natural frequencies:	
	View mode shapes:	

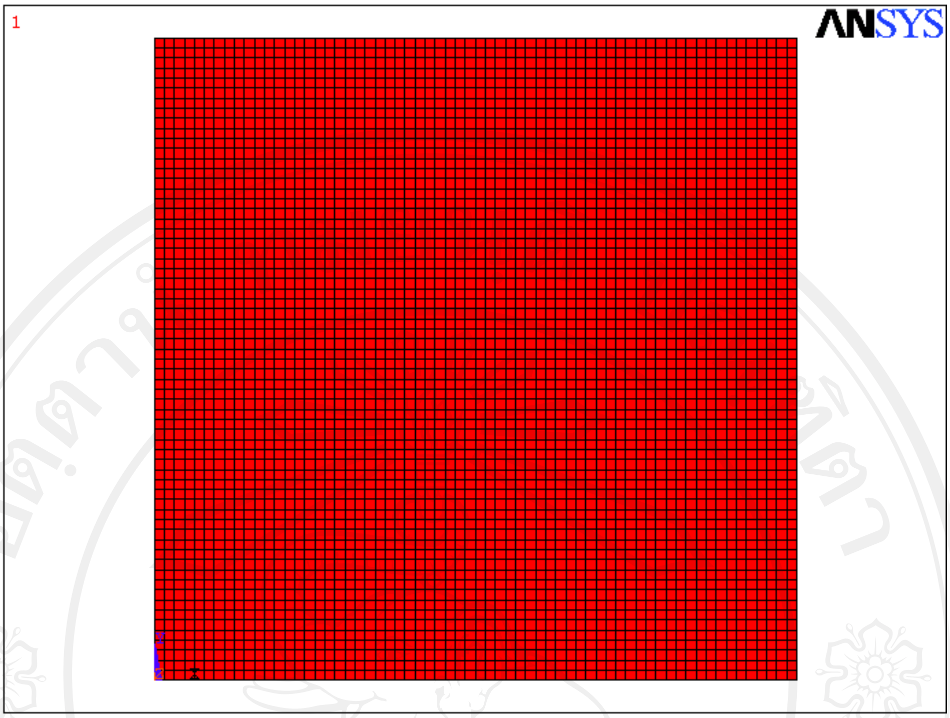


Figure 3.8 Mesh element by ANSYS.

Table 3.6 The convergence of the number of element of the first natural frequency of [0/90/90/0] laminated composite square plates.

Boundary conditions	Number of element					
	2x2	4x4	8x8	16x16	32x32	64x64
CCCC	89.087	100.588	101.086	101.504	101.558	101.558
CCCS	86.286	95.144	96.431	96.872	96.927	96.927
CCSS	67.547	71.164	71.726	71.924	71.949	71.949
CFCC	78.816	87.636	89.978	90.526	90.605	90.605
CFCF	78.110	87.030	88.901	89.349	89.405	89.405
CFCS	78.664	87.383	89.556	90.062	90.132	90.132
CFSC	57.985	62.094	63.124	63.374	63.404	63.404
CFSF	57.651	60.838	61.570	61.748	61.771	61.771
CFSS	57.754	61.583	62.498	62.718	62.742	62.742
CSCS	83.499	91.839	93.552	94.004	94.059	94.059
CSSS	63.207	67.047	67.846	68.048	68.072	68.072
FSCS	24.904	25.671	25.971	26.047	26.063	26.063
FSFS	17.935	17.965	17.981	17.986	17.986	17.986
SSFS	19.866	20.534	20.702	20.744	20.752	20.752
SSSS	45.264	47.607	48.044	48.133	48.144	48.144

By Table 3.6, the number of element is 64x64 elements. In general, a large number of elements provide a better approximation of the solution. Therefore, it is important that the mesh is adequately fine or coarse in the appropriate regions. An analysis with an initial mesh is performed first and then reanalyzed by using twice as many elements. The two solutions are compared. If the results are close to each other, the initial mesh configuration is considered to be adequate. If there are substantial differences between the two, the analysis should continue with a more-refined mesh and a subsequent comparison until the convergence is established. Table 3.5 shows the convergence of the number of element of the first natural frequency of [0/90/90/0] laminated composite square plates.

3.5 Comparison of the mode shapes and the natural frequencies

The mode shapes and the natural frequencies are compared with the known solutions and the finite element method. The comparison is divided into 2 steps

(1) The comparison of the extended Kantorovich method with the known solutions such as Sakata *et al* (1996), Rajalingham *et at* (1997) and Reddy (2004).

(2) The comparison of the application of the extended Kantorovich method with the finite element method.

The first step is used to verify the accuracy of the extended Kantorovich method, if the difference satisfies the specified tolerance level, the extended Kantorovich method can be applied to the interesting work. After that the extended Kantorovich method is compared with the finite element method to verify the accuracy of the interesting work.

As a matter of the fact that the exact mode shapes of plates have straight nodal lines and curved nodal lines. The nodal line is the zero lateral displacement which is defined by the dashed line. For straight nodal lines (Figure 3.9), the notation of mode shape is (i, j) which is a number of i^{th} and j^{th} sine waves in the x and y coordinate direction, respectively. For curved nodal lines, it is complicated to classify the number of sine waves in the x or y coordinates direction. Therefore their mode shapes are conventionally ordered as 1, 2, 3,... .

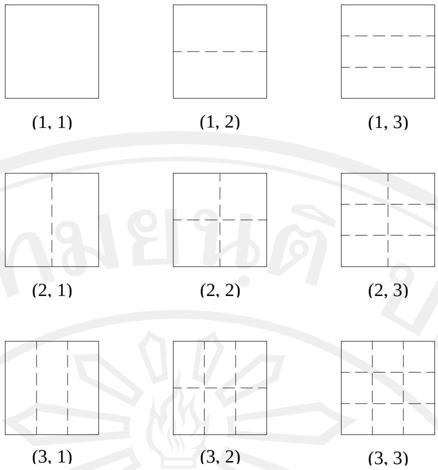


Figure 3.9 The straight nodal lines of square plates with CCCC boundary conditions.