Chapter 1

Introduction

Generalized functions have of late been commanding constantly expanding interest in several different branches of mathematics. In somewhat nonrigorous form, they have already long been used in essence by physicists and opened up a new area of mathematical research, which in turn provided an impetus in the development of a number of mathematical disciplines, such as ordinary and partial differential equations, operational calculus, transformation theory, and functional analysis.

The Fourier transform is one of the important tools used in solving differential equations and associated problems. The Fourier transform of Generalized functions as developed in the Gel'fand and Shilov book [5] requires no assumptions concerning the growth of the functions treated, and can be used for functions of any number of variables. It is thus evident that this method can be used to solve, in particular, all types of problems.

In 1988-2000, S. E. Trione [15-17] showed that the $n-$dimensional ultrahyperbolic equation, $\Box^k u(x) = \delta$, has a unique elementary solution $u(x) = R^H_{2k}(x)$, where $\Box^k$ is the ultra-hyperbolic operator iterated $k-$times, defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k, \tag{1.1}$$

$p + q = n$ is the $n-$dimensional of Euclidean space $\mathbb{R}^n$ and k is a positive integer. The function $R^H_{2k}(x)$ is called the ultra-hyperbolic kernel of the Marcel Riesz, defined by

$$R^H_{2k}(x) = \begin{cases} \frac{V^{\frac{2k-n}{2}}}{K_n(2k)} & \text{for } x \in \Gamma_+, \\ 0 & \text{for } x \notin \Gamma_+, \end{cases} \tag{1.2}$$

where $V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$, Γ_+ is the interior of the forward cone, defined by

$$\Gamma_+ = \{x \in \mathbb{R}_n : x_1 > 0 \text{ and } V > 0\},$$

and

$$K_n(2k) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{2k+2-n}{2})\Gamma(\frac{1-2k}{2})\Gamma(2k)}{\Gamma(\frac{2k+2-p}{2})\Gamma(\frac{p-2k}{2})}.$$

If $p = n$ and $q = 0$, then the ultra-hyperbolic operator reduce to the Laplace operator, that is,

$$\triangle^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)^k. \tag{1.3}$$

The elementary solution of the Laplace operator is $R^e_{2k}(x)$ and is defined by

$$R^e_{2k}(x) = 2^{-2k}\pi^{-n/2}\Gamma\left(\frac{n-2k}{2}\right)\frac{|x|^{(2k-n)}}{\Gamma(k)}, \tag{1.4}$$

where $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

Furthermore, S. E. Trione studied the elementary solution of the ultra-hyperbolic Klein-Gordon operator iterated $k-$times, defined by

$$(\square + m^2)^k = \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} + m^2\right)^k, \tag{1.5}$$

and the elementary solution of the ultra-hyperbolic Klein-Gordon operator is $W^H_{2k}(x)$, where

$$W^H_{2k}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r\Gamma(k+r)}{r!\Gamma(k)}\left(m^2\right)^r R^H_{2k+2r}(x). \tag{1.6}$$

If $p = n$ and $q = 0$, then the ultra-hyperbolic Klein-Gordon operator reduce to the Helmholtz operator, that is,

$$\triangle + m^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + m^2. \tag{1.7}$$

In 1997-2001, A. Kananthai [7-9] first introduced the Diamond operator iterated k times $\diamondsuit^k$ and the Diamond operator can be expressed by $\diamondsuit = \triangle\square = \square\triangle$ where $\triangle$ is the Laplace operator and $\square$ is the ultra-hyperbolic operator. He has proved that distribution related to the n dimensional ultra-hyperbolic equation, the solutions of the n dimensional classical diamond operator and Fourier transformation of the diamond kernel of Marcel Riesz have the solution of the convolution form $u(x) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ which is an unique elementary solution of the equation $\diamondsuit^k u(x) = \delta$. The equation $\diamondsuit^k u(x) = \sum_{r=1}^m C_r \diamondsuit^r \delta$ has been already studied and obtained the type of these solutions of such equation depends on the relationship between the values of k and m. He also studied the solution of the equation $\diamondsuit^k u(x) = f(x)$ and the nonlinear equation $\diamondsuit^k u(x) = f(x, \triangle^{k-1}\square^k u(x))$ related to wave equation.

By keeping on studying such operator continuously, we define the new operator that so call the L_m^k operator. We study the solution of the generalized heat equation and wave equation related the L_m^k operator. The such solutions play important role in constructing the high technology in the present such as telecommunication and heat transfer.

This thesis is organized as follow.

In chapter 2, we give some useful definitions and properties of the special functions, partial differential equations, distributions and elementary solutions.

In chapter 3, firstly we define by

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^m\right]^k$$

and study the elementary solution of the operator L_m^k related to the generalized heat equation and spectrum.

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\triangle u(x,t)$$

with the initial condition $u(x,0) = f(x)$, where $\triangle$ is the Laplace operator defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, we obtain the solution in the convolution form $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}e^{-\frac{|x|^2}{4c^2t}}.$$

$E(x,t)$ is call the heat kernel, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [6].

In [9], K. Nonlaopon and A. Kananthai have studied the generalized ultra-hyperbolic heat kernel of the equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\Box^k u(x,t)$$

with the initial condition $u(x,0) = f(x)$, where $\Box^k$ is the ultra-hyperbolic operator iterated k time, defined by (1.1) and c is a positive constant. They obtained $u(x,t) = E(x,t) * f(x)$ as a solution of such equation where $E(x,t)$ is the kernel of such equation, defined by

$$E(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_\Omega \exp\left[c^2t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k + i(\xi,x)\right]d\xi,$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n), x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$, $d\xi = d\xi_1\, d\xi_2 \cdots d\xi_n$ and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$.

In [14], J. Tariboon has studied the generalized diamond heat kernel of the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \diamondsuit^k u(x,t)$$

with the initial condition $u(x,0) = f(x)$, where $\diamondsuit^k$ is the diamond operator iterated k time, defined by

$$\diamondsuit^k = \left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right]^k,$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and c is a positive constant. He obtained $u(x,t) = E(x,t) * f(x)$ as a solution of such equation where $E(x,t)$ is the kernel of such equation, defined by

$$E(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega \exp\left[c^2 t \left(\left(\sum_{i=1}^{p} \xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^2\right)^k + i(\xi, x)\right] d\xi$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n), x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$, $d\xi = d\xi_1\, d\xi_2 \cdots d\xi_n$ and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$.

In this thesis we propose to study the equation

$$\frac{\partial}{\partial t} u(x,t) + c^2 L_m^k u(x,t) = 0,$$

with initial condition $u(x,0) = f(x)$ for $x \in \mathbb{R}^n$ where the operator L_m^k is defined by

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^m\right]^k$$

$p + q = n$ is the dimension of the space $\mathbb{R}^n$, $u(x,t)$ is an unknown function, $f(x)$ is a given generalized function, k and m is a positive integer and c is a positive constant.
We obtain $u(x,t) = E(x,t) * f(x)$, as a solution of such equation which satisfies $u(x,0) = f(x)$, where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t > 0$. The function $E(x,t)$ is called the kernel or elementary solution.

In chapter 4, we study the operator L_m^k related to the generalized wave equation by using ϵ approximation.

It is well known that for the 1-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t),$$

we obtain $u(x,t) = f(x+ct) + g(x-ct)$ as a solution of the equation where f and g are continuous. Also for the n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2\triangle u(x,t) = 0,$$

with the initial condition

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x)$$

where f and g are given continuous functions. By solving the Cauchy problem for such equation, the Fourier transform has been applied and the solution is given by

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos(2\pi|\xi|)\,t + \widehat{g}(\xi)\frac{\sin(2\pi|\xi|)\,t}{2\pi|\xi|},$$

where $|\xi|^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_n^2$ (see [4], p177). By using the inverse Fourier transform, we obtain $u(x,t)$ in the convolution form, that is

$$u(x,t) = f(x) * \psi(x,t) + g(x) * \phi(x,t),$$

where $\phi(\xi,t)$ is an inverse Fourier transform of $\widehat{\phi}(\xi,t) = \dfrac{\sin(2\pi|\xi|)\,t}{2\pi|\xi|}$ and $\psi(\xi,t)$ is an inverse Fourier transform of $\widehat{\psi}(\xi,t) = \cos(2\pi|\xi|)\,t = \dfrac{\partial}{\partial t}\widehat{\phi}(\xi,t)$. Sritantatana and Kananthai studied the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(-\triangle)^k u(x,t) = 0,$$

see [12], where $\triangle$ is Laplacian iterated k times, defined by

$$\triangle = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}.$$

Next, Satsanit and Kananthai studied the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2(\diamondsuit)^k u(x,t) = 0,$$

see [11], where $\Diamond$ is Diamond operator iterated k times, defined by

$$\Diamond = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2, \quad p+q=n.$$

In this thesis, we study the equation

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2 L_m^k u(x,t) = 0,$$

where L_m^k defined by

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^m\right]^k,$$

with $u(x,0) = f(x)$ and $\frac{\partial}{\partial t}u(x,0) = g(x)$ where c is a positive constant, k and m are positive integer, f and g are continuous functions and absolutely integrable. We obtain

$$u(x,t) = f(x) * \psi(x,t) + g(x) * \phi(x,t)$$

as a solution of such equation where $\phi(x,t)$ is an inverse Fourier transform of $\widehat{\phi}(\xi,t) = \dfrac{\sin ct\sqrt{(r^{2m}-s^{2m})^k}}{c\sqrt{(r^{2m}-s^{2m})^k}}$ and $\psi(x,t)$ is an inverse Fourier transform of $\widehat{\psi}(\xi,t) = \cos ct\sqrt{(r^{2m}-s^{2m})^k} = \frac{\partial}{\partial t}\widehat{\phi}(\xi,t)$ where $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$ and $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$.

Moreover, if we put $m = k = 1$ and $q = 0$, then it become the generalized n-dimensional wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2\triangle u(x,t) = 0.$$

We also study the asymptotic form of $u(x,t)$ by using ϵ approximation and obtain $u(x,t) = \mathrm{O}(\epsilon^{\frac{-n}{mk}})$.

In chapter 5, we study the nonlinear product of Laplacian related to the nonhomogeneous Biharmonic equation.

Gelfand and Shilov [5] have shown that the iterated Laplace equation $\triangle^k u(x) = f(x)$ will be solved when we have obtained an elementary solution $E(x)$. Kananthai [11], [14] has shown that $u(x) = (-1)^k R_{2k}^e(x)$ be the elementary solution of the equation $\triangle^k u(x) = \delta(x)$, where $R_{2k}^e(x)$ defined by below equation and $u(x) = ((-1)^{k-1} R_{2(k-1)}^e(x))^{(l)}$ be a solution of $\triangle^k u(x) = 0$.

R.Courant and D.Hilbert [2] have studied the nonlinear equation of the form $\triangle u(x) = f(x, u(x))$ with f defined and continuous function for all $x \in \Omega \cup \partial\Omega$ where Ω is an open set in $\mathbb{R}^n$, $\partial\Omega$ denotes the boundary of Ω and $\triangle$ is the Laplace operator, defined by

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

They found that the solution $u(x)$ of such equation is unique under the condition $|f(x, u(x))| \leq N$ for all $x \in \Omega$ where N is a constant and the boundary condition $u(x) = 0$ for all $x \in \partial\Omega$.

In [7], A. Kananthai first introduced the diamond operator $\diamondsuit^k$ iterated k times, defined by

$$\diamondsuit^k = \left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2\right]^k.$$

The equation $\diamondsuit^k u(x) = \delta(x)$ has the convolution $u(x) = (-1)^k R^e_{2k}(x) * R^H_{2k}(x)$ as an elementary solution and is called the Diamond Kernel of Marcel Riesz where $R^e_{2k}(x)$ and $R^H_{2k}(x)$ are defined by

$$R^e_\alpha(x) = 2^{-\alpha}\pi^{-n/2}\Gamma\left(\frac{n-\alpha}{2}\right)\frac{\|x\|^{\alpha-n}}{\Gamma\left(\frac{\alpha}{2}\right)}$$

and

$$R^H_\alpha(x) = \begin{cases} \frac{V^{(\alpha-n)/2}}{K_n(\alpha)} & \text{for } x \in \Gamma_+ \\ 0 & \text{for } x \notin \Gamma_+, \end{cases}$$

for

$$K_n(\alpha) = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{\alpha+2-n}{2})\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}{\Gamma(\frac{\alpha+2-p}{2})\Gamma(\frac{p-\alpha}{2})},$$

with $\alpha = 2k$,

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,$$

$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } V > 0\}$, $p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and α is a complex number.

In [13], G. Sritanratana and A. Kananthai have studied the solution of the nonlinear equation $\diamondsuit^k u(x) = f(x, \triangle^{k-1}\square^k u(x))$ where $\diamondsuit^k$ is the Diamond operator iterated k times and $\triangle^{k-1}$ is the Laplace operator iterated $k-1$ times, defined by

$$\triangle^{k-1} = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right]^{k-1},$$

and $\square^k$ is the ultra-hyperbolic operator iterated k times. They obtained that the existence of the solution $u(x)$ of such equation depends on the conditions of f and

$\triangle^{k-1}\Box^k u(x)$. Moreover such solution $u(x)$ related to the wave equation depends on the conditions of p, q and k.

In this thesis, we study the nonlinear equation of the form

$$\triangle^k(\triangle + m^2)^k u(x) = f(x, \triangle^{k-1}(\triangle + m^2)^k u(x)),$$

where $\triangle + m^2$ is Helmholtz operator, k is a positive integer, f defined and continuous for all $x \in \Omega \cup \partial\Omega$ where Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω. We can find the solution $u(x)$ which is unique under the condition $|f(x, \triangle^{k-1}(\triangle + m^2)^k u(x))| \leq N$ where N is a constant for all $x \in \Omega$ and the boundary condition $\triangle^{k-1}(\triangle + m^2)^k u(x) = 0$ for $x \in \partial\Omega$. Moreover the solution $u(x)$ related to the nonhomogeneous biharmonic equation depends on the conditions of k.