Chapter 2

Preliminaries and Basic Concepts

The aim of this chapter is to give some definitions, notations and properties of the spacial functions, distributions, elementary solutions and partial differential equations of the partial differential operators which will be used in the later chapters.

2.1 The Special Functions

In this section, we shall present the definitions and some properties of the gamma function, the Dirac-delta function.

2.1.1 The Gamma Function

Definition 2.1.1. The *gamma function* is denoted by Γ and is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \tag{2.1}$$

where z is a complex number with $\text{Re}\, z > 0$.

A result that yields an immediate analytic continuation from the left haft plane is the following properties.

Proposition 2.1.2. *Let z be a complex number. Then*

(1) $\Gamma(z) = \dfrac{\Gamma(z+1)}{z}\ , \qquad z \neq 0, -1, -2, \ldots$

(2) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}\ , \qquad z \neq 0, \pm 1, \pm 2, \ldots .$

Proposition 2.1.3. *Let z be a complex number. Then*

$$\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z), \tag{2.2}$$

for $z \neq 0, -1, -2, \ldots .$

Definition 2.1.4. The *beta function* is denote by B and is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt, \tag{2.3}$$

where x and y are complex number with Re $x>0$ and Re $y>0$.

Now, our first result is the formula

$$\int_0^\infty e^{-pt}t^{z-1}dt=\frac{\Gamma(z)}{p^z},\quad \text{Re } p>0,\ \text{Re } z>0, \tag{2.4}$$

which is easily proved for positive real p by making the change of variables $s=pt$, and then using the integral representation (2.1). The extension of (2.4) to arbitrary complex p with Re $p>0$ is accomplished by using the principle of analytic continuation.

Next consider the beta function, if we introduce the new variable of integration

$$u=\frac{t}{1-t}$$

then (2.3) becomes

$$B(x,y)=\int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}du,\quad \text{Re } x>0,\ \text{Re } y>0. \tag{2.5}$$

Let $p=1+u$ and $z=x+y$ in (2.4), we find that

$$\frac{1}{(1+u)^{x+y}}=\frac{1}{\Gamma(x+y)}\int_0^\infty e^{-(1+u)t}t^{x+y-1}dt,$$

and substituting the result into (2.5), we obtain

$$\begin{aligned} B(x,y) &= \frac{1}{\Gamma(x+y)}\int_0^\infty\left(\int_0^\infty e^{-ut}u^{x-1}du\right)e^{-t}t^{x+y-1}dt \\ &= \frac{\Gamma(x)}{\Gamma(x+y)}\int_0^\infty e^{-t}t^{y-1}dt \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \end{aligned} \tag{2.6}$$

2.1.2 The Dirac-delta Function

In mathematical physics we often encounter functions which have non-zero values in vary short intervals. For example, an impulsive force is envisaged as acting for only a very short interval of time. The Dirac delta function, which is used extensively in quantum mechanics and classical applied mathematics, may me thought of as a generalization of this concept.

If we consider the function

$$\delta_a(x) = \begin{cases} \dfrac{1}{2a}, & \text{for } |x| < a, \\ 0, & \text{for } |x| > a \end{cases} \tag{2.7}$$

then it is easily to show that

$$\int_{-\infty}^{\infty} \delta_a(x)\, dx = 1. \tag{2.8}$$

Also, if $f(x)$ is any function which is integrable in the interval $(-a, a)$ then, by using the mean value theorem of the integral calculus, we see that

$$\int_{-\infty}^{\infty} f(x)\delta_a(x)\, dx = \frac{1}{2a}\int_{-a}^{a} f(x)\, dx = f(\theta a), \quad \text{for } |\theta| \leq 1.$$

We now define

$$\delta(x) = \lim_{a\to 0} \delta_a(x). \tag{2.9}$$

Letting a tend to zero in equation (2.7) and (2.8) we see that $\delta(x)$ satisfies the relations

$$\delta(x) = 0, \quad \text{if } x \neq 0, \tag{2.10}$$

and

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{2.11}$$

The function $\delta(x)$ is known as the Dirac-delta function. For this reason Dirac has called the delta function an *improper function* and has emphasized that it may be used in mathematical analysis only when no inconsistency can possibly arise from its use. The delta function could be dispensed with entirely by using a limiting procedure involving ordinary functions of the kind $\delta(x)$, but the function $\delta(x)$ and it derivatives play such a useful role in the formulation and solution of boundary value problem is classical mathematical physics as well as in quantum mechanics that it is important to derive the formal properties of the Dirac delta function. It should be emphasized, however, that these properties are purely formal.

First of all it should be observed that the precise variation of $\delta(x)$ in the neighborhood of the origin is not important provided that its oscillations, if it has any, are not too violent. For instance, the function

$$\delta(x) = \lim_{n\to\infty} \frac{\sin 2n\pi x}{\pi x}$$

satisfies equations (2.10) and (2.11) and has the same formal properties as the function defined by equation (2.9). If we let a tend to zero in equation (2.8), we obtain the relation

$$\int_{-\infty}^{\infty} f(x)\delta(x)\, dx = f(0),$$

which a simple change of variable transforms to

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,dx = f(a).$$

In other words the operation of multiplying $f(x)$ by $\delta(x-a)$ and integrating over all x is merely equivalent to substituting a for x in the original function. Symbolically we may write

$$f(x)\delta(x-a) = f(a)\delta(x-a)$$

if we remember that this equation has meaning only in the sense that its two sides give equivalent results when used as factors in an integrand. As a special case we have

$$x\delta(x) = 0.$$

In similar way we can prove that relations

$$\begin{aligned} \delta(-x) &= \delta(x), \\ \delta(ax) &= \frac{1}{a}\delta(x) \quad \text{for } a > 0, \\ \delta(a^2 - x^2) &= \frac{1}{2a}\left(\delta(x-a) + \delta(x+a)\right), \quad \text{for } a > 0. \end{aligned}$$

Let us now consider the interpretation we must put upon the derivatives of $\delta(x)$. If we assume that $\delta'(x)$ exists and that both it and $\delta(x)$ can be regarded as ordinary functions in the rule for integrating by parts we see that

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\delta'(x)\,dx &= f(x)\delta(x)\Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\delta(x)\,dx \\ &= -f'(0). \end{aligned}$$

Replacing this process we find that

$$\int_{-\infty}^{\infty} f(x)\delta^{(n)}(x)\,dx = (-1)^n f^{(n)}(0).$$

Next we shall study about delta-convergent sequence. There are many ways to construct a sequence of regular functions which converge to the delta function. All that is needed is that the corresponding ordinary functions $f_n(x)$, from which we shall call a *delta-convergent sequence*, must possess the following two properties:

(a) For any $M > 0$ and for $|a| \le M$ and $|b| \le M$, the quantities

$$\left| \int_a^b f_n(\xi)\,d\xi \right|$$

must be bounded by a constant independent on a, b or n (in other words, depending only on M).

(b) For any fixed non-vanishing a and b, we must have

$$\lim_{n\to\infty}\int_a^b f_n(\xi)\,d\xi = \begin{cases} 0, & \text{for } a<b<0 \text{ and } 0<a<b, \\ 1, & \text{for } a<0<b. \end{cases}$$

Let $f_n(x)$ be such a delta-convergent sequence. Consider also the sequence of primitive functions

$$F_n(x) = \int_{-1}^{x} f_n(\xi)\,d\xi.$$

It follows simply from the two properties of a delta-convergent sequence that as n is followed to increase the $F_n(x)$ converge to zero for $x<0$ and to one for $x>0$. Moreover, these functions are uniformly bounded (in n) in every interval. This implies that the $F_n(x)$ converge in the sense of generalized functions to $H(x)$, which defined by

$$H(x) = \begin{cases} 1, & \text{for } x>0, \\ 0, & \text{for } x<0. \end{cases}$$

Then in the sense of generalized functions the sequence $f_n(x) = F_n(x)$ converges to $H'(x) = \delta(x)$, asserted.

2.2 Distribution

In this section, we give some definitions and properties of the distribution which will be used in the later chapters.

Definition 2.2.1. Let $\Omega \subset \mathbb{R}^n$ and $f:\Omega\to\mathbb{R}$. The *support of* f is defined to be the closure of the set $S=\{x\in\Omega : f(x)\neq 0\}$. And support of f denote by Supp$f$.

Definition 2.2.2. A set $\Omega\subset\mathbb{R}^n$ is *compact* if every sequence in Ω has a convergence subsequence whose limit is an element of Ω.

Definition 2.2.3. Let $\Omega\subset\mathbb{R}^n$, define $\mathcal{D}=C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions on Ω with compact support, $\varphi\in\mathcal{D}$ is called a test function.

Definition 2.2.4. A sequence of testing function $\varphi_i(x)_{i=1}^\infty$ is said to *converge* to $\varphi(x)$ in $\mathcal{D}$ if all $\varphi_i(x)$ are zero outside a certain region in $\mathbb{R}^n$ and if for every nonnegative integers $m_1, m_2, \ldots, m_n$ the sequence $\left\{\frac{\partial^{m_1+m_2+\cdots+m_n}\varphi_i(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}}\right\}_{i=1}^\infty$ converges uniformly to $\frac{\partial^{m_1+m_2+\cdots+m_n}\varphi(x)}{\partial x_1^{m_1}\partial x_2^{m_2}\ldots\partial x_n^{m_n}}$ on $\mathbb{R}^n$.

Proposition 2.2.5. *$\mathcal{D}$ is closed under convergence, that is, the limit of every sequence that converge in $\mathcal{D}$ is also in $\mathcal{D}$.*

Definition 2.2.6. A *distribution* is a mapping $f:\mathcal{D}\to\mathbb{C}$ such that

(1) $\langle f,\varphi\rangle$ is a well defined complex number for every $\varphi\in\mathcal{D}$,

(2) for any $\varphi_1, \varphi_2 \in \mathcal{D}$ and any scalars a_1, a_2,

$$\langle f, a_1\varphi_1 + a_2\varphi_2\rangle = a_1\langle f, \varphi_1\rangle + a_2\langle f, \varphi_2\rangle,$$

(3) for any sequence $\{\varphi_n\}$ in $\mathcal{D}$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ then $\lim_{n\to\infty} \langle f, \varphi_n\rangle = \langle f, \varphi\rangle$.

We note that each continuous(or even locally integrable) function $f(x)$ generates a distribution

$$\langle f, \varphi\rangle = \int f(x)\varphi(x)dx.$$

Definition 2.2.7. A *regular distribution* is a distribution which is generated by a locally integrable function.

Definition 2.2.8. A *singular distribution* is a distribution which is not generated by a locally integrable function.

Definition 2.2.9. The *Dirac-delta* distribution with singularity $\xi \in \mathbb{R}^n$, denoted by $\delta(x - \xi)$, which is defined by

$$\langle \delta(x - \xi), \phi\rangle = \phi(\xi).$$

Definition 2.2.10. Let $x = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$, define $\mathcal{S}$ is the set of all real value functions $\varphi(x)$ that are infinitely smooth and are such that, all nonnegative integer m and $k = (k_1, k_2, \ldots, k_n)$,

$$\|x\|^m |D^k\varphi(x)| \leq C_{mk},$$

for some a constant C_{mk} and denote D^k by $D^k = \frac{\partial^{k_1+k_2+\ldots+k_n}}{\partial x_1^{k_1}\partial x_2^{k_2}\ldots\partial x_n^{k_n}}$.

Definition 2.2.11. A *tempered distribution* is a mapping $f : \mathcal{S} \to \mathbb{C}$ such that

(1) $\langle f, \varphi\rangle$ is a well defined complex number for every $\varphi \in \mathcal{S}$,

(2) for any $\varphi_1, \varphi_2 \in \mathcal{S}$ and any scalars a_1, a_2,

$$< f, a_1\varphi_1 + a_2\varphi_2 >= a_1 < f, \varphi_1 > +a_2 < f, \varphi_2 >,$$

(3) for any sequence $\{\varphi_n\}$ in $\mathcal{S}$ such that $\lim_{n\to\infty} \varphi_n = \varphi$ then $\lim_{n\to\infty} \langle f, \varphi_n\rangle = \langle f, \varphi\rangle$.

Definition 2.2.12. A space $\mathcal{C}$ of distributions is said to be a *convolution algebra* if it possesses the following properties:

(1) $\mathcal{C}$ is a linear space.

(2) $\mathcal{C}$ is closed under convolution.

(3) Convolution is associative for any three distributions in $\mathcal{C}$.

Definition 2.2.13. Let f be a distribution. The *derivative* $\dfrac{\partial f}{\partial x_k}$ as the distribution given by

$$\langle \frac{\partial f}{\partial x_k}, \phi \rangle = -\langle f, \frac{\partial \phi}{\partial x_k} \rangle,$$

and more generally $D^k f$ denoted by

$$\langle D^k f, \phi \rangle = (-1)^{|k|} \langle f, D^k \phi \rangle,$$

where $|k| = k_1 + k_2 + \cdots + k_n$.

Proposition 2.2.14. ([20]) *Let x be an n-dimensional real variable and y an m-dimensional real variable. Also, let $\varphi(x, y)$ be a testing function in $\mathcal{D}$ define over $\mathbb{R}^{n+m}$. If $f(x)$ is a distribution defined over $\mathbb{R}^n$, then $\theta(y) = \langle f(x), \varphi(x, y) \rangle$ is a testing function of y in $\mathcal{D}$.*

Proposition 2.2.15. ([5]) *Let f be a distribution in m dimensions and g be a distribution in n dimensions. Then the functional h defined by*

$$\langle h(x, y), \varphi(x, y) \rangle = \langle f(x), \langle g(y), \varphi(x, y) \rangle \rangle$$

is a distribution in $m + n$ dimensions.

Definition 2.2.16. The distribution h in Proposition (2.2.15) is called the *tensor (or direct) product* of $f(x)$ and $g(y)$ and is denoted by $h(x, y) = f(x) \times g(y)$, that is,

$$\langle f(x) \times g(y), \varphi(x, y) \rangle = \langle f(x), \langle g(y), \varphi(x, y) \rangle \rangle.$$

Definition 2.2.17. The *support of a distribution* f is defined as the complement of the largest open set on which f is zero.

Proposition 2.2.18. ([5]) *Let f and g be distributions in n dimensions. Then the function h defined by*

$$\langle h, \varphi \rangle = \langle f(x) \times g(y), \varphi(x + y) \rangle$$

is a distribution provided that it satisfies either of the following conditions:

(1) *Either f or g has bounded support, or*

(2) *In one dimension the supports of both f and g are bounded on the same side (for instance, $f = 0$ for $x < a$, and $g = 0$ for $y < b$).*

Definition 2.2.19. The distribution h in Proposition (2.2.15) is called the *convolution* of f and g and is denoted by $h = f * g$, that is,

$$\langle f * g, \varphi \rangle = \langle f(x) \times g(y), \varphi(x + y) \rangle.$$

Now we shall give some helpful properties of convolutions.

Proposition 2.2.20. ([5],[20]) *Let f, g and h be distributions.*

(1) *For δ is the Dirac-delta function, we have*

$$f * \delta = f.$$

(2) *If f and g satisfy at least one of the (1) and (2) of proposition 2.2.18, then*

$$f * g = g * f.$$

(3) *If $P(D)$ is a linear partial differential operator with constant coefficients and f and g satisfy at least one of the (1) and (2) of proposition 2.2.18, then*

$$P(D)f * g = P(D)(f * g) = f * P(D)g.$$

Lemma 2.2.21. *Let $\diamondsuit$ and u be a distribution defined for all $\phi \in C_0^\infty(\Omega)$, where Ω is an open subset of $\mathbb{R}^n$. Then $\langle \diamondsuit u, \phi \rangle = \langle u, \diamondsuit \phi \rangle$*

Proof. By Definition 2.2.13 we have

$$\begin{aligned} \left\langle \frac{\partial^2 u}{\partial x_k^2}, \phi \right\rangle &= (-1)^2 \left\langle u, \frac{\partial^2 \phi}{\partial x_k^2} \right\rangle \\ &= \left\langle u, \frac{\partial^2 \phi}{\partial x_k^2} \right\rangle. \end{aligned}$$

Thus,

$$\left\langle \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}, \phi \right\rangle = \left\langle u, \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2} \right\rangle$$

that is,

$$\langle \triangle u, \phi \rangle = \langle u, \triangle \phi \rangle.$$

Similarly,

$$\langle \Box u, \phi \rangle = \langle u, \Box \phi \rangle.$$

Therefore,

$$\begin{aligned} \langle \diamondsuit u, \phi \rangle &= \langle \Box(\triangle u), \phi \rangle \\ &= \langle \triangle u, \Box \phi \rangle \\ &= \langle u, \triangle(\Box)\phi \rangle \\ &= \langle u, \diamondsuit \phi \rangle. \end{aligned}$$

2.3 Partial Differential Equations

A partial differential equation (PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Definition 2.3.1. *For an integer $k \geq 1$ and let U denote an open subset of R^n An expression of the form*

$$F\left(D^k u(x), D^{k-1}u(x), \ldots, Du(x), u(x), x\right) = 0 \tag{2.12}$$

$(x \in U)$ *is called a k^{th}-order partial differential equation, where*

$$F : R^{n^k} \times R^{n^{k-1}} \times \cdots \times R^n \times R \times U \longrightarrow R$$

is given and $u : U \longrightarrow R$ is the unknown.

We solve the PDE if we find all u verifying (2.12), possibly only among those functions satisfying certain auxiliary boundary conditions on some part Γ of ∂U.

Definition 2.3.2. .

- *(i) The partial differential equation (2.12) is called linear if it has the form*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

for given function $a_\alpha(|\alpha| \leq k), f$.

- *(ii) The partial differential equation (2.12) is called semilinear if it has the form*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0\left(D^{k-1}u, \ldots, Du, u, x\right) = 0.$$

- *(iii) The partial differential equation (2.12) is called quasilinear if it has the form*

$$\sum_{|\alpha| = k} a_\alpha\left(D^{k-1}u, \ldots, Du, u, x\right) D^\alpha u + a_0\left(D^{k-1}u, \ldots, Du, u, x\right) = 0.$$

- *(iv) The partial differential equation (2.12) is called nonlinear if it depends nonlinearly upon the highest order derivatives.*

2.3.1 Elementary Solutions

We shall mainly be interested in the equations where in the coefficients are constants. The theory of partial differential equations stems from the intensive and extensive study of a few basic equations of mathematical physics, and the coefficients in all of these are constants. Such equations arise in the study of gravitation, electromagnetism, perfect fluids, elasticity, heat transfer, and quantum mechanics. Of great importance in the study of these equations are their elementary solutions. Recall that a elementary solution $E(x)$ is a generalized function that satisfies the equation

$$LE(x) = \delta(x). \tag{2.13}$$

This solution is not unique, because we can add to it any solution of the homogeneous equation. This understood, in the sequel we shall select the elementary solution from among the particular solutions according to its behavior at infinity or other appropriate criteria. In the study of these solutions the following interesting concept is helpful. It is called Hadamard's method of descent:

Given the solution of a partial differential equation in $\mathbb{R}^{n+1}$, we can find its solution in $\mathbb{R}^n$ or in a still lower dimension. In doing so, we descend from the higher-dimensional problem to a lower-dimensional one. For instance, the solution of the initial value problem for the wave equation in two dimensions can be obtained from that in three dimensions. Specifically, let us consider a linear partial differential equation

$$L\left(D, \frac{\partial}{\partial x_{n+1}}\right) u = f(x) \otimes \delta(x_{n+1}), \tag{2.14}$$

in the space $\mathbb{R}^{n+1}$ of variable (x, x_{n+1}), where $x = (x_1, \ldots, x_n)$, D is $\partial/\partial x_j$, $j = 1, \ldots, n$, $f \in \mathcal{D}'(\mathbb{R}^n)$,

$$L\left(D, \frac{\partial}{\partial x_{n+1}}\right) u = \sum_{q=1}^{p} \frac{\partial^q}{\partial x_{n+1}} L_q(D) + L_0(D), \tag{2.15}$$

and $L_q(D)$ are partial differential operators involving the variables $x_1, \ldots, x_n$.

When we say that the generalized function $g \in \mathcal{D}'(\mathbb{R}^{n+1})$ allows the continuation over functions of the form $\varphi(x)1(x_{n+1})$ where $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we mean the following: Given an arbitrary sequence of functions $\psi_m(x_{n+1})$, $m = 1, 2, \ldots$, belonging to $\mathcal{D}(\mathbb{R}^1)$, where $\mathbb{R}^1$ is the space with variables x_{n+1} and converging to 1 in $\mathbb{R}^1$[i.e.$1(x_{n+1})$], then there is the limit

$$\lim_{m\to\infty} \langle g, \varphi(x)\psi_m(x_{n+1})\rangle = \langle g, \varphi(x)1(x_{n+1})\rangle = \langle g_0, \varphi\rangle \tag{2.16}$$

$\varphi \in \mathcal{D}(\mathbb{R}^n)$. In view of the completeness of $\mathcal{D}'$, we find that $g_0 \in \mathcal{D}'(\mathbb{R}^n)$.

Specifically, for $g(x)$ such that $g(x) = f(x) \otimes \delta(x_{n+1})$, the inhomogeneous term in (2.14), we have

$$\begin{aligned}
\langle g_0, \varphi \rangle &= \lim_{m\to\infty} \langle g(x), \varphi(x)\psi_m(x_{n+1})\rangle \\
&= \lim_{m\to\infty} \langle f(x) \otimes \delta(x_{n+1}), \varphi(x)\psi_m(x_{n+1})\rangle \\
&= \lim_{m\to\infty} \langle f(x), \delta(x_{n+1})\varphi(x)\psi_m(x_{n+1})\rangle \\
&= \lim_{m\to\infty} \langle f(x), \varphi(x)\psi_m(0)\rangle \\
&= \langle f(x), \varphi(x)\rangle, \qquad\qquad \varphi \in \mathcal{D}.
\end{aligned}$$

Accordingly, the method of descent can be stated as follows: If the solution $u \in \mathcal{D}'(\mathbb{R}^{n+1})$ of (2.13) allows the continuation (2.16), then the distribution $u_0 \in \mathcal{D}'(\mathbb{R}^n)$ is the solution of the equation

$$L_0(D)u_0 = f(x). \tag{2.17}$$

For instance, if the locally integrable function $E(x,t)$ is the elementary solution of the operator $L(D, \partial/\partial t)$, then the distribution

$$E_0(x) = \int_{-\infty}^{\infty} E(x,t)\ dt, \tag{2.18}$$

is the elementary solution of the operator L_0. Indeed, in view of the Lebesgue theorem on the passage of the limit under the integral sign, we have

$$\begin{aligned}
\lim_{m\to\infty} \langle E(x,t), \varphi(x)\psi_m(t)\rangle &= \lim_{m\to\infty} \int E(x,t)\varphi(x)\psi_m(t)dxdt \\
&= \int E(x,t)\varphi(x)dxdt \\
&= \int \varphi(x) \left(\int_{-\infty}^{\infty} E(x,t)dt\right) dx \\
&= \langle E_0(x), \varphi(x)\rangle,
\end{aligned}$$

where E_0 is defined in (2.18) and $\varphi \in \mathcal{D}$. Moreover, this limit does not depend on the sequence $\psi_m(t)$. Here $E_0(x)$ is the elementary solution of the operator L_0, as required.

2.3.2 Fourier Transform

Definition 2.3.3. *Let $f(x) \in L_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ defined by*

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)dx. \tag{2.19}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ *and* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ *and* $dx = dx_1, dx_2, \ldots, dx_n$

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \hat{f}(\xi) d\xi. \tag{2.20}$$

Definition 2.3.4. *The spectrum of the kernel* $E(x,t)$*, defined by*

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi, \tag{2.21}$$

is the bounded support of the Fourier transform $\widehat{E(x,t)}$*, for any fixed* $t > 0$.

Definition 2.3.5. *Let* $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ *and denote*

$$\Gamma_+ = \{\xi \in \mathbb{R}^n : \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \cdots - \xi_{p+q}^2 > 0 \text{ and } \xi_1 > 0\}$$

to be the set of an interior of the forward cone and $\overline{\Gamma}_+$ *denotes the closure of* Γ_+*. Let* Ω *be the spectrum of* $E(x,t)$ *defined by definition* (2.3.4) *and* $\Omega \subset \overline{\Gamma}_+$*. Let* $\widehat{E(\xi,t)}$ *be the Fourier transform of* $E(x,t)$ *which is defined by*

$$\widehat{E(\xi,t)} = \begin{cases} \dfrac{1}{(2\pi)^{\frac{n}{2}}} e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m\right)^k\right]} & \text{for } \xi \in \Gamma^+; \\ 0 & \text{for } \xi \notin \Gamma^+. \end{cases} \tag{2.22}$$

Lemma 2.3.6. *Let The operator* L *defined by*

$$L = \frac{\partial}{\partial t} + c^2 L_m^k, \tag{2.23}$$

where L_m^k *is the operator iterated* k*-times and is defined by*

$$L_m^k = (-1)^{mk} \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^m - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^m \right]^k$$

$p + q = n$ *is the dimension of the* $\mathbb{R}^n$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, k *and* m *are a positive integer and* c *is a positive constant. Then we obtain*

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p} \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi \tag{2.24}$$

is the elementary solution of (2.23) *in the spectrum* $\Omega \subset \mathbb{R}^n$ *for* $t > 0$.

Proof. Let $E(x,t)$ be the kernel or elementary solution of L_m^k operator and let δ be the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t} E(x,t) + c^2 L_m^k E(x,t) = \delta(x)\delta(t).$$

Applying the Fourier transform to the both sides of the above equation, we have

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} + c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right]^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Hence, we obtain

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}} e^{-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k}$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}} e^{-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k}$$

which has been already by (2.22). By inverse Fourier transform, we have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi$$

Since Ω is the spectrum of $E(x,t)$, we obtain

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega} e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi$$

for $t > 0$. □

Definition 2.3.7. *Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and write*

$$v = x_1^2 + x_2^2 + ... + x_n^2. \tag{2.25}$$

For any complex number β, define

$$R_\beta^e(v) = 2^{-\beta}\pi^{\frac{-n}{2}}\Gamma\left(\frac{n-\beta}{2}\right)\frac{v^{\frac{\beta-n}{2}}}{\Gamma(\frac{\beta}{2})}. \tag{2.26}$$

The function $R_\beta^e(v)$ is called the elliptic kernel of Marcel Riesz and is ordinary function for $Re(\beta) \geq n$ and is a distribution of β for $Re(\beta) < n$.

Definition 2.3.8. *For any complex number β, define*

$$W_{2k}^e(v,m) = \sum_{r=0}^{\infty}\frac{(-1)^r\Gamma(k+r)}{r!\Gamma(k)}(m^2)^r(-1)^{k+r}R_{2k+2r}^e(v), \tag{2.27}$$

where $R_{2k+2r}^e(v)$ is defined by (2.26) with $\beta = 2k + 2r$, m is a nonnegative real number.

Lemma 2.3.9. *Given the equation*

$$\triangle^k u(x) = 0, \tag{2.28}$$

where $\triangle^k$ is the Laplacian iterated k-times defined by equation (1) we obtain $u(x) = ((-1)^{k-1}R^e_{2(k-1)}(x))^{(l)}$ as a solutions of (2.28) where $l = (n-4)/2, n \geq 4$ is nonnegative integer and n is even and $R^e_{2(k-1)}(x)$ defined by equation (2.26) with m derivatives and $\beta = 2(k-1)$.

Proof. see [8].

Lemma 2.3.10. *Given the equation $\triangle^k u(x) = \delta(x)$ for $x \in \mathbb{R}^n$, where $\triangle^k$ is the Laplace operator iterated k-times defined by (1). Then $u(x) = (-1)^k R^e_{2k}(x)$ is an elementary solution of the operator $\triangle^k$.*

Proof. See [7].

Lemma 2.3.11. *The function $W^e_{2k}(v, m)$ is an elementary solutions of the operator $(\triangle + m^2)^k$ where $(\triangle + m^2)^k$ is the Helmholtz operator iterated k-times, $\triangle$ is the Laplacian, and $W^e_{2k}(v, m)$ defined by equation (2.27)*

Proof. At first, the following formula is valid ([1] p.3)

$$\Gamma\left(\frac{\eta}{2} + r\right) = \frac{\eta}{2}\left(\frac{\eta}{2} + 1\right)\cdots\left(\frac{\eta}{2} + r - 1\right)\Gamma\left(\frac{\eta}{2}\right).$$

Equivalently,

$$\begin{aligned}\frac{(-1)^r}{r!}\Gamma\left(\frac{\eta}{2} + r\right) &= \frac{(-1)^r}{r!}\frac{\eta}{2}\left(\frac{\eta}{2} + 1\right)\cdots\left(\frac{\eta}{2} + r - 1\right)\Gamma\left(\frac{\eta}{2}\right) \\ &= \frac{1}{r!}\left(\frac{-\eta}{2}\right)\left(\frac{-\eta}{2} - 1\right)\cdots\left(\frac{-\eta}{2} - r + 1\right)\Gamma\left(\frac{\eta}{2}\right).\end{aligned}$$

We have,

$$\frac{(-1)^r}{r!}\Gamma\left(\frac{\eta}{2} + r\right) = \binom{-\frac{\eta}{2}}{r}\Gamma\left(\frac{\eta}{2}\right).$$

Then, we obtain the function $W^e_{2k}(v, m)$ is defined by (2.27) become

$$W^e_{2k}(v, m) = \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r(-1)^{k+r}R^e_{2k+2r}(v).$$

Since the operator $\triangle$ is a linearly continuous and have $1-1$ mapping, it has an inverse. By Lemma 2.3.10, we obtain

$$W^e_{2k}(v, m) = \sum_{r=0}^{\infty}\binom{-k}{r}(m^2)^r\triangle^{-k-r}\delta(x) = (\triangle + m^2)^{-k}\delta(x),$$

where $(\triangle + m^2)^{-k}$ is the inverse operator of the operator $(\triangle + m^2)^k$. By applying the operator $(\triangle + m^2)^k$ to both sides of the above equation, we have

$$(\triangle + m^2)^k W^e_{2k}(v, m) = (\triangle + m^2)^k(\triangle + m^2)^{-k}\delta(x) = \delta(x).$$

Lemma 2.3.12. *Given the equation*

$$\triangle u(x) = f(x, u(x)), \tag{2.29}$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega, \Omega$ is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ denotes the boundary of Ω. Assume f is a bounded, that is $|f(x, u)| \leq N$ and the boundary condition $u(x) = 0$ for $x \in \partial\Omega$. Then we obtain $u(x)$ as a unique solution of (2.29).

Proof. We can prove this lemma by the method of iterations and the Schauder's estimates, see [2].