Chapter 3

Generalized Heat Kernel Related to the Operator L_m^k and Spectrum

In this chapter, we study the generalized solution of the operator L_m^k related to the generalized heat equation and spectrum. Moreover, such heat kernel has interesting properties and also related to the kernel of an extension of the heat equation.

Theorem 3.1 *Given the equation*

$$\frac{\partial}{\partial t}u(x,t) + c^2 L_m^k u(x,t) = 0 \tag{3.1}$$

with initial condition

$$u(x,0) = f(x) \tag{3.2}$$

where L_m^k is the operator iterated k-times and defined by

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^m\right]^k$$

$p+q=n$ is the dimension of the space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t)=(x_1,x_2,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, $f(x)$ is a given generalized function, k and m are positive integers and c is a positive constant. Then

$$u(x,t) = E(x,t) * f(x) \tag{3.3}$$

is a solution of (3.1) *which satisfies* (3.2), *where $E(x,t)$ is given by* (2.24).

Proof. Taking the Fourier transform to the both sides of the (3.1), we obtain

$$\frac{\partial}{\partial t}\hat{u}(\xi,t) = -c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right]^k \hat{u}(\xi,t).$$

Thus

$$\hat{u}(\xi,t) = K(\xi)e^{-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k}, \tag{3.4}$$

where $K(\xi)$ is constant and $\hat{u}(\xi,0) = K(\xi)$.

Now, by (3.2) we have

$$K(\xi) = \hat{u}(\xi,0) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i(\xi,x)}f(x)dx. \tag{3.5}$$

And by the inversion in (2.20), (3.4), (3.5) we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\hat{u}(\xi,t)d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-i(\xi,y)}f(y)e^{-c^2t\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right]^k}d\xi dy. \end{aligned}$$

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\left[-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x-y)\right]}f(y)d\xi dy. \tag{3.6}$$

Set

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\left[-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]}d\xi. \tag{3.7}$$

Since the integral in (3.7) is divergent, therefore we choose $\Omega \subset \mathbb{R}^n$ be the spectrum of $E(x,t)$ and by (2.24), we have

$$\begin{aligned} E(x,t) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\left[-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]}d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\Omega} e^{\left[-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]}d\xi. \end{aligned} \tag{3.8}$$

Thus (3.6) can be written in the convolution form

$$u(x,t) = E(x,t) * f(x).$$

Moreover, since $E(x,t)$ exists, we see that

$$\begin{aligned} \lim_{t\to 0} E(x,t) &= \frac{1}{(2\pi)^n}\int_{\Omega} e^{i(\xi,x)}d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}d\xi \\ &= \delta(x), \quad \text{for } \ x \in \mathbb{R}^n. \end{aligned} \tag{3.9}$$

holds (see [6],p.396, equation (10.2.19b)).

Thus for the solution $u(x,t) = E(x,t) * f(x)$ of (3.1), then we have

$$u(x,0) = \lim_{t\to 0} u(x,t) = \lim_{t\to 0} E(x,t) * f(x) = \delta(x) * f(x) = f(x)$$

which satisfies (3.2). This complete the proof. $\square$

Theorem 3.2 *The kernel $E(x,t)$ is defined by (3.8) has the following properties:*

(1) $E(x,t) \in C^{\infty}(\mathbb{R}^n \times (0,\infty))$ *the space of continuous with infinitely differentiable,*

(2) $(\frac{\partial}{\partial t} + c^2 L_m^k)E(x,t) = 0$, *for* $t > 0$,

(3) $|E(x,t| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$, *for* $t > 0$,
where $M(t) = \int_0^R \int_0^T e^{-c^2t\left(r^{2m}-s^{2m}\right)^k} r^{p-1}s^{q-1}drds$ *is a function of* $t > 0$ *in the the spectrum* Ω *and* Γ *denote the Gamma function. Thus* $E(x,t)$ *is bounded for any fixed* $t > 0$.

(4) $\lim_{t\to 0} E(x,t) = \delta(x)$.

Proof. (1) From (3.8), since

$$\frac{\partial^n}{\partial x^n}E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \frac{\partial^n}{\partial x^n} e^{\left[-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]}d\xi$$

Thus $E(x,t) \in C^\infty$ for $x \in \mathbb{R}^n$ and $t > 0$.
(2) By computing directly, we obtain $\left(\frac{\partial}{\partial t} + c^2L_m^k\right)E(x,t) = 0$.
(3) We have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega e^{\left[-c^2t\left(\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]}d\xi$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n}\int_\Omega e^{-c^2t\left[\left(\sum_{i=1}^p \xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right]^k}d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p$$

and

$$\xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q},$$

where $\sum_{i=1}^p \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q}\omega_j^2 = 1$. Thus

$$|E(x,t)| \leq \frac{1}{(2\pi)^n}\int_\Omega e^{-c^2t\left(r^{2m}-s^{2m}\right)^k} r^{p-1}s^{q-1}drdsd\omega_pd\omega_q$$

where $d\xi = drdsd\omega_pd\omega_q$ and $d\omega_p$, $d\omega_q$ are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively. Since $\omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \leq r \leq R$ and $0 \leq s \leq T$ where R and T are constants. Thus

$$\begin{aligned}|E(x,t)| &\leq \frac{\omega_p\omega_q}{(2\pi)^n}\int_0^R\int_0^T e^{-c^2t\left(r^{2m}-s^{2m}\right)^k} r^{p-1}s^{q-1}drds\\ &= \frac{\omega_p\omega_q}{(2\pi)^n}M(t) \quad \text{for any fixed } t>0 \text{ in the spectrum } \Omega\\ &= \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},\end{aligned}$$

where

$$M(t) = \int_0^R\int_0^T e^{-c^2t\left(r^{2m}-s^{2m}\right)^k} r^{p-1}s^{q-1}drds$$

is a function of t, $\omega_p = \frac{2\pi^{p/2}}{\Gamma(\frac{p}{2})}$ and $\omega_q = \frac{2\pi^{p/2}}{\Gamma(\frac{q}{2})}$. Thus, for any fixed $t > 0$, $E(x,t)$ is bounded.

(4) Obvious by (3.9). $\square$

For the example, If we put $n = 1, q = 0, m = 2, k = 1, c = 1/3$ in (3.1) and $u(x,0) = \sin(\sqrt{3}x)$, we have

$$\frac{\partial}{\partial t}u(x,t) + \frac{1}{9}\frac{\partial^4}{\partial x^4}u(x,t) = 0. \tag{3.10}$$

From (3.8) we have $E(x,t) = \frac{1}{2\pi}\int_\Omega e^{-\frac{1}{9}t\xi^4+i\xi x}d\xi$ and $u(x,t) = e^{-t}\sin(\sqrt{3}x)$ is the solution of (3.10). Graphical solution shows below.

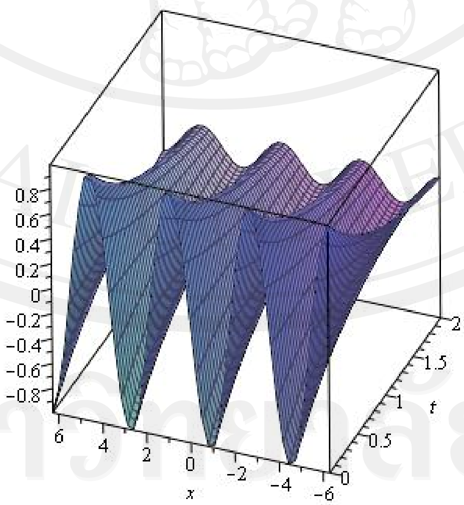

Figure 3.1: The solution $u(x,t) = e^{-t}\sin(\sqrt{3}x)$.

For another example, we put $n = 2, q = 0, m = 2, k = 1, c = 1/2$ in (3.1) and $u(x, 0) = \sin x \sin y$, we have the equation

$$\frac{\partial}{\partial t}u(x,y,t) + \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 u(x,y,t) = 0. \qquad (3.11)$$

From (3.8), we have $E(x,t) = \frac{1}{(2\pi)^2}\int_\Omega e^{-\frac{1}{4}t(\xi_1^2+\xi_2^2)^2+i\xi x}d\xi$ and $u(x,y,t) = e^{-t}\sin x \sin y$ is the solution of (3.11). Graphical solution shows below.

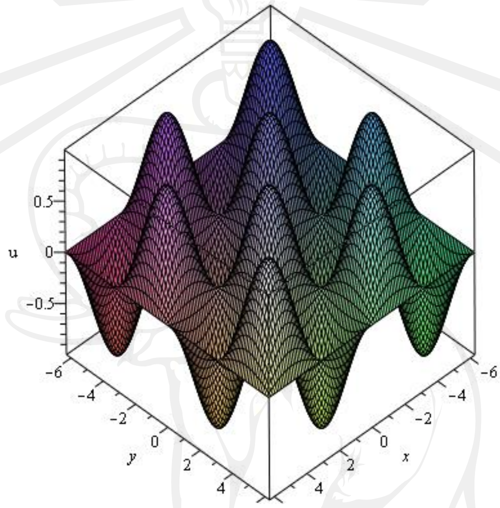

Figure 3.2: The solution $u(x,t)$ at $t = 0$.

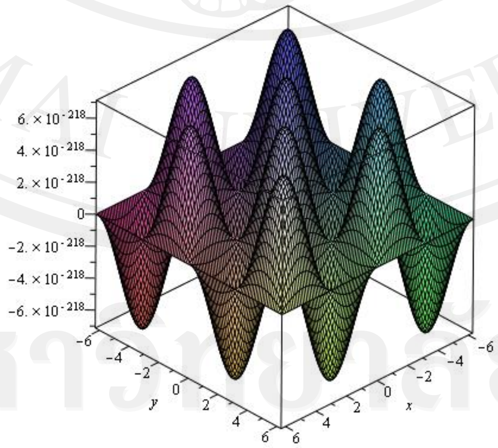

Figure 3.3: The solution $u(x,t)$ at $t = 500$.