# Chapter 4

# Generalized Wave Equation Related to the $L_m^k$ Operator

In this chapter, we study the operator $L_m^k$ related to the generalized wave equation by using $\epsilon$ approximation.

**Theorem 4.1** *Given the equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) + c^2 L_m^k, u(x,t) = 0, \tag{4.1}$$

*where $L_m^k$ is the product operator iterated $k$-times and is defined by*

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^m\right]^k$$

*$p+q=n$ is the dimension of the $\mathbb{R}^n$, $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$ with initial conditions*

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x), \tag{4.2}$$

*where $u(x,t)\in\mathbb{C}$, $c$ is a positive constant, $k$ and $m$ are nonnegative integers, $f$ and $g$ are continuous functions and absolutely integrable for $x\in\mathbb{R}^n$. Then (4.1) has a unique solution*

$$u(x,t) = f(x) * \psi(x,t) + g(x) * \phi(x,t) \tag{4.3}$$

*and satisfy the condition (4.2) where $\phi(x,t)$ is an inverse Fourier transform of*

$$\widehat{\phi}(\xi,t) = \frac{\sin ct\sqrt{(r^{2m}-s^{2m})^k}}{c\sqrt{(r^{2m}-s^{2m})^k}}$$

*and $\psi(x,t)$ is an inverse Fourier transform of*

$$\widehat{\psi}(\xi,t) = \cos ct\sqrt{(r^{2m}-s^{2m})^k} = \frac{\partial}{\partial t}\widehat{\phi}(\xi,t),$$

*where* $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$, $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$ *and* $r > s > 0$. *Moreover, if we put* $m = k = 1$ *and* $q = 0$ *in* (4.1)*, then it become the generalized* $n$*-dimensional wave equation*

$$\frac{\partial^2}{\partial t^2}u(x,t) - c^2\triangle u(x,t) = 0.$$

**Proof.** By applying the Fourier transform defined by (2.19) to both side of (4.1), we obtain

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + c^2\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k \widehat{u}(\xi,t) = 0. \tag{4.4}$$

Now, put $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$, $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$ and let $r > s > 0$. Thus (4.4) becomes

$$\frac{\partial^2}{\partial t^2}\widehat{u}(\xi,t) + c^2\left(r^{2m} - s^{2m}\right)^k \widehat{u}(\xi,t) = 0, \tag{4.5}$$

with the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t}u(x,0) = g(x).$$

Thus by (4.2), we have

$$u(\xi,0) = \widehat{f}(\xi) \quad \text{and} \quad \frac{\partial}{\partial t}u(\xi,0) = \widehat{g}(\xi). \tag{4.6}$$

Now, we are solving the solution of (4.5) satisfies (4.6). Then

$$\widehat{u}(\xi,t) = A(\xi)\cos ct\sqrt{(r^{2m} - s^{2m})^k} + B(\xi)\sin ct\sqrt{(r^{2m} - s^{2m})^k}$$

and

$$\begin{aligned}\frac{\partial \widehat{u}(\xi,t)}{\partial t} &= -c\sqrt{(r^{2m} - s^{2m})^k}A(\xi)\sin ct\sqrt{(r^{2m} - s^{2m})^k} \\ &\quad + c\sqrt{(r^{2m} - s^{2m})^k}B(\xi)\cos ct\sqrt{(r^{2m} - s^{2m})^k}.\end{aligned}$$

By (4.6), we obtain $\widehat{u}(\xi,0) = A(\xi) = \widehat{f}(\xi)$ and

$$\begin{aligned}\frac{\partial \widehat{u}(\xi,0)}{\partial t} &= 0 + c\sqrt{(r^{2m} - s^{2m})^k}B(\xi) = \widehat{g}(\xi) \\ B(\xi) &= \frac{\widehat{g}(\xi)}{c\sqrt{(r^{2m} - s^{2m})^k}}.\end{aligned}$$

Thus the solution of (4.5) satisfies (4.6) is

$$\widehat{u}(\xi,t) = \widehat{f}(\xi)\cos ct\sqrt{(r^{2m}-s^{2m})^k} + \frac{\widehat{g}(\xi)}{c\sqrt{(r^{2m}-s^{2m})^k}}\sin ct\sqrt{(r^{2m}-s^{2m})^k} \tag{4.7}$$

or in the convolution form

$$u(x,t) = f(x) * \psi(x,t) + g(x) * \phi(x,t). \tag{4.8}$$

Thus (4.8) is a solution of (4.1) where $\phi(x,t)$ is an inverse Fourier transform of $\widehat{\phi}(\xi,t) = \dfrac{\sin ct\sqrt{(r^{2m}-s^{2m})^k}}{c\sqrt{(r^{2m}-s^{2m})^k}}$ and $\psi(x,t)$ is an inverse Fourier transform of $\widehat{\psi}(\xi,t) = \cos ct\sqrt{(r^{2m}-s^{2m})^k} = \dfrac{\partial}{\partial t}\widehat{\phi}(\xi,t)$. Since $\widehat{\phi}(\xi,t)$ and $\widehat{\psi}(\xi,t)$ can not be Lebesgue integrable, that is $\widehat{\phi},\widehat{\psi} \notin L^1(\mathbb{R}^n)$. Thus we can not find the inverse $\phi$ and $\psi$ directly . But we can compute the inverse $\phi$ and $\psi$ by using the method of $\epsilon-$approximation. Let us defined $\widehat{\phi}_\epsilon(\xi,t) = e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}\widehat{\phi}(\xi,t)$ and $\widehat{\psi}_\epsilon(\xi,t) = e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}\widehat{\psi}(\xi,t)$. Clearly, $\widehat{\phi}_\epsilon(\xi,t) \to \widehat{\phi}(\xi,t)$, $\widehat{\psi}_\epsilon(\xi,t) \to \widehat{\psi}(\xi,t)$ uniformly as $\epsilon \to 0$. Since $\widehat{\phi}_\epsilon,\widehat{\psi}_\epsilon \in L^1(\mathbb{R}^n)$, then we can obtain the inverse $\phi_\epsilon$ and $\psi_\epsilon$ by applying (2.20) and we obtain $\phi_\epsilon \to \phi$ and $\psi_\epsilon \to \psi$ as $\epsilon \to 0$. Now, by (2.20) we have

$$\begin{aligned}\phi_\epsilon(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{\phi}_\epsilon(\xi,t)d\xi \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{i(\xi,x)}e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}\frac{\sin ct\sqrt{(r^{2m}-s^{2m})^k}}{c\sqrt{(r^{2m}-s^{2m})^k}}d\xi\end{aligned}$$

and

$$|\phi_\epsilon(x,t)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}}{c\sqrt{(r^{2m}-s^{2m})^k}}d\xi.$$

By changing to bipolar coordinates, we put

$$\xi_1 = rw_1, \xi_2 = rw_2, \ldots, \xi_p = rw_p$$

and $$\xi_{p+1} = sw_{p+1}, \xi_{p+2} = sw_{p+2}, \ldots, \xi_p = sw_{p+q},\ p+q=n$$

where $w_1^2+w_2^2+\cdots+w_p^2=1$ and $w_{p+1}^2+w_{p+2}^2+\cdots+w_{p+q}^2=1$. Thus

$$\begin{aligned}
|\phi_\epsilon(x,t)| &\leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\frac{e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}}{c\sqrt{(r^{2m}-s^{2m})^k}}r^{p-1}s^{q-1}drdsd\Omega_p d\Omega_q \\
&= \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_0^r\frac{e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}}{c\sqrt{(r^{2m}-s^{2m})^k}}r^{p-1}s^{q-1}dsdr,
\end{aligned}$$

where $\Omega_p=\dfrac{2\pi^{p/2}}{\Gamma(p/2)}$, $\Omega_q=\dfrac{2\pi^{q/2}}{\Gamma(q/2)}$ are the surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Now, put $s^m=r^m\sin\theta$, thus $0\leq\theta\leq\frac{\pi}{2}$ and $ds=\frac{r}{m}\cos\theta(\sin\theta)^{\frac{1-m}{m}}d\theta$. Then we obtain

$$\begin{aligned}
|\phi_\epsilon(x,t)| &\leq \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_0^{\pi/2}\frac{e^{-\epsilon c(r^m\cos\theta)^k}}{c(r^m\cos\theta)^k}\left[r(\sin\theta)^{-m}\right]^{q-1}\times \\
&\qquad \frac{r}{m}\cos\theta(\sin\theta)^{\frac{1-m}{m}}r^{p-1}d\theta dr \\
&= \frac{\Omega_p\Omega_q}{m(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_0^{\pi/2}\frac{e^{-\epsilon c(r^m\cos\theta)^k}}{c(r^m\cos\theta)^k}r^{p+q-1}(\sin\theta)^{\frac{q-m}{m}}\times \\
&\qquad \cos\theta d\theta dr.
\end{aligned}$$

Put $y=\epsilon c\left(r^m\cos\theta\right)^k=\epsilon cr^{mk}\cos^k\theta$, $r^{mk}=\dfrac{y}{c\epsilon\cos^k\theta}$, $dr=\dfrac{rdy}{mky}$, it follows that

$$\begin{aligned}
|\phi_\epsilon(x,t)| &\leq \frac{\Omega_p\Omega_q}{m(2\pi)^{\frac{n}{2}}}\int_0^{\pi/2}\int_0^\infty\frac{\epsilon e^{-y}r^{n-1}}{y}(\sin\theta)^{\frac{q-m}{m}}\cos\theta\frac{r}{mky}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{m^2(2\pi)^{\frac{n}{2}}}\int_0^{\pi/2}\int_0^\infty\frac{e^{-y}\epsilon}{ky^2}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{\frac{n}{mk}}(\sin\theta)^{\frac{q-m}{m}}\cos\theta dyd\theta \\
&= \frac{\Omega_p\Omega_q}{m^2(2\pi)^{\frac{n}{2}}}\int_0^{\pi/2}\int_0^\infty\frac{e^{-y}y^{\frac{n}{mk}-2}}{c^{\frac{n}{mk}}k\epsilon^{\frac{n}{mk}-1}}(\sin\theta)^{\frac{q}{m}-1}(\cos\theta)^{1-\frac{n}{m}}dyd\theta \\
&= \frac{\Omega_p\Omega_q}{m^2(2\pi)^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{mk}-1\right)}{k\epsilon^{\frac{n}{mk}-1}c^{\frac{n}{mk}}}\int_0^{\pi/2}(\sin\theta)^{2\left(\frac{q}{2m}\right)-1}(\cos\theta)^{2\left(1-\frac{n}{2m}\right)-1}d\theta \\
&= \frac{\Omega_p\Omega_q}{2m^2(2\pi)^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{mk}-1\right)}{k\epsilon^{\frac{n}{mk}-1}c^{\frac{n}{mk}}}\cdot\beta\left(\frac{q}{2m},\frac{2m-n}{2m}\right) \\
&= \frac{\Omega_p\Omega_q}{2m^2(2\pi)^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{mk}-1\right)}{k\epsilon^{\frac{n}{mk}-1}c^{\frac{n}{mk}}}\frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{\Gamma\left(\frac{2m-p}{2m}\right)}. \qquad (4.9)
\end{aligned}$$

Similarly, we defined $\widehat{\psi}_\epsilon(\xi,t)=e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}\cos ct\sqrt{(r^{2m}-s^{2m})^k}$ and

$$\begin{aligned}
\psi_\epsilon(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\widehat{\psi}_\epsilon(\xi,t)d\xi \\
&= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}e^{i(\xi,x)}e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}\cos ct\sqrt{(r^{2m}-s^{2m})^k}d\xi.
\end{aligned}$$

Thus

$$\begin{aligned}|\psi_\epsilon(x,t)| &\leq \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}}d\xi \\ &= \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_0^r e^{-\epsilon c\sqrt{(r^{2m}-s^{2m})^k}} r^{p-1}s^{q-1}dsdr,\end{aligned}$$

Now, put $s^m = r^m\sin\theta$, thus $0\leq\theta\leq\frac{\pi}{2}$ and $ds = \frac{r}{m}\cos\theta(\sin\theta)^{\frac{1-m}{m}}d\theta$. Then we obtain

$$\begin{aligned}|\psi_\epsilon(x,t)| &\leq \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-\epsilon c(r^m\cos\theta)^k}\left[r(\sin\theta)^{-m}\right]^{q-1}\times \\ &\qquad \frac{r}{m}\cos\theta(\sin\theta)^{\frac{1-m}{m}}r^{p-1}d\theta dr \\ &= \frac{\Omega_p\Omega_q}{m(2\pi)^{\frac{n}{2}}}\int_0^\infty\int_0^{\frac{\pi}{2}} e^{-\epsilon c(r^m\cos\theta)^k}r^{p+q-1}(\sin\theta)^{\frac{q-m}{m}}\times \\ &\qquad \cos\theta d\theta dr.\end{aligned}$$

Put $y = \epsilon c\left(r^m\cos\theta\right)^k = \epsilon c r^{mk}\cos^k\theta$, $r^{mk} = \dfrac{y}{c\epsilon\cos^k\theta}$, $dr = \dfrac{rdy}{mky}$, it follows that

$$\begin{aligned}|\psi_\epsilon(x,t)| &\leq \frac{\Omega_p\Omega_q}{m(2\pi)^{\frac{n}{2}}}\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-y}r^{n-1}(\sin\theta)^{\frac{q-m}{m}}\cos\theta\frac{r}{mky}dyd\theta \\ &= \frac{\Omega_p\Omega_q}{m^2(2\pi)^{\frac{n}{2}}}\int_0^{\frac{\pi}{2}}\int_0^\infty \frac{e^{-y}}{ky}\left(\frac{y}{c\epsilon\cos^k\theta}\right)^{\frac{n}{mk}}(\sin\theta)^{\frac{q-m}{m}}\cos\theta dyd\theta \\ &= \frac{\Omega_p\Omega_q}{m^2(2\pi)^{\frac{n}{2}}}\int_0^{\frac{\pi}{2}}\int_0^\infty \frac{e^{-y}y^{\frac{n}{mk}-1}}{k(c\epsilon)^{\frac{n}{mk}}}(\sin\theta)^{\frac{q}{m}-1}(\cos\theta)^{1-\frac{n}{m}}dyd\theta \\ &= \frac{\Omega_p\Omega_q}{m^2(2\pi)^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{mk}\right)}{k(c\epsilon)^{\frac{n}{mk}}}\int_0^{\frac{\pi}{2}}(\sin\theta)^{2\left(\frac{q}{2m}\right)-1}(\cos\theta)^{2\left(1-\frac{n}{2m}\right)-1}d\theta \\ &= \frac{\Omega_p\Omega_q}{2m^2(2\pi)^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{mk}\right)}{k(c\epsilon)^{\frac{n}{mk}}}\cdot\beta\left(\frac{q}{2m},\frac{2m-n}{2m}\right) \\ &= \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}}\frac{\Gamma\left(\frac{n}{mk}\right)}{k(c\epsilon)^{\frac{n}{mk}}}\frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{2m^2\Gamma\left(\frac{2m-p}{2m}\right)}. \qquad (4.10)\end{aligned}$$

Now, from (4.8), we define

$$u_\epsilon(x,t) = f(x)*\psi_\epsilon(x,t) + g(x)*\phi_\epsilon(x,t). \qquad (4.11)$$

Thus $u_\epsilon(x,t) = \int_{\mathbb{R}^n} \psi_\epsilon(y,t) f(x-y)dy + \int_{\mathbb{R}^n} \phi_\epsilon(y,t) g(x-y)dy$

$$\begin{aligned}
|u_\epsilon(x,t)| &\leq \int_{\mathbb{R}^n} |\psi_\epsilon(y,t)||f(x-y)|dy + \int_{\mathbb{R}^n} |\phi_\epsilon(y,t)||g(x-y)|dy \\
&\leq \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{mk}\right)}{k(c\epsilon)^{\frac{n}{mk}}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{2m^2\Gamma\left(\frac{2m-p}{2m}\right)} \int_{\mathbb{R}^n} |f(x-y)|dy \\
&\quad + \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{mk}-1\right)}{k\epsilon^{\frac{n}{mk}-1}c^{\frac{n}{mk}}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{2m^2\Gamma\left(\frac{2m-p}{2m}\right)} \int_{\mathbb{R}^n} |g(x-y)|dy,
\end{aligned}$$

by (4.9) and (4.10). Since $f, g \in L^1(\mathbb{R})$ and let $M = \int_{\mathbb{R}^n} |f|dy$ and $N = \int_{\mathbb{R}^n} |g|dy$ where $M$ and $N$ are constants. Thus

$$\begin{aligned}
|u_\epsilon(x,t)| &\leq \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{mk}\right)}{k(c\epsilon)^{\frac{n}{mk}}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{2m^2\Gamma\left(\frac{2m-p}{2m}\right)} M \\
&\quad + \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{mk}-1\right)}{k\epsilon^{\frac{n}{mk}-1}c^{\frac{n}{mk}}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{2m^2\Gamma\left(\frac{2m-p}{2m}\right)} N
\end{aligned}$$

$$\begin{aligned}
\epsilon^{\frac{n}{mk}}|u_\epsilon(x,t)| &\leq \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{mk}\right)}{kc^{\frac{n}{mk}}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{2m^2\Gamma\left(\frac{2m-p}{2m}\right)} M \\
&\quad + \frac{\epsilon\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{mk}-1\right)}{kc^{\frac{n}{mk}}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{2m^2\Gamma\left(\frac{2m-p}{2m}\right)} N
\end{aligned}$$

$$\lim_{\epsilon\to 0} \epsilon^{\frac{n}{mk}}|u_\epsilon(x,t)| \leq \frac{\Omega_p\Omega_q}{(2\pi)^{\frac{n}{2}}} \frac{\Gamma\left(\frac{n}{mk}\right)}{kc^{\frac{n}{mk}}} \frac{\Gamma\left(\frac{q}{2m}\right)\Gamma\left(\frac{2m-n}{2m}\right)}{2m^2\Gamma\left(\frac{2m-p}{2m}\right)} M = K \quad \text{say},$$

where $K$ is positive constant. Now $u_\epsilon(x,t) \to u(x,t)$ as $\epsilon \to 0$. Thus we obtain $u(x,t) = \mathrm{O}(\epsilon^{\frac{-n}{mk}})$ as the solution of (4.1) which is bounded by the $\epsilon-$approximation.

Now,if we put $m = k = 1$ and $q = 0$ in (4.1), then it become the generalized $n$-dimensional wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) - c^2 \triangle u(x,t) = 0.$$

This complete the proof. □

For the example, If we put $n = 1, q = 0, m = 2, k = 1, c = 1/3$ in (4.1) and $u(x,0) = \sin(\sqrt{3}x)$, we have the equation

$$\frac{\partial^2}{\partial t^2} u(x,t) + \frac{1}{9}\frac{\partial^4}{\partial x^4} u(x,t) = 0. \tag{4.12}$$

From (3.8) we obtain $u(x,t) = \cos t \sin(\sqrt{3}x)$ is the solution of (4.12). Graphical solution shows below.

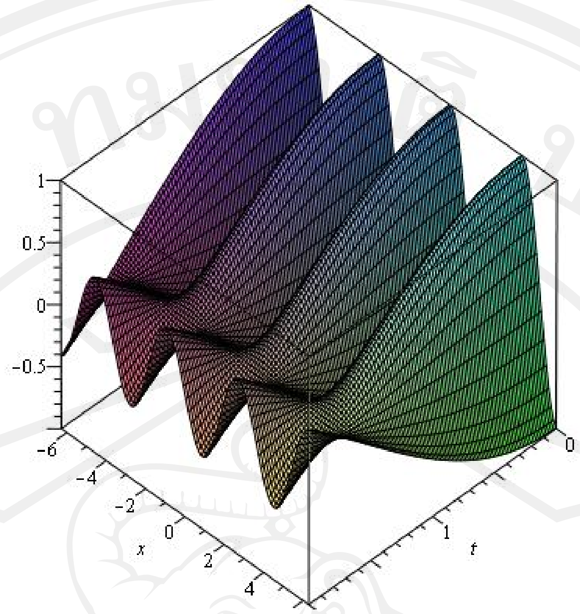


Figure 4.1: The solution $u(x,t) = \cos t \sin(\sqrt{3}x)$.

For another example, we put $n = 2, q = 0, m = 2, k = 1, c = 1/2$ in (4.1) and $u(x, 0) = 0$, we have the equation

$$\frac{\partial^2}{\partial t^2}u(x,y,t) + \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 u(x,y,t) = 0. \tag{4.13}$$

then $u(x,y,t) = \sin t \sin x \sin y$ is the solution of (4.13).