# Chapter 5

# On the Nonlinear Product of Laplacian Related to the Biharmonic Equation

In this chapter, we study the nonlinear product of Laplacian related to the nonhomogeneous Biharmonic equation.

**Theorem 5.1** *Given the nonlinear equation*

$$\triangle^k(\triangle + m^2)^k u(x) = f(x, \triangle^{k-1}(\triangle + m^2)^k u(x)) \tag{5.1}$$

*where* $\triangle^k$ *is the Laplacian iterated* $k$ *times, defined by*

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{5.2}$$

*and* $(\triangle + m^2)^k$ *is the Helmholtz operator iterated* $k$ *times, defined by*

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + m^2. \tag{5.3}$$

*Let* $f$ *be defined and have continuous first derivatives for all* $x \in \Omega \cup \partial\Omega, \Omega$ *is an open subset of* $\mathbb{R}^n$ *and* $\partial\Omega$ *denotes the boundary of* $\Omega$ *and* $n$ *is even with* $n \geq 4$. *Let* $f$ *be a bounded function, that is*

$$|f(x, \triangle^{k-1}(\triangle + m^2)^k u(x))| \leq N, \tag{5.4}$$

*where* $N$ *is a positive constant and the boundary condition*

$$\triangle^{k-1}(\triangle + m^2)^k u(x) = 0, \quad \textit{for } x \in \partial\Omega \tag{5.5}$$

*then we obtain*

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * W^e_{2k}(v, m) * W(x) \tag{5.6}$$

*as a solution of* (5.1) *with the boundary condition*

$$u(x) = (-1)^{k-2}(R^e_{2(k-2)}(x))^{(l)} * W^e_{2k}(v, m)$$

*for* $x \in \partial\Omega, l = (n-4)/2,\ k = 2, 3, 4, \ldots$ *and* $v$ *is given by* (2.25)*,* $W(x)$ *is a continuous function for* $x \in \Omega \cup \partial\Omega$*.* $R^e_{2(k-2)}(x)$*, with* $\beta = 2(k-2)$*, and* $W^e_{2k}(v, m)$ *are given by* (2.26) *and* (2.27)*, respectively. Moreover, for* $k = 1$ *then* (5.1) *becomes*

$$\triangle(\triangle + m^2)u(x) = f(x, (\triangle + m^2)u(x)) \tag{5.7}$$

*with boundary condition*

$$(\triangle + m^2)u(x) = 0, \quad for\ x \in \partial\Omega; \tag{5.8}$$

*we have*

$$u(x) = W^e_2(v, m) * W(x) \tag{5.9}$$

*as a solution of* (5.7) *and we can write* (5.7) *as*

$$\triangle^2 u(x) = g(x, \triangle u(x)),$$

*which is called the nonhomogeneous biharmonic equation, where* $g(x, \triangle u(x)) = f(x, (\triangle + m^2)u(x)) - m^2 \triangle u(x)$*.*

**Proof.** From equation (5.1), we have

$$\begin{aligned}\triangle^k(\triangle + m^2)^k u(x) &= \triangle(\triangle^{k-1}(\triangle + m^2)^k u(x)) \\ &= f(x, \triangle^{k-1}(\triangle + m^2)^k u(x)).\end{aligned} \tag{5.10}$$

Since $u(x)$ has continuous derivatives up to order $2k$ for $k = 1, 2, 3, \ldots$ we can assume

$$\triangle^{k-1}(\triangle + m^2)^k u(x) = W(x), \quad for\ x \in \partial\Omega. \tag{5.11}$$

Thus, (5.10) can be written in the form

$$\triangle^k u(x) = \triangle W(x) = f(x, W(x)). \tag{5.12}$$

by (5.4)

$$|f(x, W(x))| \leq N. \tag{5.13}$$

and by (5.5), $W(x)$=0 or

$$\triangle^{k-1}(\triangle + m^2)^k u(x) = 0, \quad for\ x \in \partial\Omega. \tag{5.14}$$

Thus by Lemma (2.3.12) there exists a unique solution $W(x)$ of (5.12) which satisfies (5.13).

Now consider the Equation (5.11); we have $(-1)^{k-1}R^e_{2(k-1)}(x)$ and $W^e_{2k}(v, m)$ are the elementary solutions of the operators $\triangle^{k-1}$ and $(\triangle+m^2)^k$, respectively. Thus, convolving both sides of (5.11) by $(-1)^{k-1}R^e_{2(k-1)}(x) * W^e_{2k}(v,m)$ we obtain

$$\begin{aligned}
[(-1)^{k-1}R^e_{2(k-1)}(x) * W^e_{2k}(v,m)] * \triangle^{k-1}(\triangle+m^2)^k u(x) \\
= (-1)^{k-1}R^e_{2(k-1)}(x) * W^e_{2k}(v,m) * W(x).
\end{aligned}$$

By properties of convolution, we obtain

$$\begin{aligned}
[\triangle^{k-1}(-1)^{k-1}R^e_{2(k-1)}(x)][(\triangle+m^2)^k W^e_{2k}(v,m)] * u(x) \\
= (-1)^{k-1}R^e_{2(k-1)}(x) * W^e_{2k}(v,m) * W(x) \\
\delta * \delta * u(x) = (-1)^{k-1}R^e_{2(k-1)}(x) * W^e_{2k}(v,m) * W(x).
\end{aligned}$$

Thus

$$u(x) = (-1)^{k-1}R^e_{2(k-1)}(x) * W^e_{2k}(v,m) * W(x) \tag{5.15}$$

as required. Consider $\triangle^{k-1}(\triangle+m^2)^k u(x) = 0,\ for\ x \in \partial\Omega$. By Lemma (2.3.9), we have

$$\begin{aligned}
(\triangle+m^2)^k u(x) = (-1)^{k-2}(R^e_{2(k-2)}(x))^{(l)} \\
u(x) = (-1)^{k-2}(R^e_{2(k-2)}(x))^{(l)} * W^e_{2k}(v,m)
\end{aligned}$$

$for\ x \in \partial\Omega$ and $k = 2, 3, 4, \ldots$.

Moreover, if we put $k = 1$ in (5.1), then

$$\triangle(\triangle+m^2)u(x) = f(x, (\triangle+m^2)u(x)) \tag{5.16}$$

with boundary condition

$$(\triangle+m^2)u(x) = 0, \quad for\ x \in \partial\Omega,$$

respectively, we obtain

$$u(x) = W^e_{2k}(v,m) * W(x).$$

From (5.16) we can write

$$\triangle^2 u(x) = g(x, (\triangle)u(x)), \tag{5.17}$$

where $g(x,(\triangle)u(x)) = f(x,(\triangle+m^2)u(x)) - m^2\triangle u(x)$ and (5.17) is called the nonhomogeneous biharmonic equation. This completes the proof. □

# Chapter 6

# Conclusion

In this thesis, we study the generalized solution of the operator $L_m^k$ related to the generalized heat equation and spectrum. Next, we study the operator $L_m^k$ related to the generalized wave equation by using $\epsilon$ approximation. Finally, we study the nonlinear product of Laplacian related to the nonhomogeneous Biharmonic equation. The results obtained in this thesis extend and improve several results obtained in this area. The results are summarized as follows.

**Theorem 1** *Given the equation*

$$\frac{\partial}{\partial t}u(x,t) + c^2 L_m^k u(x,t) = 0 \tag{6.1}$$

*with initial condition*

$$u(x,0) = f(x) \tag{6.2}$$

*where $L_m^k$ is the operator iterated $k$-times and defined by*

$$L_m^k = (-1)^{mk}\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^m - \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^m\right]^k$$

*$p+q=n$ is the dimension of the space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is a given generalized function, $k$ and $m$ are positive integers and $c$ is a positive constant. Then*

$$u(x,t) = E(x,t) * f(x) \tag{6.3}$$

*is a solution of* (6.1) *which satisfies* (6.2), *where $E(x,t)$ is given by*

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega} e^{\left[-c^2 t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^m - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^m\right)^k + i(\xi,x)\right]} d\xi \tag{6.4}$$

.

**Theorem 2** *The kernel $E(x,t)$ is defined by (6.4) has the following properties:*

(1) $E(x,t) \in C^{\infty}(\mathbb{R}^n \times (0,\infty))$ *the space of continuous with infinitely differentiable,*

(2) $(\frac{\partial}{\partial t} + c^2 L_m^k)E(x,t) = 0$, *for* $t > 0$,

(3) $|E(x,t| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$, *for* $t > 0$,

*where* $M(t) = \int_0^R \int_0^T e^{-c^2 t \left(r^{2m} - s^{2m}\right)^k} r^{p-1} s^{q-1} drds$ *is a function of* $t > 0$ *in the the spectrum* $\Omega$ *and* $\Gamma$ *denote the Gamma function. Thus* $E(x,t)$ *is bounded for any fixed* $t > 0$.

(4) $\lim_{t \to 0} E(x,t) = \delta(x)$.

**Theorem 3** *Given the equation*

$$\frac{\partial^2}{\partial t^2} u(x,t) + c^2 L_m^k, u(x,t) = 0, \tag{6.5}$$

*where* $L_m^k$ *defined by*

$$L_m^k = (-1)^{mk} \left[ \left( \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} \right)^m - \left( \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^m \right]^k,$$

*with initial conditions*

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial}{\partial t} u(x,0) = g(x), \tag{6.6}$$

*where* $u(x,t) \in \mathbb{C}$, $c$ *is a positive constant,* $k$ *and* $m$ *are nonnegative integer,* $f$ *and* $g$ *are continuous functions and absolutely integrable for* $x \in \mathbb{R}^n$. *Then* (6.5) *has a unique solution*

$$u(x,t) = f(x) * \psi(x,t) + g(x) * \phi(x,t) \tag{6.7}$$

*and satisfy the condition* (6.6) *where* $\phi(x,t)$ *is an inverse Fourier transform of*

$$\widehat{\phi}(\xi,t) = \frac{\sin ct\sqrt{\left(r^{2m} - s^{2m}\right)^k}}{c\sqrt{\left(r^{2m} - s^{2m}\right)^k}}$$

*and* $\psi(x,t)$ *is an inverse Fourier transform of*

$$\widehat{\psi}(\xi,t) = \cos ct\sqrt{\left(r^{2m} - s^{2m}\right)^k} = \frac{\partial}{\partial t}\widehat{\phi}(\xi,t),$$

*where* $r^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_p^2$, $s^2 = \xi_{p+1}^2 + \xi_{p+2}^2 + \cdots + \xi_{p+q}^2$ *and* $r > s > 0$. *Moreover, if we put* $m = k = 1$ *and* $q = 0$ *in* (6.5), *then it become the generalized* $n$*-dimensional wave equation*

$$\frac{\partial^2}{\partial t^2} u(x,t) - c^2 \triangle u(x,t) = 0.$$

**Theorem 4** *Given the nonlinear equation*

$$\triangle^k(\triangle + m^2)^k u(x) = f(x, \triangle^{k-1}(\triangle + m^2)^k u(x)) \tag{6.8}$$

*where* $\triangle^k$ *is the Laplacian iterated* $k$ *times, defined by*

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}. \tag{6.9}$$

*and* $(\triangle + m^2)^k$ *is the Helmholtz operator iterated* $k$ *times, defined by*

$$\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2} + m^2. \tag{6.10}$$

*Let* $f$ *be defined and have continuous first derivatives for all* $x \in \Omega \cup \partial\Omega, \Omega$ *is an open subset of* $\mathbb{R}^n$ *and* $\partial\Omega$ *denotes the boundary of* $\Omega$ *and* $n$ *is even with* $n \geq 4$. *Let* $f$ *be a bounded function, that is*

$$|f(x, \triangle^{k-1}(\triangle + m^2)^k u(x))| \leq N \tag{6.11}$$

*where* $N$ *is a positive constant and the boundary condition*

$$\triangle^{k-1}(\triangle + m^2)^k u(x) = 0, \quad for\ x \in \partial\Omega \tag{6.12}$$

*then we obtain*

$$u(x) = (-1)^{k-1} R^e_{2(k-1)}(x) * W^e_{2k}(v, m) * W(x) \tag{6.13}$$

*as a solution of* (6.8) *with the boundary condition*

$$u(x) = (-1)^{k-2}(R^e_{2(k-2)}(x))^{(l)} * W^e_{2k}(v, m)$$

*for* $x \in \partial\Omega, l = (n-4)/2,\ k = 2, 3, 4, \ldots$ *and* $v$ *is given by* (2.25), $W(x)$ *is a continuous function for* $x \in \Omega \cup \partial\Omega$. $R^e_{2(k-2)}(x)$, *with* $\beta = 2(k-2)$, *and* $W^e_{2k}(v, m)$ *are given by* (2.26) *and* (2.27), *respectively. Moreover, for* $k = 1$ *then* (6.8) *becomes*

$$\triangle(\triangle + m^2)u(x) = f(x, (\triangle + m^2)u(x)) \tag{6.14}$$

*with boundary condition*

$$(\triangle + m^2)u(x) = 0, \quad for\ x \in \partial\Omega; \tag{6.15}$$

*we have*

$$u(x) = W^e_2(v, m) * W(x) \tag{6.16}$$

*as a solution of* (6.14) *and we can write* (6.14) *as*

$$\triangle^2 u(x) = g(x, \triangle u(x)),$$

*which is called the nonhomogeneous biharmonic equation, where* $g(x, \triangle u(x)) = f(x, (\triangle + m^2)u(x)) - m^2 \triangle u(x)$.