

Chapter 2

Preliminaries and basic concepts

In this chapter, we give some basic definitions, notations, lemmas and results which will be used in the later chapters.

2.1 Notations

The following notations that will be used in this thesis.

$\mathbb{R}^n$ denote the n dimensional Euclidean space,

$\mathbb{R}^{n \times m}$ denote the set of all $n \times m$ real matrices,

$\mathbb{R}^+$ denote the set of positive real numbers,

$\mathbb{R}_0^+$ denote the set of nonnegative real numbers,

$\mathbb{Z}$ denote the set of integers,

$\mathbb{Z}^{-r} := \{z \in \mathbb{Z} : z \geq -r\}$,

$\overline{[M, N]} := \{M, M + 1, \dots, N\}$, where $M < N \in \mathbb{Z}$

$\|x\|$ denote the Euclidean norm of vector x ,

$\text{diag}\{\cdot\}$ denote the block diagonal matrix,

I denote the identity matrix,

A^T denote the transpose of matrix A ,

A^{-1} denote the inverse of matrix A ,

$\begin{bmatrix} A & B \\ * & C \end{bmatrix} - *$ represents the symmetric form of matrix, namely $* = B^T$,

$A < B$ denote the $A - B$ matrix is a square symmetric negative definite matrix,

$\det(A)$ denote determinant of the square matrix A .

2.2 Types of matrix

Definition 2.2.1 (Symmetric matrix) A real $n \times n$ matrix A is called *symmetric* if

$$A^T = A.$$

Definition 2.2.2 (Positive Definite Matrix) A real $n \times n$ matrix A is called *positive definite* if

$$x^T A x > 0,$$

for all nonzero vectors $x \in \mathbb{R}^n$. It is called *positive semidefinite* if

$$x^T A x \geq 0.$$

Definition 2.2.3 (Negative Definite Matrix) A real $n \times n$ matrix A is called *negative definite* if

$$x^T A x < 0,$$

for all nonzero vectors $x \in \mathbb{R}^n$. It is called *negative semidefinite* if

$$x^T A x \leq 0.$$

The follows result are well known.

Lemma 2.2.4 A symmetric matrix is positive semidefinite (definite) matrix if all of its eigenvalues are nonnegative (positive).

Lemma 2.2.5 A symmetric matrix is negative semidefinite (definite) matrix if all of its eigenvalues are nonpositive (negative).

2.3 Several discrete-time neural networks system with time-varying delay

1. Discrete-time LPD and stochastic LPD neural networks with time-varying delay

We propose to study the robust stability problem of uncertain discrete-time neural networks with time-varying delay of the form

$$x(k+1) = -[A(\xi) + \Delta A]x(k) + [W(\xi) + \Delta W]f(x(k)) + [W_1(\xi) + \Delta W_1]g(x(k - \tau(k))), \quad (2.1)$$

and uncertain stochastic discrete-time neural networks with time-varying delay:

$$\begin{aligned} x(k+1) = & -[A(\xi) + \Delta A(k)]x(k) + [W(\xi) + \Delta W(k)]f(x(k)) + [W_1(\xi) \\ & + \Delta W_1(k)]g(x(k - \tau(k))) + \sigma(x(k), x(k - \tau(k)), k)w(k), \end{aligned} \quad (2.2)$$

where $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]$ is the state vector, $\tau(k)$ is a positive integer denotes the time-varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2,$$

where $\tau_1 \geq 0$ and $\tau_2 \geq 0$ are known integers, $A(\xi)$, $W(\xi)$ and $W_1(\xi)$ (the inter-connection matrices) are of polytopic types where

$$\begin{aligned} [A(\xi) \quad W(\xi) \quad W_1(\xi)] &\in \Omega, \\ \Omega = \left\{ [A(\xi) \quad W(\xi) \quad W_1(\xi)] = \sum_{i=1}^N \xi_i [A_i \quad W_i \quad W_{1i}], \sum_{i=1}^N \xi_i = 1, \xi_i \geq 0 \right\}, \end{aligned} \quad (2.3)$$

where A_i , W_i and W_{1i} are known constant matrices and ΔA , ΔW and ΔW_1 are uncertain matrices which are of the form

$$\Delta A = H_0 F_0 E_0, \Delta W = H F E \text{ and } \Delta W_1 = H_1 F_1 E_1, \quad (2.4)$$

where H_0 , H , H_1 , E_0 , E and E_1 are known constant matrices F_0 , F and F_1 are unknown matrices which satisfy

$$F_0^T F_0 \leq I, F^T F \leq I \text{ and } F_1^T F_1 \leq I, \quad (2.5)$$

where I is the identity matrix of appropriate dimension, $w(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$\mathbb{E}[w(k)] = 0, \mathbb{E}[w^2(k)] = 1, \mathbb{E}[w(i)w(j)] = 0 \ (i \neq j) \quad (2.6)$$

and $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ is a continuous function, and is assumed to satisfy

$$\sigma^T(x, y, k) \sigma(x, y, k) \leq \rho_1 x^T x + \rho_2 y^T y, \quad x, y \in \mathbb{R}^n, \quad (2.7)$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are known constant scalars.

2. Discrete-time neural networks with time-varying delays

We propose to study the global exponential stability for discrete-time neural networks with time-varying delays:

$$x(k+1) = Cx(k) + Ag(x(k)) + \sum_{j=1}^M B_j g(x(k - \tau_j(k))), \quad (2.8)$$

where $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]$ is the state vector, $g_j(x_j(\cdot)) = \{g_1(x_1(\cdot)), \dots, g_n(x_n(\cdot))\}$, $j = 1, 2, \dots, n$ and the transformed activation functions satisfy the conditions

$$|g_j(x)| \leq L_j |x|,$$

and $\tau_j(k)$ is a positive integer denotes the time-varying delay satisfying

$$0 \leq \tau_j(k) \leq \tau_j,$$

where τ_j is known positive integer.

3. H_∞ filter for nonlinear discrete-time neural networks with time-varying delay

We propose to study the H_∞ filtering problem of discrete-time neural networks with time-varying delay of the form:

$$\begin{cases} x(k+1) = A_1x(k) + A_2f(x(k)) + A_3g(x(k-\tau(k))) + B\omega(k) \\ y(k) = C_1x(k) + C_2f(x(k)) + C_3g(x(k-\tau(k))) + D\omega(k) \\ z(k) = K_1x(k) + K_2f(x(k)) + K_3g(x(k-\tau(k))) + G\omega(k) \\ x(k) = \phi(k), \quad k = -\tau_2, \tau_2 + 1, \dots, 0, \end{cases} \quad (2.9)$$

where $x(k) \in \mathbb{R}^n$ is the neuron state vector, $y(k) \in \mathbb{R}^m$ is the measurement vector, the noise signal vector $\omega(k) \in \mathbb{R}^q$ belongs to $l_2[0, +\infty)$, $\tau(k)$ is time-varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2, \quad (2.10)$$

where $\tau_1, \tau_2 \geq 0$ are known integers, $A_1, A_2, A_3, B, C_1, C_2, C_3, D, K_1, K_2, K_3$ and G are the constant matrices with appropriate dimensions (the interconnection matrices) and the activation function $f_i(\cdot)$ $i = 1, \dots, n$ and $g_i(\cdot)$ $i = 1, \dots, n$ satisfy the following conditions

$$l_j^- \leq \frac{f_j(x) - f_j(y)}{x - y} \leq l_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (2.11)$$

$$v_j^- \leq \frac{g_j(x) - g_j(y)}{x - y} \leq v_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (2.12)$$

where $l_j^-, l_j^+, v_j^-, v_j^+, j = 1, 2, \dots, n$ are known constants.

2.4 Preliminaries

2.4.1 Autonomous systems

Consider the autonomous system of difference equation of the form

$$x(k+1) = f(x), \quad x(k_0) = x_0, \quad (2.13)$$

where $x \in \mathbb{R}^n$, $x_i = x_i(k)$, $i = 1, 2, \dots, n$, $k \geq k_0$, $k \in \mathbb{Z}^+$ and $f = (f_1, f_2, \dots, f_n) : \mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition 2.4.1 A point $\bar{x}$ is called an *equilibrium point* of equation (2.13) if $f(\bar{x}) = \bar{x}$ for all $k \geq k_0$. For all purposes of the stability theory we can assume that $\bar{0}$ is an equilibrium of (2.13).

Definition 2.4.2 [16] The equilibrium point $x^* = 0$ of the system (2.13) is

- (i) *stable* at $k = k_0$ if $\forall \epsilon > 0, \exists \delta = \delta(\epsilon) > 0$ such that

$$\|x_0\| < \delta \Rightarrow \|x\| < \epsilon, \quad \forall k \geq k_0;$$

- (ii) *unstable* if not stable at $k = k_0$, that is $\exists \epsilon > 0$ such that, $\forall \delta > 0$ so that

$$\|x_0\| < \delta \Rightarrow \|x(k_1)\| \geq \epsilon, \quad \exists k_0 > 0.$$

If this holds for every x_0 in $\|x_0\| < \delta$ the equilibrium point is completely unstable ;

- (iii) *asymptotically stable* if $x^* = 0$ is stable and $\exists \delta(k_0)$ such that

$$\|x_0\| < \delta \Rightarrow \lim_{k \rightarrow \infty} x(k) = 0;$$

- (iv) *exponentially stable* if there exist three positive real constants ϵ, K and λ such that

$$\|x(k)\| \leq K\|x_0\|e^{-\lambda(k-k_0)}, \quad \forall \|x_0\| < \epsilon, \quad k \geq k_0;$$

The largest constant λ which may be utilized in above inequality is called the rate of convergence.

- (v) *globally asymptotically stable* if $x^* = 0$ is stable and $\forall x_0 \in \mathbb{R}^n$

$$\lim_{k \rightarrow \infty} x(k) = 0.$$

Definition 2.4.3 (Lyapunov Function [16]) Let D be a domain $\mathbb{R}^n$ such that $\bar{0} \in D$ $V : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, we say that $V(x)$ is a Lyapunov function of system (2.13) if the following conditions hold :

- (1) $V(x)$ is continuous on $D \subseteq \mathbb{R}^n$.
- (2) $V(x)$ is positive definite such that $V(\bar{0}) = 0$ and $V(x(k)) > 0$ for $x(k) \neq \bar{0}$.
- (3) $\Delta V(x(k)) = V(x(k+1)) - V(x(k))$ is negative semidefinite such that $\Delta V(\bar{0}) = 0$ and $\Delta V(x(k)) \leq 0$ for $x(k) \neq \bar{0}$.

Theorem 2.4.4 [16] Let $x^* = 0$ be the equilibrium point of the system (2.13) and D be a domain $\mathbb{R}^n$ such that $\bar{0} \in D$. Let $V(x(k)) : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function, such that

$$V(0) = 0 \text{ and } V(x(k)) > 0 \text{ in } D - \{0\},$$

$$\Delta V(x(k)) \leq 0 \text{ in } D - \{0\}.$$

Then, $x^* = 0$ is stable. Moreover, if

$$\Delta V(x(k)) < 0 \text{ in } D - \{0\},$$

then $x^* = 0$ is asymptotically stable.

Theorem 2.4.5 [16] Let $x^* = 0$ be the equilibrium point of the system (2.13) and D be a domain $\mathbb{R}^n$ such that $\bar{0} \in D$. Let $V(x(k)) : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function, such that

$$V(0) = 0 \text{ and } V(x(k)) > 0 \text{ in } \forall x \neq 0,$$

$$\|x\| \rightarrow \infty \Rightarrow V(x(k)) \rightarrow \infty,$$

$$\Delta V(x(k)) < 0, \quad \forall x \neq 0,$$

then $x^* = 0$ is globally asymptotically stable.

2.4.2 Nonautonomous systems

Consider the nonautonomous system of difference equation of the form

$$x(k+1) = f(k, x(k)), \quad x(k_0) = x_0 \text{ and } f(k, \bar{0}) = \bar{0}, \quad k \in \mathbb{Z}^+, \quad k \geq k_0 \quad (2.14)$$

where $x \in \mathbb{R}^n$, $x_i = x_i(k)$, $i = 1, 2, \dots, n$, $k \geq k_0$, $k \in \mathbb{Z}^+$
and $f = (f_1, f_2, \dots, f_n) : \mathbb{Z} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

Definition 2.4.6 A point $\bar{x}$ is called an *equilibrium point* of equation (2.14) if $f(k, \bar{x}) = \bar{x}$ for all $k \geq k_0$. For all purposes of the stability theory we can assume that $\bar{0}$ is an equilibrium of (2.14).

Definition 2.4.7 [16] The equilibrium point $x^* = 0$ of the system (2.14) is

- (i) *stable* at $k = k_0$ if $\forall \epsilon > 0, \exists \delta = \delta(k_0, \epsilon) > 0$ such that

$$\|x(k_0)\| < \delta \Rightarrow \|x(k)\| < \epsilon, \quad \forall k \geq k_0 \geq 0;$$

- (ii) *unstable* if not stable ;

(iii) *asymptotically stable* if $x^* = 0$ is stable and $\exists \delta(k_0) > 0$ such that

$$\|x(k_0)\| < \delta \Rightarrow \lim_{k \rightarrow \infty} x(k) = 0;$$

(iv) *exponentially stable* if there exist three positive real constants ϵ, K and λ such that

$$\|x(k)\| \leq K\|x_0\|e^{-\lambda(k-k_0)}, \quad \forall \|x_0\| < \epsilon, \quad k \geq k_0;$$

The largest constant λ which may be utilized in above inequality is called the rate of convergence.

(v) *globally asymptotically stable* if $x^* = 0$ is stable and $\forall x_0 \in \mathbb{R}^n$

$$\lim_{k \rightarrow \infty} x(k) = 0.$$

Definition 2.4.8 (Lyapunov Function) Let D be a domain $\mathbb{R}^n$ such that $\bar{0} \in D$. $V : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, we say that $V(x)$ is a Lyapunov function of system (2.13) if the following conditions hold :

- (1) $V(k, x(k))$ is continuous on $D \subseteq \mathbb{R}^n$.
- (2) $V(k, x(k))$ is positive definite such that $V(\bar{0}) = 0$ and $V(k, x(k)) > 0$ for $x(k) \neq \bar{0}$.
- (3) $\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k))$ is negative semidefinite such that $\Delta V(0, \bar{0}) = 0$ and $\Delta V(k, x(k)) < -\lambda \|x(k)\|^2$, for some $\lambda > 0$.

Definition 2.4.9 [16] (Robustly Stable) Consider the system

$$x(k+1) = (A + \Delta A)f(k, x(k)), \quad x(k_0) = x_0 \quad \text{and} \quad f(k, \bar{0}) = \bar{0}, \quad k \in \mathbb{Z}^+, \quad k \geq k_0, \quad (2.15)$$

where $A = \{a_{ij}\} \in \mathbb{R}^{n \times n}$, ΔA is uncertain matrix. The system (2.15) is said to be *robustly stable* if for all ΔA which is the form $\Delta A = HFE$, where H and E are known constant matrices and F is unknown matrix satisfying $F^T F \leq I$, $\bar{0}$ is A.S.

Definition 2.4.10 [16] The zero solution of discrete-time system (2.1) and (2.2) with $\Delta A = \Delta W = \Delta W_1 = 0$, is asymptotically stable if there exists a positive definite scalar function $V(k, x(k)) : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)) < 0,$$

along any trajectory of solution of the system.

Definition 2.4.11 [27] The discrete-time system (2.1) and (2.2) is robustly stable if there exists a positive definite scalar function $V(k, x(k)) : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)) < 0,$$

along the solution of the system for all uncertainties which satisfy (2.3), (2.4) and (2.5).

Definition 2.4.12 [20] The discrete-time system (2.1) and (2.2) is robustly exponentially stable if there exists $\kappa > 0$ and $0 < \alpha < 1$ such that every solution of the system (2.1) satisfies

$$\|x(k)\| \leq \kappa \alpha^k \max_{-\tau_2 \leq j \leq 0} \|x(j)\|, \quad k \geq 0.$$

Definition 2.4.13 The discrete-time system (2.2) is robustly stable in the mean square if there exists a positive definite scalar function $V(k, x(k)) : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$\mathbb{E}\{\Delta V(k, x(k))\} = \mathbb{E}\{V(k+1, x(k+1)) - V(k, x(k))\} < 0,$$

along any trajectory of solution of the system for all uncertainties which satisfy (2.3), (2.4) and (2.5).

Definition 2.4.14 [35] The solution of (2.8) is globally exponentially stable if for any solution $x(k, \phi_i)$ with the initial condition $x_i(l) = \phi_i(l)$ for $l \in [-\tau_M, 0]$ from $\mathbb{R}^m$ there exist constant $\epsilon \in (0, 1)$ and $D \geq 1$ such that

$$\|x(k)\| \leq D \|\phi\| \epsilon^n \quad \forall n \leq 0,$$

where $\|\phi\| = \max_{l \in [-\tau_M, 0]} \{\|\phi(l)\|\}$.

The following Lemmas will be used throughout this thesis.

Lemma 2.4.15 (Schur Complement [5]) Given constant symmetric matrices Q , S and $R \in \mathbb{R}^{n \times n}$ where $R(x) > 0$, $Q(x) = Q^T(x)$ and $R(x) = R^T(x)$, we have

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} < 0 \Leftrightarrow Q(x) - S(x)R^{-1}(x)S^T(x) < 0.$$

Lemma 2.4.16 (S-procedure [5]) Let $T_i \in \mathbb{R}^{n \times n}$ ($i = 1, 2, \dots, p$) be matrices. The conditions on T_i ($i = 1, 2, \dots, p$),

$$\zeta^T T_0 \zeta > 0, \quad \forall \zeta \neq 0 \text{ such that } \zeta^T T_i \zeta \geq 0 \quad (i = 1, 2, \dots, p),$$

hold if there exist $\tau_i \geq 0$ ($i = 1, 2, \dots, p$) such that

$$T_0 - \sum_{i=1}^p \tau_i T_i > 0.$$