

# Chapter 3

## Robust stability criteria of discrete-time neural networks with time-varying delay

In this chapter, we study the problem of robust stability for discrete-time neural networks with time-varying delay and stochastic neural networks with time-varying delay with norm-bounded uncertainties as well as polytopic type uncertainties. The linear system with polytopic type uncertainties is called linear parameter dependent system. Based on Lyapunov stability theory and the S-procedure, we derive robust stability criteria in terms of linear matrix inequalities which are solvable by several available algorithms. Numerical examples are presented to illustrate the effectiveness of the theoretical results.

### 3.1 Robust Stability Criteria of LPD Neural Networks with time-varying delay

Consider the following uncertain discrete-time LPD neural networks with time-varying delay:

$$\begin{aligned} u(k+1) = & -[A(\xi) + \Delta A]u(k) + [W(\xi) + \Delta W]\hat{f}(u(k)) \\ & + [W_1(\xi) + \Delta W_1]\hat{g}(u(k - \tau(k))) + b, \end{aligned} \quad (3.1)$$

where  $u(k) = [u_1(k), \dots, u_n(k)]^T \in \mathbb{R}^n$  is the neuron state vector,  $b = [b_1, \dots, b_n]^T$  is constant input vector,  $\tau(k)$  is a positive integer denotes the time-varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2,$$

where  $\tau_1 \geq 0$  and  $\tau_2 \geq 0$  are known integers,  $A(\xi)$ ,  $W(\xi)$  and  $W_1(\xi)$  (the inter-connection matrices) are of polytopic types where

$$[A(\xi) \quad W(\xi) \quad W_1(\xi)] \in \Omega,$$

$$\Omega = \left\{ [A(\xi) \quad W(\xi) \quad W_1(\xi)] = \sum_{i=1}^N \xi_i [A_i \quad W_i \quad W_{1i}], \sum_{i=1}^N \xi_i = 1, \xi_i \geq 0 \right\}, \quad (3.2)$$

where  $A_i$ ,  $W_i$  and  $W_{1i}$  are known constant matrices and  $\Delta A$ ,  $\Delta W$  and  $\Delta W_1$  are uncertain matrices which are of the form

$$\Delta A = H_0 F_0 E_0, \Delta W = H F E \text{ and } \Delta W_1 = H_1 F_1 E_1, \quad (3.3)$$

where  $H_0$ ,  $H$ ,  $H_1$ ,  $E_0$ ,  $E$  and  $E_1$  are known constant matrices  $F_0$ ,  $F$  and  $F_1$  are unknown matrices which satisfy

$$F_0^T F_0 \leq I, F^T F \leq I, \text{ and } F_1^T F_1 \leq I, \quad (3.4)$$

where  $I$  is the identity matrix of appropriate dimension,  $\hat{f}(u(\cdot)) = [\hat{f}_1(u_1(\cdot)), \dots, \hat{f}_n(u_n(\cdot))]^T$  and  $\hat{g}(u(\cdot)) = [\hat{g}_1(u_1(\cdot)), \dots, \hat{g}_n(u_n(\cdot))]^T$  are the activation functions satisfying the following conditions

$$l_j^- \leq \frac{\hat{f}_j(x) - \hat{f}_j(y)}{x - y} \leq l_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (3.5)$$

$$v_j^- \leq \frac{\hat{g}_j(x) - \hat{g}_j(y)}{x - y} \leq v_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (3.6)$$

where  $l_j^-$ ,  $l_j^+$ ,  $v_j^-$ ,  $v_j^+$  for  $j = 1, 2, \dots, n$  are some constants. By Proposition 1. in [20] guarantees the existence of an equilibrium point of system (3.1). Let  $u^* = [u_1^*, u_2^*, \dots, u_n^*]$  be an equilibrium point of system (3.1). We shift the equilibrium point  $u^*$  to the origin by the transformation  $x(\cdot) = u(\cdot) - u^*$ . Then we obtain the new system

$$x(k+1) = -[A(\xi) + \Delta A]x(k) + [W(\xi) + \Delta W]f(x(k)) + [W_1(\xi) + \Delta W_1]g(x(k - \tau(k))), \quad (3.7)$$

where  $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]$  is the state vector of the transformed system,  $\tau(k)$  is a positive integer denotes the time-varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2,$$

where  $\tau_1 \geq 0$  and  $\tau_2 \geq 0$  are known integers, with  $f_j(x_j(k)) = \hat{f}_j(x_j(k) + u_j^*) - \hat{f}_j(u_j^*)$ ,  $j = 1, 2, \dots, n$  and the transformed activation function satisfies the condition

$$l_j^- \leq \frac{f_j(x) - f_j(y)}{x - y} \leq l_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (3.8)$$

$$v_j^- \leq \frac{g_j(x) - g_j(y)}{x - y} \leq v_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (3.9)$$

where  $l_j^-$ ,  $l_j^+$ ,  $v_j^-$ ,  $v_j^+$  for  $j = 1, 2, \dots, n$ .

**Theorem 3.1.1** *The origin of system (3.7) with polytopic type uncertainties (3.2) is robustly stable if there exist  $P_i = P_i^T > 0$ ,  $Q_i = Q_i^T > 0$ ,  $T_i = \text{diag}\{t_{1i}, t_{2i}, \dots, t_{ni}\} \geq 0$ ,  $S_i = \text{diag}\{s_{1i}, s_{2i}, \dots, s_{ni}\} \geq 0$  and scalars  $e_{0i} > 0$ ,  $e_i > 0$  and  $e_{1i} > 0$ ,  $i = 1, 2, \dots, N$  satisfying the following conditions*

$$\left\{ \begin{array}{l} (i) \quad M_{i,i,i} + N_i < -I, \quad i = 1, 2, \dots, N; \\ (ii) \quad M_{i,i,j} + M_{j,i,i} + M_{i,j,i} + 2N_i + N_j < \frac{1}{(N-1)^2} I, \\ \quad i = 1, 2, \dots, N, \quad i \neq j, \quad j = 1, 2, \dots, N; \\ (iii) \quad M_{i,j,l} + M_{i,l,j} + M_{j,i,l} + M_{j,l,i} + M_{l,i,j} + M_{l,j,i} \\ \quad + 2N_i + 2N_j + 2N_l < \frac{6}{(N-1)^2} I, \\ \quad i = 1, 2, \dots, N-2, \quad j = i+1, 2, \dots, N-1, \quad l = 1, 2, \dots, N, \end{array} \right. \quad (3.10)$$

where

$$M_{i,j,l} = \begin{bmatrix} \varphi_{11} & 0 & \varphi_{13} & \varphi_{14} & \varphi_{15} & \varphi_{16} & \varphi_{17} \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \varphi_{34} & \varphi_{35} & \varphi_{36} & \varphi_{37} \\ * & * & * & * & \varphi_{45} & \varphi_{46} & \varphi_{47} \\ * & * & * & * & * & \varphi_{56} & \varphi_{57} \\ * & * & * & * & * & \varphi_{66} & \varphi_{67} \\ * & * & * & * & * & * & \varphi_{77} \end{bmatrix},$$

$$N_i = \begin{bmatrix} \Pi_{11}(i) & 0 & -2L_2T_i & 0 & 0 & 0 & 0 \\ 0 & -Q_i - 2V_1S_i & 0 & -2V_2S_i & 0 & 0 & 0 \\ -2L_2T_i & 0 & \Pi_{33}(i) & 0 & 0 & 0 & 0 \\ 0 & -2V_2S_i & 0 & \Pi_{44}(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Pi_{55}(i) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Pi_{66}(i) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Pi_{77}(i) \end{bmatrix},$$

and  $\varphi_{11} = A_i^T P_j A_l$ ,  $\varphi_{13} = -A_i^T P_j W_l$ ,  $\varphi_{14} = -A_i^T P_j W_{1l}$ ,  $\varphi_{15} = A_i^T P_j H_0$ ,

$\varphi_{16} = -A_i^T P_j H$ ,  $\varphi_{17} = -A_i^T P_j H_1$ ,  $\varphi_{34} = i^T P_j W_{1l}$ ,  $\varphi_{35} = -W_i^T P_j H_0$ ,

$\varphi_{36} = W_i^T P_j H$ ,  $\varphi_{37} = W_i^T P_j H_1$ ,  $\varphi_{45} = -W_{1i}^T P_j H_0$ ,  $\varphi_{46} = W_{1i}^T P_j H$ ,

$\varphi_{47} = W_{1i}^T P_j H_1$ ,  $\varphi_{56} = -H_0^T P_i H$ ,  $\varphi_{57} = -H_0^T P_i H_1$ ,  $\varphi_{66} = H^T P_i H$ ,

$\varphi_{67} = H^T P_i H_1$ ,  $\varphi_{77} = H_1^T P_i H_1$ ,

$\hat{\tau} = \tau_2 - \tau_1 + 1$ ,  $\Pi_{11}(i) = e_{0i} E_0^T E_0 - P_i + \hat{\tau} Q_i - 2L_1 T_i$ ,  $\Pi_{33}(i) = -2T_i + e_i E^T E$ ,

$$\begin{aligned} \Pi_{44}(i) &= -2S_i + e_{1i}E_1^TE_1, \quad \Pi_{55}(i) = -e_{0i}I, \quad \Pi_{66}(i) = -e_iI, \quad \Pi_{77}(i) = -e_{1i}I, \\ L_1 &= \text{diag}(l_1^+l_1^-, l_2^+l_2^-, \dots, l_n^+l_n^-), \quad L_2 = \text{diag}\left(-\frac{l_1^++l_1^-}{2}, -\frac{l_2^++l_2^-}{2}, \dots, -\frac{l_n^++l_n^-}{2}\right), \\ V_1 &= \text{diag}(v_1^+v_1^-, v_2^+v_2^-, \dots, v_n^+v_n^-) \text{ and } V_2 = \text{diag}\left(-\frac{v_1^++v_1^-}{2}, -\frac{v_2^++v_2^-}{2}, \dots, -\frac{v_n^++v_n^-}{2}\right). \end{aligned}$$

*Proof.* Consider the Lyapunov function candidate

$$V(x(k), \xi) = V_1(x(k), \xi) + V_2(x(k), \xi) + V_3(x(k), \xi),$$

where

$$\begin{aligned} V_1(x(k), \xi) &= \sum_{i=1}^N x^T(k) \xi_i P_i x(k), \quad V_2(x(k), \xi) = \sum_{i=1}^N \sum_{l=k-\tau(k)}^{k-1} x^T(l) \xi_i Q_i x(l), \\ V_3(x(k), \xi) &= \sum_{i=1}^N \sum_{j=k-\tau(k)}^{-\tau_1+1} \sum_{l=k+j-1}^{k-1} x^T(l) \xi_i Q_i x(l). \end{aligned}$$

The Lyapunov difference of system along trajectory of solution of (3.7) is given by

$$\begin{aligned} \Delta V_2(x(k), \xi) &= \sum_{i=1}^N \sum_{l=k+1-\tau(k+1)}^k x^T(l) \xi_i Q_i x(l) - \sum_{i=1}^N \sum_{l=k-\tau(k)}^{k-1} x^T(l) \xi_i Q_i x(l) \\ &= \sum_{i=1}^N \sum_{l=k-\tau(k+1)}^{k-\tau_1} x^T(l) \xi_i Q_i x(l) + \sum_{i=1}^N x^T(k) \xi_i Q_i x(k) \\ &\quad - \sum_{i=1}^N x^T(k - \tau(k)) \xi_i Q_i x(k - \tau(k)) + \sum_{i=1}^N \sum_{k+1-\tau_1}^{k-1} x^T(l) \xi_i Q_i x(l) \\ &\quad - \sum_{i=1}^N \sum_{l=k+1-\tau(k)}^{k-1} x^T(l) \xi_i Q_i x(l). \end{aligned}$$

Since  $\tau(k) \geq \tau_1$ , we get

$$\sum_{i=1}^N \sum_{k+1-\tau_1}^{k-1} x^T(l) \xi_i Q_i x(l) - \sum_{i=1}^N \sum_{l=k+1-\tau(k)}^{k-1} x^T(l) \xi_i Q_i x(l) \leq 0.$$

Therefore,

$$\begin{aligned} \Delta V_2(x(k), \xi) &\leq \sum_{i=1}^N \sum_{l=k-\tau(k+1)}^{k-\tau_1} x^T(l) \xi_i Q_i x(l) + \sum_{i=1}^N x^T(k) \xi_i Q_i x(k) \\ &\quad - \sum_{i=1}^N x^T(k - \tau(k)) \xi_i Q_i x(k - \tau(k)). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\Delta V_3(x(k), \xi) &= \sum_{i=1}^N \sum_{l=-\tau_2+2}^{-\tau_1+1} \left[ x^T(k) \xi_i Q_i x(k) + \sum_{m=k+j}^{k-1} x^T(m) \xi_i Q_i x(m) \right. \\ &\quad \left. - \sum_{m=k+j-1}^{k-1} x^T(m) \xi_i Q_i x(m) \right] \\ &= \sum_{i=1}^N (\tau_2 - \tau_1) x^T(k) \xi_i Q_i x(k) - \sum_{i=1}^N \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(k) \xi_i Q_i x(k).\end{aligned}$$

From  $\Delta V_2(x(k), \xi)$  and  $\Delta V_3(x(k), \xi)$ , we get

$$\begin{aligned}\Delta V_2(x(k), \xi) + \Delta V_3(x(k), \xi) &\leq \sum_{i=1}^N \hat{\tau} x^T(k) \xi_i Q_i x(k) \\ &\quad - \sum_{i=1}^N x^T(k - \tau(k)) \xi_i Q_i x(k - \tau(k)) \\ &\quad + \sum_{i=1}^N \sum_{l=k+1}^{k-\tau_1} x^T(l) \xi_i Q_i x(l) \\ &\quad - \sum_{i=1}^N \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(k) \xi_i Q_i x(k),\end{aligned}$$

where  $\hat{\tau} = \tau_2 - \tau_1 + 1$ . Since  $\tau(k) \leq \tau_2$ , we have

$$\sum_{i=1}^N \sum_{l=k+1-\tau(k+1)}^{k-\tau_1} x^T(l) \xi_i Q_i x(l) - \sum_{i=1}^N \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(k) \xi_i Q_i x(k) \leq 0.$$

Thus,

$$\Delta V_2(x(k), \xi) + \Delta V_3(x(k), \xi) \leq \sum_{i=1}^N \hat{\tau} x^T(k) \xi_i Q_i x(k) - \sum_{i=1}^N x^T(k - \tau(k)) \xi_i Q_i x(k - \tau(k)).$$

As a result, we obtain

$$\begin{aligned}\Delta V(x(k), \xi) &\leq \left[ -(A(\xi) + \Delta A)x(k) + (W(\xi) + \Delta W)f(x(k)) + (W_1(\xi) + \Delta W_1) \right. \\ &\quad \left. \times g(x(k - \tau(k))) \right]^T P(\xi) \left[ -(A(\xi) + \Delta A)x(k) + (W(\xi) + \Delta W) \right. \\ &\quad \left. \times f(x(k)) + (W_1(\xi) + \Delta W_1)g(x(k - \tau(k))) \right] - x^T(k) P(\xi) x(k) \\ &\quad + \hat{\tau} x^T(k) Q(\xi) x(k) - x^T(k - \tau(k)) Q(\xi) x(k - \tau(k)).\end{aligned}$$

By condition (3.3), we get

$$\begin{aligned} \Delta V(x(k), \xi) \leq & \left[ -(A(\xi) + H_0 F_0 E_0)x(k) + (W(\xi) + HFE)f(x(k)) \right. \\ & \left. + (W_1(\xi) + H_1 F_1 E_1)g(x(k - \tau(k))) \right]^T P(\xi) \left[ -(A(\xi) + H_0 F_0 E_0) \right. \\ & \left. \times x(k) + (W(\xi) + HFE)f(x(k)) + (W_1(\xi) + H_1 F_1 E_1) \right. \\ & \left. \times g(x(k - \tau(k))) \right] - x^T(k)P(\xi)x(k) + \hat{\tau}x^T(k)Q(\xi)x(k) \\ & - x^T(k - \tau(k))Q(\xi)x(k - \tau(k)). \end{aligned} \quad (3.11)$$

By condition (3.8) and (3.9), we have

$$(f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \leq 0, \quad j = 1, 2, \dots, n, \quad (3.12)$$

and

$$\begin{aligned} (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) - v_j^- x_j(k - \tau(k))) \leq 0, \\ j = 1, 2, \dots, n, \end{aligned} \quad (3.13)$$

which are equivalent to

$$\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} l_j^+ l_j^- d_j d_j^T & -\frac{l_j^+ + l_j^-}{2} d_j d_j^T \\ -\frac{l_j^+ + l_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \leq 0, \quad k = 1, 2, \dots, n, \quad (3.14)$$

$$\begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} v_j^+ v_j^- d_j d_j^T & -\frac{v_j^+ + v_j^-}{2} d_j d_j^T \\ -\frac{v_j^+ + v_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \leq 0, \quad (3.15)$$

$$k = 1, 2, \dots, n,$$

where  $d_k$  denotes the unit column vector having “1” element on its  $k$ th row and zeros elsewhere. By (3.12), (3.13), and the S-procedure in Lemma 2.4.16,  $\Delta V(x(k), \xi) \leq 0$  if the following inequality (3.16) holds:

$$\begin{aligned} \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} (f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \\ - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) \\ - v_j^- x_j(k - \tau(k))) \leq 0. \end{aligned} \quad (3.16)$$

We will now show that, under assumptions of the theorem, (3.16) holds. From (3.14) and (3.15), we have

$$\begin{aligned}
& \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} (f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \\
& - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) \\
& - v_j^- x_j(k - \tau(k))) \\
& = \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} l_j^+ l_j^- d_j d_j^T & -\frac{l_j^+ + l_j^-}{2} d_j d_j^T \\ -\frac{l_j^+ + l_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\
& - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} v_j^+ v_j^- d_j d_j^T & -\frac{v_j^+ + v_j^-}{2} d_j d_j^T \\ -\frac{v_j^+ + v_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \\
& = \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\
& - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}, \tag{3.17}
\end{aligned}$$

where  $L_1 = \text{diag}(l_1^+ l_1^-, l_2^+ l_2^-, \dots, l_n^+ l_n^-)$ ,  $L_2 = \text{diag}(-\frac{l_1^+ + l_1^-}{2}, -\frac{l_2^+ + l_2^-}{2}, \dots, -\frac{l_n^+ + l_n^-}{2})$ ,  $V_1 = \text{diag}(v_1^+ v_1^-, v_2^+ v_2^-, \dots, v_n^+ v_n^-)$  and  $V_2 = \text{diag}(-\frac{v_1^+ + v_1^-}{2}, -\frac{v_2^+ + v_2^-}{2}, \dots, -\frac{v_n^+ + v_n^-}{2})$ . From (3.4), for  $e_0 > 0$ ,  $e > 0$ ,  $e_1 > 0$ , we have

$$\begin{cases} e_0 [F_0 E_0 x(k)]^T [F_0 E_0 x(k)] \leq e_0 x^T(k) E_0^T E_0 x(k), \\ e [F E f(x(k))]^T [F E f(x(k))] \leq e f^T(x(k)) E^T E f(x(k)), \\ e_1 [F_1 E_1 g(x(k - \tau(k)))]^T [F_1 E_1 g(x(k - \tau(k)))] \\ \leq e_1 g^T(x(k - \tau(k))) E_1^T E_1 g(x(k - \tau(k))). \end{cases} \tag{3.18}$$

By adding and subtracting the terms on the left handside of (3.18) into (3.17) and then using (3.11), we get

$$\begin{aligned}
& \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\
& - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \\
& \leq \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ x^T(k) \left( A_i^T P_j A_l - P_i + \hat{\tau} Q_i - 2 L_1 T_i \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ x^T(k) \left( -A_i^T P_j W_l - 2 L_2 T_i \right) f(x(k)) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ x^T(k) \left( -A_i^T P_j W_{1l} \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ x^T(k) \left( A_i^T P_j H_0 \right) (F_0 E_0(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ x^T(k) \left( -A_i^T P_j H \right) (F E f(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ x^T(k) \left( -A_i^T P_j H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ x^T(k - \tau(k)) \left( Q_i - 2V_1 S_i \right) x(k - \tau(k)) \right] \\
& - \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( 2V_2 S_i \right) x(k - \tau(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( -W_l^T P_j A_i - 2L_2 T_i \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( W_i^T P_j W_l - 2T_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( W_i^T P_j W_{1l} \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( -W_i^T P_j H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( W_i^T P_j H \right) (F E f(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( W_i^T P_j H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( -W_{1j}^T P_j A_i \right) x(k) \right] \\
& - \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( 2V_2 S_i \right) x(k - \tau(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( -W_{1i}^T P_j H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( W_{1l}^T P_j W_i \right) f(x(k)) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( W_{1i}^T P_j W_{1l} - 2S_i \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( W_{1i}^T P_j H \right) (FEf(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( W_{1i}^T P_j H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( H_0^T P_j A_i \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( -H_0^T P_j W_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( -H_0^T P_j W_{1i} \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( H_0^T P_i H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( -H_0^T P_i H \right) (FEf(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( -H_0^T P_i H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (FEf(x(k))) \left( -H^T P_j A_i \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (FEf(x(k))) \left( H^T P_j W_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (FEf(x(k))) \left( H^T P_j W_{1i} \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (FEf(x(k))) \left( -H^T P_i H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (FEf(x(k))) \left( H^T P_i H \right) (FEf(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (FEf(x(k))) \left( H^T P_i H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( -H_1^T P_j A_i \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( H_1^T P_j W_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( H_1^T P_j W_{1i} \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( -H_1^T P_i H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( H_1^T P_i H \right) (F E f(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( H_1^T P_i H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& \leq \sum_{i=1}^N \xi_i^3 y^T(k) [M_{i,i,i} + N_i] y(k) + \sum_{i=1}^N \xi_i \sum_{j \neq i; j=1}^N \xi_j y^T(k) \left[ \begin{matrix} M_{i,j,i} + M_{i,i,j} + M_{j,i,i} \\ + 2N_i + N_j \end{matrix} \right] y(k) \\
& + \sum_{i=1}^{N-2} \xi_i \sum_{j=i+1}^{N-1} \xi_j \sum_{l=j+1}^N \xi_l y^T(k) \left[ \begin{matrix} M_{i,j,l} + M_{i,l,j} + M_{j,i,l} + M_{j,l,i} \\ + M_{l,i,j} + M_{l,j,i} + 2N_i + 2N_j + 2N_l \end{matrix} \right] y(k).
\end{aligned}$$

where

$$y(k) = [x^T(k) \ x^T(k - \tau(k)) \ f^T(x(k)) \ g^T(x(k - \tau(k))) \ (F_0 E_0 x(k))^T \ (F E f(x(k)))^T \ (F_1 E_1 g(x(k - \tau(k))))^T]^T.$$

By conditions (3.10), we have

$$\begin{aligned}
\Delta V(x(k), \xi) & - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\
& - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \\
& < - y^T(k) \left[ \sum_{i=1}^N \xi_i^3 - \frac{1}{(N-1)^2} \sum_{i=1}^N \sum_{j \neq i; j=1}^N \xi_i^2 \xi_j I \right. \\
& \quad \left. - \frac{6}{(N-1)^2} \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^N \xi_i \xi_j \xi_l I \right] y(k).
\end{aligned}$$

Now let  $\Theta$  and  $\Lambda$  be defined as

$$\begin{aligned}\Theta &\equiv \sum_{i=1}^N \sum_{j=1}^N \xi_i (\xi_i - \xi_j)^2 = (N-1) \sum_{i=1}^N \xi_i^3 - \sum_{i=1}^{N-2} \sum_{j \neq i; j=1}^N \xi_i^2 \xi_j, \\ \Lambda &\equiv \sum_{i=1}^N \sum_{j=i+1}^{N-1} \sum_{l=j+1}^N \xi_i (\xi_j - \xi_l)^2 = (N-2) \sum_{i=1}^N \sum_{j \neq i; j=1}^N \xi_i^2 \xi_j - 6 \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^N \xi_i \xi_j \xi_l.\end{aligned}$$

Then, we obtain

$$(N-1)\Theta + \Lambda = (N-1)^2 \sum_{i=1}^N \xi_i^3 - \sum_{i=1}^N \sum_{j \neq i; j=1}^N \xi_i^2 \xi_j - 6 \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^N \xi_i \xi_j \xi_l \geq 0.$$

Thus, we have  $\Delta V(x(k), \xi) < 0$ . By Definition 2.4.11, the system (3.7) is robustly stable. The proof is complete.  $\square$

We give a couple of corollaries below in order to show that our main result are more general than some of the previous known results.

**Case 1.** In this case, we consider (3.7) without polytopic uncertainties. Let  $A(\xi) = A$ ,  $W(\xi) = W$  and  $W_1(\xi) = W_1$  be given constant matrices, then the system (3.7) is reduced to

$$x(k+1) = -[A + \Delta A]x(k) + [W + \Delta W]f(x(k)) + [W_1 + \Delta W_1]g(x(k - \tau(k))). \quad (3.19)$$

In this case, we have the following corollary (cf. Theorem 1 in [22]).

**Corollary 3.1.2** *The origin of system (3.19) is robustly exponentially stable if there exist  $P = P^T > 0$ ,  $Q = Q^T > 0$ ,  $T = \text{diag}\{t_1, t_2, \dots, t_n\} \geq 0$ ,  $S = \text{diag}\{s_1, s_2, \dots, s_n\} \geq 0$  and scalars  $e_0 > 0$ ,  $e > 0$  and  $e_1 > 0$  such that the following LMI holds :*

$$M = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{13} & -A^T P W_1 & A^T P H_0 & -A^T P H & -A^T P H_1 \\ * & \Pi_{22} & 0 & -2V_2 S & 0 & 0 & 0 \\ * & * & \Pi_{33} & W^T P W_1 & -W^T P H_0 & W^T P H & W^T P H_1 \\ * & * & * & \Pi_{44} & -W_1^T P H_0 & W_1^T P H & W_1^T P H_1 \\ * & * & * & * & \Pi_{55} & -H_0^T P H & -H_0^T P H_1 \\ * & * & * & * & * & \Pi_{66} & H^T P H_1 \\ * & * & * & * & * & * & \Pi_{77} \end{bmatrix} < 0, \quad (3.20)$$

where

$$\begin{aligned}\Pi_{11} &= A^T P A - P + \hat{\tau} Q + e_0 E_0^T E_0 - 2L_1 T, & \Pi_{13} &= -A^T P W - 2L_2 T, \\ \Pi_{22} &= -Q - 2V_1 S & \Pi_{33} &= W^T P W - 2T + e E^T E, \\ \Pi_{44} &= W_1^T P W_1 - 2S + e_1 E_1^T E_1, & \Pi_{55} &= H_0^T P H_0 - e_0 I, \\ \Pi_{66} &= H^T P H - e I, & \Pi_{77} &= H_1^T P H_1 - e_1 I, \text{ and } \hat{\tau} = \tau_2 - \tau_1 + 1.\end{aligned}$$

*Proof.* From Theorem 3.1.1, with  $N = 1$ , the LMI (3.10) becomes (3.20), Let

$$V(x(k)) = V_1(x(k)) + V_2(x(k)) + V_3(x(k))$$

$$\text{where } V_1(x(k)) = x^T(k)Px(k), \quad V_2(x(k)) = \sum_{i=k-\tau(k)}^{k-1} x^T(i)Qx(i),$$

and  $V_3(x(k)) = \sum_{j=k-\tau_2+1}^{k-\tau_1} \sum_{i=j}^{k-1} x^T(i)Qx(i)$ . One may show, with the same argument as in the proof of Theorem 3.1.1, that the Lyapunov difference along any trajectory of solution of (3.19) satisfies

$$\Delta V(x(k)) = \Lambda_{\max}(M_1)\|x(k)\|^2.$$

Then, by a similar argument used in the proof of Theorem 1 of [20], one may prove that if the LMI (3.20) holds, then (3.19) is robustly exponentially stable. The proof is complete.  $\square$

**Case 2.** In this case, we suppose that there are neither polytopic type uncertainties nor uncertain matrices in (3.7), namely,  $A(\xi) = A$ ,  $W(\xi) = W$  and  $W_1(\xi) = W_1$  are given constant matrices and  $\Delta A = 0$ ,  $\Delta W = 0$  and  $\Delta W_1 = 0$ , then (3.7) is reduced to

$$x(k+1) = -Ax(k) + Wf(x(k)) + W_1g(x(k-\tau(k))). \quad (3.21)$$

We have the following main result (Theorem 1) in [20] as the following corollary.

**Corollary 3.1.3** *The origin of system (3.21) is globally exponentially stable if there exist  $P = P^T > 0$ ,  $Q = Q^T > 0$ ,  $T = \text{diag}\{t_1, t_2, \dots, t_n\} \geq 0$  and  $S = \text{diag}\{s_1, s_2, \dots, s_n\} \geq 0$  such that the following LMI holds*

$$M = \begin{bmatrix} \Pi & 0 & -A^T P W - 2L_2 T & -A^T P W_1 \\ * & -Q - 2V_1 S & 0 & -2V_2 S \\ * & * & W^T P W - 2T & W^T P W_1 \\ * & * & * & W_1^T P W_1 - 2S \end{bmatrix} < 0,$$

where

$$\begin{aligned} \Pi &= A^T P A - P + \hat{\tau} Q - 2L_1 T, \\ \hat{\tau} &= \tau_2 - \tau_1 + 1. \end{aligned}$$

*Proof.* Similar to Corollary 3.1.2 and will be omitted (see also [20]).

**Remark 3.1.4** We now compare our obtained results with those obtained in [22]. In our main results, we derived robust stability conditions for general discrete-time LPD neural networks with both norm-bounded uncertainties and polytopic

type uncertainties. When neither norm-bounded uncertainties nor polytopic type uncertainties are considered, our system is in the following form

$$x(k+1) = Ax(k) + Wf(x(k)) + W_1g(x(k-\tau(k)))$$

As discussed previously, Corollary 3.1.3 of our main result is exactly the same as the main result (Theorem 1) of [22], and in this circumstances, Corollary 3.1.3 also gives global exponential stability condition (see [22]). Therefore, our main result is much better than [22] as it can be used for systems with both norm-bounded uncertainties and polytopic type uncertainties.

**Example 3.1.5** Consider the CNNs (3.7) with polytopic type uncertainties with  $N = 2$  where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, A_2 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, W_1 = \begin{bmatrix} 0.3 & -0.1 & 0.2 \\ 0 & -0.3 & 0.2 \\ -0.1 & -0.1 & -0.2 \end{bmatrix}, \\ W_2 &= \begin{bmatrix} 0.2 & -0.2 & 0.1 \\ 0 & -0.3 & 0.2 \\ -0.2 & -0.1 & -0.2 \end{bmatrix}, W_{11} = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}, \\ W_{12} &= \begin{bmatrix} -0.2 & 0.1 & 0 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}, H_0 = \begin{bmatrix} 0.005 & -0.011 & 0.002 \\ 0.004 & 0.005 & -0.006 \\ 0 & 0.009 & 0.012 \end{bmatrix}, \\ H &= \begin{bmatrix} -0.005 & -0.003 & 0.012 \\ 0.008 & 0 & -0.014 \\ 0.001 & 0.008 & -0.007 \end{bmatrix}, H_1 = \begin{bmatrix} 0.022 & -0.016 & 0.007 \\ -0.006 & 0 & 0.037 \\ 0.008 & 0.005 & -0.004 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} L_1 = V_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} -0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, V_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}, \\ F_0 &= \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & -1 \end{bmatrix}, F = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, F_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

$$\tau(k) = 4 + \sin\left(\frac{k\pi}{2}\right), b = \begin{bmatrix} -0.3 & 0.2 & -0.1 \end{bmatrix}^T, \tau_1 = 3, \tau_2 = 5, E_0 = H_0, E = H,$$

$$E_1 = H_1, \hat{f}_1(s) = \tanh(0.6s), \hat{f}_2(s) = \tanh(-0.4s), \hat{f}_3(s) = \tanh(-0.2s),$$

$$\hat{g}_1(s) = \tanh(-0.4s), \hat{g}_2(s) = \tanh(0.2s), \hat{g}_3(s) = \tanh(0.4s).$$

By using the Matlab LMI toolbox, we can solve  $P_i, Q_i, S_i$  and  $T_i$  for  $i = 1, 2$  which satisfy the conditions (3.10) in Theorem 3.1.1. Thus, the system (3.7) is

robustly stable. A set of solutions of (3.7) are given below

$$\begin{aligned}
 P_1 &= \begin{bmatrix} 31.4088 & -3.8603 & 0.5752 \\ -3.8603 & 20.9466 & 5.1187 \\ 0.5752 & 5.1187 & 27.7678 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4.1676 & -0.7573 & 0.0712 \\ -0.7573 & 2.1780 & 0.2434 \\ 0.0712 & 0.2434 & 3.4850 \end{bmatrix}, \\
 S_1 &= \begin{bmatrix} 5.9263 & 0 & 0 \\ 0 & 5.9263 & 0 \\ 0 & 0 & 5.9263 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 6.5273 & 0 & 0 \\ 0 & 6.5273 & 0 \\ 0 & 0 & 6.5273 \end{bmatrix}, \\
 e_{01} &= 10.6009, \quad e_1 = 10.6020, \quad e_{11} = 10.6095, \\
 P_2 &= \begin{bmatrix} 38.3718 & -10.7932 & 5.9300 \\ -10.7932 & 25.1604 & 0.6578 \\ 5.9300 & 0.6578 & 25.7448 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 4.7744 & -0.5258 & 0.5766 \\ -0.5258 & 4.6576 & 0.9620 \\ 0.5766 & 0.9620 & 5.2208 \end{bmatrix}, \\
 S_2 &= \begin{bmatrix} 6.3622 & 0 & 0 \\ 0 & 6.3622 & 0 \\ 0 & 0 & 6.3622 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 6.6139 & 0 & 0 \\ 0 & 6.6139 & 0 \\ 0 & 0 & 6.6139 \end{bmatrix}, \\
 e_{02} &= 10.6048, \quad e_2 = 10.6099, \quad e_{12} = 10.6319.
 \end{aligned}$$

The trajectories of solutions of system (3.7) of example are illustrated in Fig.3.1, where  $A(\xi) = \frac{1}{2}A_1 + \frac{1}{2}A_2$ ,  $W(\xi) = \frac{2}{3}W_1 + \frac{1}{3}W_2$ ,  $W_1(\xi) = \frac{2}{5}W_{11} + \frac{3}{5}W_{12}$ , with  $u_1(k) = u_2(k) = u_3(k) = 0$ ,  $k = -4, -3, -2, -1$ , and  $u_1(0) = 0.6$ ,  $u_2(0) = 0.3$ ,  $u_3(0) = -0.5$ .

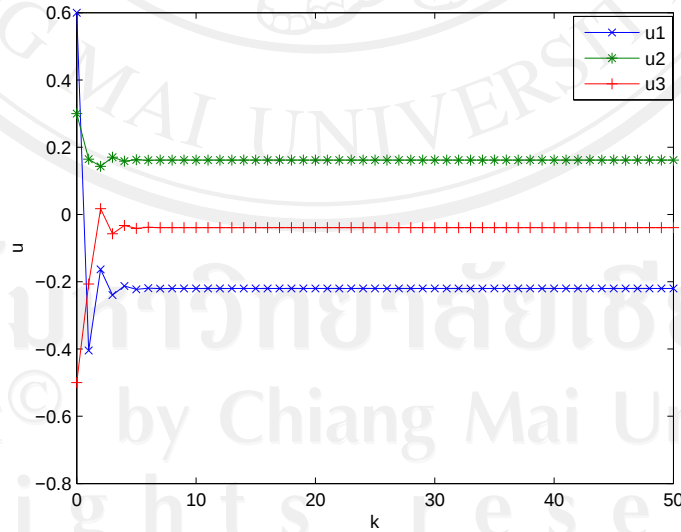


Figure 3.1: The trajectory of solutions of system (3.7) in Example 3.1.5.

### 3.2 Robust stability of stochastic LPD neural networks with time-varying delay

Consider the following uncertain stochastic LPD discrete-time neural networks with time-varying delay:

$$\begin{aligned} x(k+1) = & -[A(\xi) + \Delta A(k)]x(k) + [W(\xi) + \Delta W(k)]f(x(k)) + [W_1(\xi) \\ & + \Delta W_1(k)]g(x(k - \tau(k))) + \sigma(x(k), x(k - \tau(k)), k)w(k), \end{aligned} \quad (3.22)$$

where  $x(k) = [x_1(k), x_2(k), \dots, x_n(k)] \in \mathbb{R}^n$  is the neuron state vector,  $\tau(k)$  is a positive integer denotes the time-varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2,$$

where  $\tau_1 \geq 0$  and  $\tau_2 \geq 0$  are known integers,  $A(\xi)$ ,  $W(\xi)$  and  $W_1(\xi)$  (the inter-connection matrices) are of polytopic types where

$$\begin{aligned} [A(\xi) \quad W(\xi) \quad W_1(\xi)] & \in \mathfrak{D}, \\ \mathfrak{D} = \left\{ [A(\xi) \quad W(\xi) \quad W_1(\xi)] = \sum_{i=1}^N \xi_i [A_i \quad W_i \quad W_{1i}], \sum_{i=1}^N \xi_i = 1, \xi_i \geq 0 \right\}, \end{aligned} \quad (3.23)$$

where  $A_i$ ,  $W_i$  and  $W_{1i}$  are known constant matrices and  $\Delta A(k)$ ,  $\Delta W(k)$  and  $\Delta W_1(k)$  are uncertain matrices which are of the form

$$\Delta A(k) = H_0 F_0(k) E_0, \Delta W(k) = H F(k) E \text{ and } \Delta W_1(k) = H_1 F_1(k) E_1, \quad (3.24)$$

where  $H_0$ ,  $H$ ,  $H_1$ ,  $E_0$ ,  $E$  and  $E_1$  are known constant matrices  $F_0(k)$ ,  $F(k)$  and  $F_1(k)$  are unknown matrices which satisfy

$$F_0(k)^T F_0(k) \leq I, F^T(k) F(k) \leq I \text{ and } F_1(k)^T F_1(k) \leq I, \quad (3.25)$$

where  $I$  is the identity matrix of appropriate dimension,  $w(k)$  is a scalar Wiener process (Brownian Motion) on  $(\Omega, \mathcal{F}, \mathcal{P})$  with

$$\mathbb{E}[w(k)] = 0, \mathbb{E}[w^2(k)] = 1, \mathbb{E}[w(i)w(j)] = 0 \ (i \neq j), \quad (3.26)$$

and  $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  is a continuous function satisfying

$$\sigma^T(x, y, k) \sigma(x, y, k) \leq \rho_1 x^T x + \rho_2 y^T y, \quad x, y \in \mathbb{R}^n, \quad (3.27)$$

where  $\rho_1 > 0$  and  $\rho_2 > 0$  are known constant scalars,  $f(u(\cdot)) = [f_1(u_1(\cdot)), \dots, f_n(u_n(\cdot))]^T$  and  $g(u(\cdot)) = [g_1(u_1(\cdot)), \dots, g_n(u_n(\cdot))]^T$  are the activation functions satisfying the following conditions

$$l_j^- \leq \frac{f_j(x) - f_j(y)}{x - y} \leq l_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (3.28)$$

$$v_j^- \leq \frac{g_j(x) - g_j(y)}{x - y} \leq v_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (3.29)$$

where  $l_j^-, l_j^+, v_j^-, v_j^+$  for  $j = 1, 2, \dots, n$  are constant. As in Proposition 1 of [20], we may show that there exists an equilibrium point of system (3.22).

**Theorem 3.2.1** *The DSNs (3.22) with polytopic type uncertainties (3.23) is robustly stable in the mean square if there exist  $P_i = P_i^T > 0$ ,  $Q_i = Q_i^T > 0$ ,  $T_i = \text{diag}\{t_{1i}, t_{2i}, \dots, t_{ni}\} \geq 0$ ,  $S_i = \text{diag}\{s_{1i}, s_{2i}, \dots, s_{ni}\} \geq 0$  and scalars  $e_{0i} > 0$ ,  $e_i > 0$ ,  $e_{1i} > 0$  and  $\lambda_i^* > 0$ ,  $i = 1, 2, \dots, N$  satisfying the following conditions*

$$(i) \quad M_{i,i,i} + N_i < -I, \quad i = 1, 2, \dots, N; \quad (3.30)$$

$$(ii) \quad M_{i,i,j} + M_{j,i,i} + M_{i,j,i} + 2N_i + N_j < \frac{1}{(N-1)^2} I, \\ i = 1, 2, \dots, N, \quad i \neq j, \quad j = 1, 2, \dots, N; \quad (3.31)$$

$$(iii) \quad M_{i,j,l} + M_{i,l,j} + M_{j,i,l} + M_{j,l,i} + M_{l,i,j} + M_{l,j,i} \\ + 2N_i + 2N_j + 2N_l < \frac{6}{(N-1)^2} I, \\ i = 1, 2, \dots, N-2, \quad j = i+1, 2, \dots, N-1, \quad l = 1, 2, \dots, N; \quad (3.32)$$

$$(iv) \quad P_i < \lambda_i^* I, \quad (3.33)$$

where

$$M_{i,j,l} = \begin{bmatrix} \varphi_{11} & 0 & \varphi_{13} & \varphi_{14} & \varphi_{15} & \varphi_{16} & \varphi_{17} \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \varphi_{34} & \varphi_{35} & \varphi_{36} & \varphi_{37} \\ * & * & * & * & \varphi_{45} & \varphi_{46} & \varphi_{47} \\ * & * & * & * & * & \varphi_{56} & \varphi_{57} \\ * & * & * & * & * & \varphi_{66} & \varphi_{67} \\ * & * & * & * & * & * & \varphi_{77} \end{bmatrix},$$

$$N_i = \begin{bmatrix} \Pi_{11}(i) & 0 & -2L_2T_i & 0 & 0 & 0 & 0 \\ 0 & \Pi_{22}(i) & 0 & -2V_2S_i & 0 & 0 & 0 \\ -2L_2T_i & 0 & \Pi_{33}(i) & 0 & 0 & 0 & 0 \\ 0 & -2V_2S_i & 0 & \Pi_{44}(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Pi_{55}(i) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Pi_{66}(i) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Pi_{77}(i) \end{bmatrix},$$

$$\Pi_{11}(i) = e_{0i}E_0^TE_0 - P_i + \hat{\tau}Q_i - 2L_1T_i + \lambda_i^*\rho_1I, \quad \Pi_{22}(i) = -Q_i - 2V_1S_i + \lambda_i^*\rho_2I,$$

$$\Pi_{33}(i) = -2T_i + e_iE^TE, \quad \Pi_{44}(i) = -2S_i + e_{1i}E_1^TE_1, \quad \Pi_{55}(i) = -e_{0i}I,$$

$$\Pi_{66}(i) = -e_iI, \quad \Pi_{77}(i) = -e_{1i}I, \quad \hat{\tau} = \tau_2 - \tau_1 + 1,$$

$$L_1 = \text{diag}(l_1^+l_1^-, l_2^+l_2^-, \dots, l_n^+l_n^-), \quad L_2 = \text{diag}(-\frac{l_1^++l_1^-}{2}, -\frac{l_2^++l_2^-}{2}, \dots, -\frac{l_n^++l_n^-}{2}),$$

$$V_1 = \text{diag}(v_1^+v_1^-, v_2^+v_2^-, \dots, v_n^+v_n^-), \quad \text{and } V_2 = \text{diag}(-\frac{v_1^++v_1^-}{2}, -\frac{v_2^++v_2^-}{2}, \dots, -\frac{v_n^++v_n^-}{2}).$$

*Proof* Consider the Lyapunov function candidate

$$V(x(k), \xi) = V_1(x(k), \xi) + V_2(x(k), \xi) + V_3(x(k), \xi),$$

where

$$\begin{aligned} V_1(x(k), \xi) &= \sum_{i=1}^N x^T(k) \xi_i P_i x(k), \quad V_2(x(k), \xi) = \sum_{i=1}^N \sum_{l=k-\tau(k)}^{k-1} x^T(l) \xi_i Q_i x(l), \\ V_3(x(k), \xi) &= \sum_{i=1}^N \sum_{j=k-\tau(k)}^{-\tau_1+1} \sum_{l=k+j-1}^{k-1} x^T(l) \xi_i Q_i x(l). \end{aligned}$$

The Lyapunov difference of system along any trajectory of solution of (3.22) and taking the mathematical expectation, we obtained

$$\mathbb{E}\{\Delta V(x(k), \xi)\} = \mathbb{E}\{\Delta V_1(x(k), \xi) + \Delta V_2(x(k), \xi) + \Delta V_3(x(k), \xi)\},$$

then, we get

$$\begin{aligned} \mathbb{E}\{\Delta V_1(x(k), \xi)\} &= \mathbb{E}\left\{ \sum_{i=1}^N x^T(k+1) \xi_i P_i x(k+1) - \sum_{i=1}^N x^T(k) \xi_i P_i x(k) \right\} \\ &= \mathbb{E}\left\{ \left[ - (A(\xi) + \Delta A(k))x(k) + (W(\xi) + \Delta W(k))f(x(k)) \right. \right. \\ &\quad \left. \left. + (W_1(\xi) + \Delta W_1(k))g(x(k - \tau(k))) + \sigma(x(k), x(k - \tau(k)), k) \right. \right. \\ &\quad \left. \left. \times w(k) \right]^T P(\xi) \left[ - (A(\xi) + \Delta A(k))x(k) + (W(\xi) + \Delta W(k)) \right. \right. \\ &\quad \left. \left. \times f(x(k)) + (W_1(\xi) + \Delta W_1(k))g(x(k - \tau(k))) \right. \right. \\ &\quad \left. \left. + \sigma(x(k), x(k - \tau(k)), k)w(k) \right] \right\} \\ &= \mathbb{E}\left\{ \left[ - (A(\xi) + \Delta A(k))x(k) + (W(\xi) + \Delta W(k))f(x(k)) \right. \right. \\ &\quad \left. \left. + (W_1(\xi) + \Delta W_1(k))g(x(k - \tau(k))) \right]^T P(\xi) \left[ - (A(\xi) \right. \right. \\ &\quad \left. \left. + \Delta A(k))x(k) + (W(\xi) + \Delta W(k))f(x(k)) + (W_1(\xi) \right. \right. \\ &\quad \left. \left. + \Delta W_1(k))g(x(k - \tau(k))) \right] + \left[ - (A(\xi) + \Delta A(k))x(k) \right. \right. \\ &\quad \left. \left. + (W(\xi) + \Delta W(k))f(x(k)) + (W_1(\xi) + \Delta W_1(k))g(x(k - \tau(k))) \right]^T \right. \\ &\quad \left. \times P(\xi) \sigma(x(k), x(k - \tau(k)), k)w(k) + \sigma^T(x(k), x(k - \tau(k)), k)w(k) \right. \\ &\quad \left. \times P(\xi) \left[ - (A(\xi) + \Delta A(k))x(k) + (W(\xi) + \Delta W(k))f(x(k)) \right. \right. \\ &\quad \left. \left. + (W_1(\xi) + \Delta W_1(k))g(x(k - \tau(k))) \right]^T \right. \\ &\quad \left. \left. + \sigma^T(x(k), x(k - \tau(k)), k)w(k) P(\xi) \sigma(x(k), x(k - \tau(k)), k)w(k) \right\}. \end{aligned}$$

By assumption (3.26), we have

$$\begin{aligned} \mathbb{E}\{\Delta V_1(x(k), \xi)\} = & \mathbb{E}\left\{ \left[ - (A(\xi) + \Delta A(k))x(k) + (W(\xi) + \Delta W(k))f(x(k)) \right. \right. \\ & + (W_1(\xi) + \Delta W_1(k))g(x(k - \tau(k))) \left. \right]^T P(\xi) \left[ - (A(\xi) \right. \\ & + \Delta A(k))x(k) + (W(\xi) + \Delta W(k))f(x(k)) + (W_1(\xi) + \Delta W_1(k)) \\ & \times g(x(k - \tau(k))) \left. \right] + \sigma^T(x(k), x(k - \tau(k)), k) P(\xi) \sigma(x(k), \\ & \left. x(k - \tau(k)), k) \right\}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\{\Delta V_2(x(k), \xi)\} = & \mathbb{E}\left\{ \sum_{i=1}^N \sum_{l=k+1-\tau(k+1)}^k x^T(l) \xi_i Q_i x(l) - \sum_{i=1}^N \sum_{l=k-\tau(k)}^{k-1} x^T(l) \xi_i Q_i x(l) \right\} \\ = & \mathbb{E}\left\{ \sum_{i=1}^N \sum_{l=k-\tau(k+1)}^{k-\tau_1} x^T(l) \xi_i Q_i x(l) + \sum_{i=1}^N x^T(k) \xi_i Q_i x(k) \right. \\ & - \sum_{i=1}^N x^T(k - \tau(k)) \xi_i Q_i x(k - \tau(k)) + \sum_{i=1}^N \sum_{k+1-\tau_1}^{k-1} x^T(l) \xi_i Q_i x(l) \\ & \left. - \sum_{i=1}^N \sum_{l=k+1-\tau(k)}^{k-1} x^T(l) \xi_i Q_i x(l) \right\}. \end{aligned}$$

Since  $\tau(k) \geq \tau_1$ , we get

$$\sum_{i=1}^N \sum_{k+1-\tau_1}^{k-1} x^T(l) \xi_i Q_i x(l) - \sum_{i=1}^N \sum_{l=k+1-\tau(k)}^{k-1} x^T(l) \xi_i Q_i x(l) \leq 0.$$

Therefore,

$$\begin{aligned} \mathbb{E}\{\Delta V_2(x(k), \xi)\} \leq & \mathbb{E}\left\{ \sum_{i=1}^N \sum_{l=k-\tau(k+1)}^{k-\tau_1} x^T(l) \xi_i Q_i x(l) + \sum_{i=1}^N x^T(k) \xi_i Q_i x(k) \right. \\ & \left. - \sum_{i=1}^N x^T(k - \tau(k)) \xi_i Q_i x(k - \tau(k)) \right\}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} \mathbb{E}\{\Delta V_3(x(k), \xi)\} = & \mathbb{E}\left\{ \sum_{i=1}^N \sum_{l=-\tau_2+2}^{-\tau_1+1} \left[ x^T(k) \xi_i Q_i x(k) + \sum_{m=k+j}^{k-1} x^T(m) \xi_i Q_i x(m) \right. \right. \\ & \left. \left. - \sum_{m=k+j-1}^{k-1} x^T(m) \xi_i Q_i x(m) \right] \right\} \\ = & \mathbb{E}\left\{ \sum_{i=1}^N (\tau_2 - \tau_1) x^T(k) \xi_i Q_i x(k) - \sum_{i=1}^N \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(k) \xi_i Q_i x(k) \right\}. \end{aligned}$$

From  $\Delta V_2(x(k), \xi)$  and  $\Delta V_3(x(k), \xi)$  obtained above, we get

$$\begin{aligned} & \mathbb{E}\{\Delta V_2(x(k), \xi) + \Delta V_3(x(k), \xi)\} \\ & \leq \mathbb{E}\left\{ \sum_{i=1}^N \hat{\tau} x^T(k) \xi_i Q_i x(k) - \sum_{i=1}^N x^T(k - \tau(k)) \xi_i Q_i x(k - \tau(k)) \right. \\ & \quad \left. + \sum_{i=1}^N \sum_{l=k+1}^{k-\tau_1} x^T(l) \xi_i Q_i x(l) - \sum_{i=1}^N \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(k) \xi_i Q_i x(k) \right\}, \end{aligned}$$

where  $\hat{\tau} = \tau_2 - \tau_1 + 1$ . Since  $\tau(k) \leq \tau_2$ , we have

$$\sum_{i=1}^N \sum_{l=k+1-\tau(k+1)}^{k-\tau_1} x^T(l) \xi_i Q_i x(l) - \sum_{i=1}^N \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(k) \xi_i Q_i x(k) \leq 0.$$

Thus,

$$\begin{aligned} & \mathbb{E}\{\Delta V_2(x(k), \xi) + \Delta V_3(x(k), \xi)\} \\ & \leq \mathbb{E}\left\{ \sum_{i=1}^N \hat{\tau} x^T(k) \xi_i Q_i x(k) - \sum_{i=1}^N x^T(k - \tau(k)) \xi_i Q_i x(k - \tau(k)) \right\}. \end{aligned}$$

From (3.27) and (3.33), we have

$$\begin{aligned} & \rho^T(x(k), x(k - \tau(k)), k) P \rho(x(k), x(k - \tau(k)), k) \\ & \leq \lambda_{\max}(P_i) \rho^T(x(k), x(k - \tau(k)), k) \rho(x(k), x(k - \tau(k)), k) \\ & \leq \lambda_i^* (\rho_1 x^T(k) x(k) + \rho_2 x^T(k - \tau(k)) x(k - \tau(k))). \end{aligned}$$

As a result, we obtain

$$\begin{aligned} \mathbb{E}\{\Delta V(x(k), \xi)\} & \leq \mathbb{E}\left\{ \left[ - (A(\xi) + H_0 F_0 E_0) x(k) + (W(\xi) + H F E) f(x(k)) \right. \right. \\ & \quad \left. \left. + (W_1(\xi) + H_1 F_1 E_1) g(x(k - \tau(k))) \right]^T P(\xi) \left[ - (A(\xi) \right. \right. \\ & \quad \left. \left. + H_0 F_0 E_0) x(k) + (W(\xi) + H F E) f(x(k)) + (W_1(\xi) \right. \right. \\ & \quad \left. \left. + H_1 F_1 E_1) g(x(k - \tau(k))) \right] + \lambda^* \rho_1 x^T(k) x(k) \right. \\ & \quad \left. + \lambda^* \rho_2 x^T(k - \tau(k)) x(k - \tau(k)) - x^T(k) P(\xi) x(k) \right. \\ & \quad \left. + \hat{\tau} x^T(k) Q(\xi) x(k) - x^T(k - \tau(k)) Q(\xi) x(k - \tau(k)) \right\}. \quad (3.34) \end{aligned}$$

By conditions (3.28)-(3.29), we have

$$(f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \leq 0, \quad j = 1, 2, \dots, n, \quad (3.35)$$

and

$$\begin{aligned} & (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) - v_j^- x_j(k - \tau(k))) \leq 0, \\ & \quad j = 1, 2, \dots, n, \quad (3.36) \end{aligned}$$

which are equivalent to

$$\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} l_j^+ l_j^- d_j d_j^T & -\frac{l_j^+ + l_j^-}{2} d_j d_j^T \\ -\frac{l_j^+ + l_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \leq 0, \quad k = 1, 2, \dots, n, \quad (3.37)$$

$$\begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} v_j^+ v_j^- d_j d_j^T & -\frac{v_j^+ + v_j^-}{2} d_j d_j^T \\ -\frac{v_j^+ + v_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \leq 0, \quad (3.38)$$

$$k = 1, 2, \dots, n,$$

where  $d_k$  denotes the unit column vector having “1” element on its  $k$ th row and zeros elsewhere.

By (3.35), (3.36), and the S-procedure,  $\mathbb{E}\{\Delta V(x(k), \xi) < 0\}$  if the following inequality (3.39) holds:

$$\begin{aligned} & \mathbb{E}\left\{ \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} (f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \right. \\ & \quad - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) \\ & \quad \left. - v_j^- x_j(k - \tau(k))) \right\}. \end{aligned} \quad (3.39)$$

We will now show that, under assumptions of the theorem, (3.39) holds. From (3.37) and (3.38), we have

$$\begin{aligned} & \mathbb{E}\left\{ \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} (f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \right. \\ & \quad - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) \\ & \quad \left. - v_j^- x_j(k - \tau(k))) \right\} \\ & = \mathbb{E}\left\{ \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} l_j^+ l_j^- d_j d_j^T & -\frac{l_j^+ + l_j^-}{2} d_j d_j^T \\ -\frac{l_j^+ + l_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \right. \\ & \quad \left. - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} v_j^+ v_j^- d_j d_j^T & -\frac{v_j^+ + v_j^-}{2} d_j d_j^T \\ -\frac{v_j^+ + v_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \right\} \\ & = \mathbb{E}\left\{ \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \right. \\ & \quad \left. - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \right\}, \end{aligned} \quad (3.40)$$

where  $L_1 = \text{diag}(l_1^+ l_1^-, l_2^+ l_2^-, \dots, l_n^+ l_n^-)$ ,  $L_2 = \text{diag}(-\frac{l_1^+ + l_1^-}{2}, -\frac{l_2^+ + l_2^-}{2}, \dots, -\frac{l_n^+ + l_n^-}{2})$ ,  $V_1 = \text{diag}(v_1^+ v_1^-, v_2^+ v_2^-, \dots, v_n^+ v_n^-)$  and  $V_2 = \text{diag}(-\frac{v_1^+ + v_1^-}{2}, -\frac{v_2^+ + v_2^-}{2}, \dots, -\frac{v_n^+ + v_n^-}{2})$ . From (3.25), for  $e_0 > 0$ ,  $e > 0$ ,  $e_1 > 0$ , we have

$$\begin{cases} e_0[F_0 E_0 x(k)]^T [F_0 E_0 x(k)] \leq e_0 x^T(k) E_0^T E_0 x(k), \\ e[F E f(x(k))]^T [F E f(x(k))] \leq e f^T(x(k)) E^T E f(x(k)), \\ e_1[F_1 E_1 g(x(k - \tau(k)))]^T [F_1 E_1 g(x(k - \tau(k)))] \\ \leq e_1 g^T(x(k - \tau(k))) E_1^T E_1 g(x(k - \tau(k))). \end{cases} \quad (3.41)$$

By adding and subtracting the terms on the left handside of (3.41) into (3.40) and then using (3.34), we obtain

$$\begin{aligned} & \mathbb{E} \left\{ \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \right. \\ & \quad \left. - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \right\} \\ & \leq \mathbb{E} \left\{ \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ x^T(k) \left( A_i^T P_j A_l - P_i + \hat{\tau} Q_i - 2L_1 T_i + \lambda_i^* \rho_1 I \right) x(k) \right] \right. \\ & \quad + \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ x^T(k) \left( -A_i^T P_j W_l - 2L_2 T_i \right) f(x(k)) \right] \\ & \quad + \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ x^T(k) \left( -A_i^T P_j W_{1l} \right) g(x(k - \tau(k))) \right] \\ & \quad + \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ x^T(k) \left( A_i^T P_j H_0 \right) (F_0 E_0(x(k))) \right] \\ & \quad + \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ x^T(k) \left( -A_i^T P_j H \right) (F E f(x(k))) \right] \\ & \quad + \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ x^T(k) \left( -A_i^T P_j H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\ & \quad + \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ x^T(k - \tau(k)) \left( Q_i - 2V_1 S_i + \lambda_i^* \rho_2 I \right) x(k - \tau(k)) \right] \\ & \quad - \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( 2V_2 S_i \right) x(k - \tau(k)) \right] \\ & \quad \left. + \sum_{i=1}^N \xi_i \sum_{j=1}^N \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( -W_l^T P_j A_i - 2L_2 T_i \right) x(k) \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( W_i^T P_j W_l - 2T_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( W_i^T P_j W_{1l} \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( -W_i^T P_j H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( W_i^T P_j H \right) (F E f(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ f^T(x(k)) \left( W_i^T P_j H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( -W_{1j}^T P_j A_i \right) x(k) \right] \\
& - \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( 2V_2 S_i \right) x(k - \tau(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( -W_{1i}^T P_j H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( W_{1l}^T P_j W_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( W_{1i}^T P_j W_{1l} - 2S_i \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( W_{1i}^T P_j H \right) (F E f(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ g^T(x(k - \tau(k))) \left( W_{1i}^T P_j H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( H_0^T P_j A_i \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( -H_0^T P_j W_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( -H_0^T P_j W_{1i} \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( H_0^T P_i H_0 \right) (F_0 E_0 x(k)) \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( -H_0^T P_i H \right) (F E f(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_0 E_0 x(k)) \left( -H_0^T P_i H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F E f(x(k))) \left( -H^T P_j A_i \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F E f(x(k))) \left( H^T P_j W_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F E f(x(k))) \left( H^T P_j W_{1i} \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F E f(x(k))) \left( -H^T P_i H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F E f(x(k))) \left( H^T P_i H \right) (F E f(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F E f(x(k))) \left( H^T P_i H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( -H_1^T P_j A_i \right) x(k) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( H_1^T P_j W_i \right) f(x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( H_1^T P_j W_{1i} \right) g(x(k - \tau(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( -H_1^T P_i H_0 \right) (F_0 E_0 x(k)) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( H_1^T P_i H \right) (F E f(x(k))) \right] \\
& + \sum_{i=1}^N \xi_i \sum_{j=1}^N \xi_j \sum_{l=1}^N \xi_l \left[ (F_1 E_1 g(x(k - \tau(k)))) \left( H_1^T P_i H_1 \right) (F_1 E_1 g(x(k - \tau(k)))) \right] \Big\} \\
& \leq \mathbb{E} \left\{ \sum_{i=1}^N \xi_i^3 y^T(k) [M_{i,i,i} + N_i] y(k) + \sum_{i=1}^N \xi_i \sum_{j \neq i}^N \xi_j y^T(k) \left[ \begin{matrix} M_{i,j,i} + M_{i,i,j} \\ + M_{j,i,i} + 2N_i + N_j \end{matrix} \right] y(k) \right. \\
& \quad \left. + \sum_{i=1}^{N-2} \xi_i \sum_{j=i+1}^{N-1} \xi_j \sum_{l=j+1}^N \xi_l y^T(k) \left[ \begin{matrix} M_{i,j,l} + M_{i,l,j} + M_{j,i,l} + M_{j,l,i} \\ + M_{l,i,j} + M_{l,j,i} + 2N_i + 2N_j + 2N_l \end{matrix} \right] y(k) \right\},
\end{aligned}$$

where

$$y(k) = [x^T(k) \ x^T(k - \tau(k)) \ f^T(x(k)) \ g^T(x(k - \tau(k))) \ (F_0 E_0 x(k))^T \ (F E f(x(k)))^T \ (F_1 E_1 g(x(k - \tau(k))))^T]^T. \text{ By conditions (3.30)-(3.32), we have}$$

$$\begin{aligned} & \mathbb{E} \left\{ \Delta V(x(k), \xi) - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i t_{ji} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \right. \\ & \quad \left. - 2 \sum_{i=1}^N \sum_{j=1}^n \xi_i s_{ji} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \right\} \\ & < \mathbb{E} \left\{ -y^T(k) \left[ \sum_{i=1}^N \xi_i^3 I - \frac{1}{(N-1)^2} \sum_{i=1}^N \sum_{j \neq i; j=1}^N \xi_i^2 \xi_j I \right. \right. \\ & \quad \left. \left. - \frac{6}{(N-1)^2} \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^N \xi_i \xi_j \xi_l I \right] y(k) \right\}. \end{aligned}$$

Now let  $\Theta$  and  $\Lambda$  be defined as

$$\begin{aligned} \Theta &\equiv \sum_{i=1}^N \sum_{j=1}^N \xi_i (\xi_i - \xi_j)^2 I = (N-1) \sum_{i=1}^N \xi_i^3 I - \sum_{i=1}^{N-2} \sum_{j \neq i; j=1}^N \xi_i^2 \xi_j I, \\ \Lambda &\equiv \sum_{i=1}^N \sum_{j=i+1}^{N-1} \sum_{l=j+1}^N \xi_i (\xi_j - \xi_l)^2 I = (N-2) \sum_{i=1}^N \sum_{j \neq i; j=1}^N \xi_i^2 \xi_j I - 6 \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^N \xi_i \xi_j \xi_l I. \end{aligned}$$

Then, we obtain

$$(N-1)\Theta + \Lambda = (N-1)^2 \sum_{i=1}^N \xi_i^3 I - \sum_{i=1}^N \sum_{j \neq i; j=1}^N \xi_i^2 \xi_j I - 6 \sum_{i=1}^{N-2} \sum_{j=i+1}^{N-1} \sum_{l=j+1}^N \xi_i \xi_j \xi_l I \geq 0.$$

Thus, we have  $\mathbb{E}\{\Delta V(x(k), \xi)\} < 0$ . By Definition 2.4.13, the system (3.22) is robustly stable in the mean square. The proof is complete.  $\square$

**Case 1.** In this case, we consider (3.22) without polytopic type uncertainties, our system is in the following form

$$\begin{aligned} x(k+1) &= -[A + \Delta A(k)]x(k) + [W + \Delta W(k)]f(x(k)) + [W_1 \\ &\quad + \Delta W_1(k)]g(x(k - \tau(k))) + \sigma(x(k), x(k - \tau(k)), k)w(k). \end{aligned} \quad (3.42)$$

In this case, we can obtain the following corollary based on Theorem 3.2.1

**Corollary 3.2.2** *The DSNNs (3.22) is robustly exponentially stable if there exist  $P = P^T > 0$ ,  $Q = Q^T > 0$ ,  $T = \text{diag}\{t_1, t_2, \dots, t_n\} \geq 0$ ,  $S = \text{diag}\{s_1, s_2, \dots, s_n\} \geq 0$  and scalars  $e_0 > 0$ ,  $e > 0$ ,  $e_1 > 0$  and  $\lambda^* > 0$  such that the following two LMIs hold:*

$$M_1 = \begin{bmatrix} \Pi_{11} & 0 & \Pi_{13} & -A^T P W_1 & A^T P H_0 & -A^T P H & -A^T P H_1 \\ * & \Pi_{22} & 0 & -2V_2 S & 0 & 0 & 0 \\ * & * & \Pi_{33} & W^T P W_1 & -W^T P H_0 & W^T P H & W^T P H_1 \\ * & * & * & \Pi_{44} & -W_1^T P H_0 & W_1^T P H & W_1^T P H_1 \\ * & * & * & * & \Pi_{55} & -H_0^T P H & -H_0^T P H_1 \\ * & * & * & * & * & \Pi_{66} & H^T P H_1 \\ * & * & * & * & * & * & \Pi_{77} \end{bmatrix} < 0 \quad (3.43)$$

$$P < \lambda_* I \quad (3.44)$$

where

$$\begin{aligned} \Pi_{11} &= A^T P A - P + \hat{\tau} Q + e_0 E_0^T E_0 - 2L_1 T + \lambda_* \rho_1 I, & \Pi_{13} &= -A^T P W - 2L_2 T, \\ \Pi_{22} &= -Q - 2V_1 S + \lambda_* \rho_2 I & \Pi_{33} &= W^T P W - 2T + e E^T E, \\ \Pi_{44} &= W_1^T P W_1 - 2S + e_1 E_1^T E_1, & \Pi_{55} &= H_0^T P H_0 - e_0 I, \\ \Pi_{66} &= H^T P H - e I, & \Pi_{77} &= H_1^T P H_1 - e_1 I, \\ \hat{\tau} &= \tau_2 - \tau_1 + 1. \end{aligned}$$

**Proof.** From Theorem 3.2.1, with  $N=1$ , the LMI (3.30) and (3.33) becomes (3.43) and (3.44). The following Lypunov function candidate

$$V(x(k)) = V_1(x(k)) + V_2(x(k)) + V_3(x(k))$$

$$\text{where } V_1(x(k)) = x^T(k) P x(k), \quad V_2(x(k)) = \sum_{l=k-\tau(k)}^{k-1} x^T(l) Q x(l),$$

$$V_3(x(k)) = \sum_{j=k-\tau(k)}^{-\tau_1+1} \sum_{l=k+j-1}^{k-1} x^T(l) Q x(l)$$

we can show that the Lypyanov difference along trajectory of solution of (3.42) satisfies

$$\mathbb{E}\{\Delta V(x(k))\} \geq -\Lambda_{\max}(M_1) \mathbb{E}\{|x(k)|^2\}. \quad (3.45)$$

From (3.45), we get that the DSNs (3.42) is robustly stable in the mean square, by Lyapunov stability theory. Moreover, we can prove that the system (3.42), with the same as the proof of Theorem 1. of [22], while we may prove that if the LMIs (3.43) and (3.44) hold, then (3.42) is robustly exponentially stable. The proof is complete.  $\square$

**Case 2.** In this case, we consider (3.22) there are no stochastic disturbance, our system is in the following form

$$\begin{aligned} x(k+1) &= -[A(\xi) + \Delta A(k)]x(k) + [W(\xi) + \Delta W(k)]f(x(k)) \\ &\quad + [W_1(\xi) + \Delta W_1(k)]g(x(k - \tau(k))). \end{aligned} \quad (3.46)$$

We can obtain the corollary which is so that the main result (Theorem 3.1) obtained in [32]

**Corollary 3.2.3** *The origin of system (3.46) with polytopic type uncertainties (3.23) is robustly stable if there exist  $P_i = P_i^T > 0$ ,  $Q_i = Q_i^T > 0$ ,  $T_i = \text{diag}\{t_{1i}, t_{2i}, \dots, t_{ni}\} \geq 0$ ,  $S_i = \text{diag}\{s_{1i}, s_{2i}, \dots, s_{ni}\} \geq 0$  and scalars  $e_{0i} > 0$ ,  $e_i > 0$  and  $e_{1i} > 0$ ,  $i = 1, 2, \dots, N$  satisfying the following conditions*

- (i)  $M_{i,i,i} + N_i < -I$ ,  $i = 1, 2, \dots, N$ ;
- (ii)  $M_{i,i,j} + M_{j,i,i} + M_{i,j,i} + 2N_i + N_j < \frac{1}{(N-1)^2}I$ ,  
 $i = 1, 2, \dots, N$ ,  $i \neq j$ ,  $j = 1, 2, \dots, N$ ;
- (iii)  $M_{i,j,l} + M_{i,l,j} + M_{j,i,l} + M_{j,l,i} + M_{l,i,j} + M_{l,j,i}$   
 $+ 2N_i + 2N_j + 2N_l < \frac{6}{(N-1)^2}I$ ,  
 $i = 1, 2, \dots, N-2$ ,  $j = i+1, 2, \dots, N-1$ ,  $l = 1, 2, \dots, N$ ,

where

$$M_{i,j,l} = \begin{bmatrix} \varphi_{11} & 0 & \varphi_{13} & \varphi_{14} & \varphi_{15} & \varphi_{16} & \varphi_{17} \\ * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & \varphi_{34} & \varphi_{35} & \varphi_{36} & \varphi_{37} \\ * & * & * & * & \varphi_{45} & \varphi_{46} & \varphi_{47} \\ * & * & * & * & * & \varphi_{56} & \varphi_{57} \\ * & * & * & * & * & \varphi_{66} & \varphi_{67} \\ * & * & * & * & * & * & \varphi_{77} \end{bmatrix},$$

$$N_i = \begin{bmatrix} \Pi_{11}(i) & 0 & -2L_2T_i & 0 & 0 & 0 & 0 \\ 0 & -Q_i - 2V_1S_i & 0 & -2V_2S_i & 0 & 0 & 0 \\ -2L_2T_i & 0 & \Pi_{33}(i) & 0 & 0 & 0 & 0 \\ 0 & -2V_2S_i & 0 & \Pi_{44}(i) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Pi_{55}(i) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Pi_{66}(i) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Pi_{77}(i) \end{bmatrix},$$

and  $\varphi_{11} = A_i^T P_j A_i$ ,  $\varphi_{13} = -A_i^T P_j W_l$ ,  $\varphi_{14} = -A_i^T P_j W_{1l}$ ,  $\varphi_{15} = A_i^T P_j H_0$ ,

$\varphi_{16} = -A_i^T P_j H$ ,  $\varphi_{17} = -A_i^T P_j H_1$ ,  $\varphi_{34} = i^T P_j W_{1l}$ ,  $\varphi_{35} = -W_i^T P_j H_0$ ,

$\varphi_{36} = W_i^T P_j H$ ,  $\varphi_{37} = W_i^T P_j H_1$ ,  $\varphi_{45} = -W_{1i}^T P_j H_0$ ,  $\varphi_{46} = W_{1i}^T P_j H$ ,

$\varphi_{47} = W_{1i}^T P_j H_1$ ,  $\varphi_{56} = -H_0^T P_i H$ ,  $\varphi_{57} = -H_0^T P_i H_1$ ,  $\varphi_{66} = H^T P_i H$ ,

$$\varphi_{67} = H^T P_i H_1, \quad \varphi_{77} = H_1^T P_i H_1,$$

$$\hat{\tau} = \tau_2 - \tau_1 + 1, \quad \Pi_{11}(i) = e_{0i} E_0^T E_0 - P_i + \hat{\tau} Q_i - 2L_1 T_i, \quad \Pi_{33}(i) = -2T_i + e_i E^T E,$$

$$\begin{aligned} \Pi_{44}(i) &= -2S_i + e_{1i} E_1^T E_1, \quad \Pi_{55}(i) = -e_{0i} I, \quad \Pi_{66}(i) = -e_i I, \quad \Pi_{77}(i) = -e_{1i} I, \\ L_1 &= \text{diag}(l_1^+ l_1^-, l_2^+ l_2^-, \dots, l_n^+ l_n^-), \quad L_2 = \text{diag}\left(-\frac{l_1^+ + l_1^-}{2}, -\frac{l_2^+ + l_2^-}{2}, \dots, -\frac{l_n^+ + l_n^-}{2}\right), \\ V_1 &= \text{diag}(v_1^+ v_1^-, v_2^+ v_2^-, \dots, v_n^+ v_n^-) \text{ and } V_2 = \text{diag}\left(-\frac{v_1^+ + v_1^-}{2}, -\frac{v_2^+ + v_2^-}{2}, \dots, -\frac{v_n^+ + v_n^-}{2}\right). \end{aligned}$$

**Proof.** Similar to theorem 3.1 and will be see also [32].  $\square$

**Remark 3.2.4** We now compare our obtained results with those obtained in [22]. When the DSNNs without polytopic type uncertainties, the system (3.22) reduces to

$$\begin{aligned} x(k+1) &= -[A + \Delta A(k)]x(k) + [W + \Delta W(k)]f(x(k)) \\ &\quad + [W_1 + \Delta W_1(k)]g(x(k - \tau(k))) + \sigma(x(k), x(k - \tau(k)), k)w(k). \end{aligned}$$

In this corollary 3.1, we can show that, if the conditions (3.43) and (3.44) hold, the system is robustly exponentially stable same as the main result (Theorem 1) of [22].

**Remark 3.2.5** We now compare our obtained results with those obtained in [32]. In our main results, we derived robust stability conditions for general discrete-time LPD neural networks with stochastic disturbance. When the system without stochastic disturbance are considered, our system is in the following form

$$\begin{aligned} x(k+1) &= -[A(\xi) + \Delta A(k)]x(k) + [W(\xi) + \Delta W(k)]f(x(k)) \\ &\quad + [W_1(\xi) + \Delta W_1(k)]g(x(k - \tau(k))) \end{aligned}$$

As discussed previously, Theorem 3.1 is the same as the main result (Theorem 3.1) of [32].

**Example 3.2.6** Consider the NNs (3.22) with polytopic type uncertainties with  $N = 2$  where

$$\begin{aligned}
 A_1 &= \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, A_2 = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, W_1 = \begin{bmatrix} 0.3 & -0.1 & 0.2 \\ 0 & -0.3 & 0.2 \\ -0.1 & -0.1 & -0.2 \end{bmatrix}, \\
 W_2 &= \begin{bmatrix} 0.2 & -0.2 & 0.1 \\ 0 & -0.3 & 0.2 \\ -0.2 & -0.1 & -0.2 \end{bmatrix}, W_{11} = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}, \\
 W_{12} &= \begin{bmatrix} -0.2 & 0.1 & 0 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}, H_0 = \begin{bmatrix} 0.005 & -0.011 & 0.002 \\ 0.004 & 0.005 & -0.006 \\ 0 & 0.009 & 0.012 \end{bmatrix}, \\
 H &= \begin{bmatrix} -0.005 & -0.003 & 0.012 \\ 0.008 & 0 & -0.014 \\ 0.001 & 0.008 & -0.007 \end{bmatrix}, H_1 = \begin{bmatrix} 0.022 & -0.016 & 0.007 \\ -0.006 & 0 & 0.037 \\ 0.008 & 0.005 & -0.004 \end{bmatrix}, \\
 L_1 &= \begin{bmatrix} -0.16 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, L_2 = \begin{bmatrix} -0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, V_1 = \begin{bmatrix} -0.12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
 V_2 &= \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}, F_0 = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & -1 \end{bmatrix}, F = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.7 & 0 \\ 0 & 0 & 0.9 \end{bmatrix}, \\
 F_1 &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \rho_1 = \rho_2 = 0.02, E_0 = H_0, E = H, E_1 = H_1, \\
 \tau(k) &= 4 + \sin\left(\frac{k\pi}{2}\right), \tau_1 = 3, \tau_2 = 5, \hat{f}_1(s) = \tanh(0.6s) - 0.2 \sin s, \\
 \hat{f}_2(s) &= \tanh(-0.4s), \hat{f}_3(s) = \tanh(-0.2s), \hat{g}_1(s) = \tanh(-0.4s) + 0.2 \sin s, \\
 \hat{g}_2(s) &= \tanh(0.2s), \hat{g}_3(s) = \tanh(0.4s).
 \end{aligned}$$

By using the Matlab LMI toolbox, we can solve for  $P_i$ ,  $Q_i$ ,  $S_i$  and  $T_i$ ,  $i = 1, 2$  which satisfy the conditions (3.30)-(3.33) in Theorem 3.2.1. Thus, the system (3.22) is robustly stable. A set of solutions of (3.22) are given below

$$\begin{aligned}
 P_1 &= \begin{bmatrix} 366.5842 & -33.1641 & 6.8742 \\ -33.1641 & 256.2808 & 36.3198 \\ 6.8742 & 36.3198 & 309.9334 \end{bmatrix}, Q_1 = \begin{bmatrix} 75.9702 & -6.2950 & -0.8498 \\ -6.2950 & 33.3042 & 7.4723 \\ -0.8498 & 7.4723 & 52.1408 \end{bmatrix}, \\
 S_1 &= \begin{bmatrix} 75.4071 & 0 & 0 \\ 0 & 75.4071 & 0 \\ 0 & 0 & 75.4071 \end{bmatrix}, T_1 = \begin{bmatrix} 81.4174 & 0 & 0 \\ 0 & 81.4174 & 0 \\ 0 & 0 & 81.4174 \end{bmatrix},
 \end{aligned}$$

$$e_{01} = 113.2095, e_1 = 113.2460, e_{11} = 113.5785, \lambda_1^* = 518.8508$$

$$P_2 = \begin{bmatrix} 423.5919 & -80.2348 & 37.0438 \\ -80.2348 & 314.8939 & 45.9693 \\ 7.0438 & 45.9693 & 343.3546 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 89.2738 & -20.6012 & 7.3610 \\ -20.6012 & 62.3412 & 10.9227 \\ 7.3610 & 10.9227 & 66.9368 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 76.3904 & 0 & 0 \\ 0 & 76.3904 & 0 \\ 0 & 0 & 76.3904 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 81.7596 & 0 & 0 \\ 0 & 81.7596 & 0 \\ 0 & 0 & 81.7596 \end{bmatrix},$$

$$e_{02} = 113.2271, \quad e_2 = 113.2577, \quad e_{12} = 113.6523, \quad \lambda_2^* = 592.9554.$$

The trajectories of solutions of system (3.22) of example are illustrated in Fig.3.2, where  $A(\xi) = \frac{1}{2}A_1 + \frac{1}{2}A_2$ ,  $W(\xi) = \frac{2}{3}W_1 + \frac{1}{3}W_2$ ,  $W_1(\xi) = \frac{2}{5}W_{11} + \frac{3}{5}W_{12}$ , with  $x_1(k) = 0$ ,  $x_2(k) = 0$  and  $x_3(k) = 0$ ,  $k = -4, -3, -2, -1, 0$ , and  $x_1(0) = 0.8$ ,  $x_2(0) = 0.4$ ,  $x_3(0) = -0.7$ .

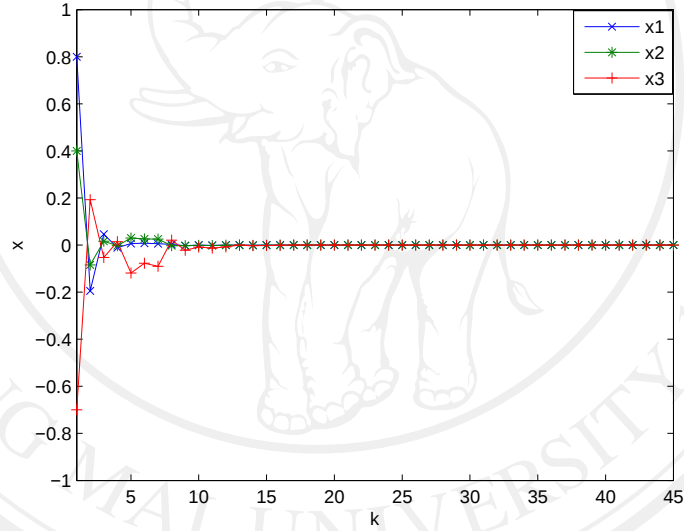


Figure 3.2: The trajectory of solutions of system (3.22) in Example 3.2.6.