

# Chapter 4

## Global stability criteria of nonlinear discrete-time systems with time-varying delay

In this chapter, we present a new approach to the global stability of nonlinear difference equation and discrete-time neural networks with time-varying delays. Based on discrete type inequalities, new global stability conditions for nonlinear difference equations are derive. Numerical examples are given to illustrate the effectiveness of our theoretical results.

### 4.1 New discrete type inequalities and global stability of nonlinear difference equations

Consider the following nonlinear difference equation:

$$\Delta x_n = f(n, x_n, x_{n-1}, \dots, x_{n-r}), \quad n \in \mathbb{Z}^+, \quad (4.1)$$

where  $\Delta x_n = x_{n+1} - x_n$ , and  $f : \mathbb{N} \times \mathbb{R}^{r+1} \rightarrow \mathbb{R}$ . The system (4.1) is called generalized difference equation and we introduce new discrete equation inequalities which will be used to derive global stability conditions in the next section.

**Theorem 4.1.1** *Let  $q_i \in \mathbb{R}_0^+$ ,  $h_i \in \mathbb{Z}^+$ ,  $i = 1, \dots, r$ ;  $p, q_r \in \mathbb{R}^+$ , where  $0 = h_0 < h_1 < \dots < h_r$  and  $\sum_{i=0}^r q_i < p \leq 1$ , and let  $\{x_j\}_{j \in \mathbb{Z}^{-h_r}}$  be a sequence of real numbers satisfying the inequality*

$$\Delta x_n \leq -px_n + \sum_{i=0}^r q_i x_{n-h_i}, \quad n \in \mathbb{Z}^0. \quad (4.2)$$

Then there exists  $\lambda_0 \in (0, 1)$  such that

$$x_n \leq \max\{0, x_0, x_{-1}, \dots, x_{-h_r}\} \lambda_0^n, \quad n \in \mathbb{Z}^0.$$

Moreover,  $\lambda_0$  might be chosen as the smallest root of the polynomial

$$P(\lambda) = \lambda^{h_r+1} - (1 - p + q_0)\lambda^{h_r} - q_1\lambda^{h_r-h_1} \dots - q_{r-1}\lambda^{h_r-h_{r-1}} - q_r \quad (4.3)$$

which lies in the interval  $(0, 1)$ .

*Proof.* Let  $\{y_n\}$  be a solution of the difference equation

$$\Delta y_n = -py_n + \sum_{i=0}^r q_i y_{n-h_i}, \quad n \in \mathbb{Z}^0. \quad (4.4)$$

Since  $q_i \in \mathbb{R}_0^+$  and  $0 < p < 1$ , it is straightforward to show that if  $\{x_n\}$  satisfies (4.2) and  $x_n \leq y_n$  for  $n = -h_r, \dots, 0$ , then  $x_n \leq y_n$  for all  $n \in \mathbb{Z}^0$ . For a given  $K > 0$  and  $\lambda \in (0, 1)$ , the sequence  $\{y_n\}$  defined by  $y_n = K\lambda^n$  is a solution of the equation (4.4) if and only if  $\lambda$  is a root of the polynomial (4.3).

Since  $\lim_{\lambda \rightarrow 0^+} P(\lambda) = -q_r < 0$  and  $P(1) = p - \sum_{i=0}^r q_i > 0$ , it follows from continuity of  $P$  that there exists the smallest real number  $\lambda_0 \in (0, 1)$  such that  $P(\lambda_0) = 0$ . Thus, for any  $K \in \mathbb{R}_0^+$ , the sequence  $\{K\lambda_0^n\}$  is a solution of (4.4). Let  $K_0 = \max\{0, x_0, x_{-1}, \dots, x_{-h_r}\}$ . Then,  $\{y_n\} = \{K_0\lambda_0^n\}$  is a solution of (4.4) and obviously we have  $x_n \leq y_n$ , for  $n = -h_r, \dots, 0$ . Therefore, by using the first part of the proof, we conclude that  $x_n \leq y_n = K_0\lambda_0^n$ ,  $n \in \mathbb{Z}^0$ .  $\square$

By a similar argument used in the proof of Theorem 4.1.1, we obtain the following result.

**Theorem 4.1.2** Let  $p, \alpha_i, \beta_i \in \mathbb{R}^+$ ,  $h_i \in \mathbb{Z}^+$ ,  $i = 1, \dots, r$ , where  $0 = h_0 < h_1 < \dots < h_r$ ,  $\sum_{i=0}^r \alpha_i = 1$  and  $\prod_{i=0}^r \beta_i < p \leq 1$ . Let  $\{x_n\}_{n \in \mathbb{Z}^{-h_r}}$  be a sequence of real numbers such that  $x_{n-h_i}^{\alpha_i}$  are defined for all  $i = 1, \dots, r$ ;  $n \in \mathbb{Z}^0$  which satisfies the inequality

$$\Delta x_n \leq -px_n + \prod_{i=0}^r \beta_i x_{n-h_i}^{\alpha_i}, \quad n \in \mathbb{Z}^0.$$

Then there exists  $\lambda_0 \in (0, 1)$  such that

$$x_n \leq \max\{0, x_0, x_{-1}, \dots, x_{-h_r}\} \lambda_0^n, \quad n \in \mathbb{Z}^0.$$

Moreover,  $\lambda_0$  might be chosen as the smallest root of the function

$$F(\lambda) = \lambda - \left( \prod_{i=0}^r \beta_i \right) \lambda^{-\sum_{i=1}^r h_i \alpha_i} + p - 1$$

which lies in the interval  $(0, 1)$ .

We derive global stability conditions for nonlinear difference equation using discrete type inequalities derived in the previous section. Consider the nonlinear difference equation:

$$\Delta x_n = -px_n + f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}), \quad (4.5)$$

$n, h_i \in \mathbb{Z}^+, i = 1, \dots, r \in \mathbb{Z}^+, p > 0$ . For any initial string  $\{x_{-r}, x_{-r+1}, \dots, x_0\}$ , (4.5) has a unique solution which can be explicitly calculated. However, it is difficult to obtain stability conditions using that form of solutions. The following result gives a global stability of solutions of (4.5) by using discrete type inequality derived in Theorem 4.1.1.

**Theorem 4.1.3** Assume that there exist  $q_i \in \mathbb{R}_0^+, h_i \in \mathbb{Z}^+, i = 1, \dots, r; q_r \in \mathbb{R}^+$ , where  $\sum_{i=0}^r q_i < p \leq 1$  such that

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq \sum_{i=0}^r q_i |x_{n-h_i}|, \quad (4.6)$$

for all  $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$ . Then, there exists  $\lambda_0 \in (0, 1)$  such that every solution  $\{x_n\}$  of (4.5) satisfies

$$|x_n| \leq \left( \max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0,$$

where  $\lambda_0$  is chosen as in Theorem 4.1.1.

*Proof.* As in [1], it is straightforward to show that every solution  $\{x_n\}$  of (4.5) is written in the form

$$x_n = x_0(1-p)^n + \sum_{i=0}^{n-1} (1-p)^{n-i-1} f(i, x_i, x_{i-h_1}, \dots, x_{i-h_r}), \quad n \in \mathbb{Z}^0.$$

By using (4.6), we obtain

$$|x_n| \leq |x_0| (1-p)^n + \sum_{i=0}^{n-1} \sum_{j=0}^r (1-p)^{n-i-1} q_j |x_{i-h_j}|, \quad n \in \mathbb{Z}^0.$$

For each  $n = -h_r, \dots, 0$ , let  $v_n = |x_n|$  and for each  $n \in \mathbb{Z}^+$ , we let

$$v_n = |x_0| (1-p)^n + \sum_{i=0}^{n-1} \sum_{j=0}^r (1-p)^{n-i-1} q_j |x_{i-h_j}|.$$

Then, we have  $|x_n| \leq v_n, n \in \mathbb{Z}^{-h_r}$ , and hence,

$$\Delta v_n = -pv_n + \sum_{i=0}^r q_i |x_{n-h_i}| \leq -pv_n + \sum_{i=0}^r q_i v_{n-h_i}, \quad n \in \mathbb{Z}^0.$$

Therefore, by Theorem 4.1.1, we obtain

$$|x_n| \leq v_n \leq \left( \max_{-h_r \leq i \leq 0} \{v_i\} \right) \lambda_0^n = \left( \max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0,$$

where  $\lambda_0$  is chosen as in Theorem 4.1.1. This completes the proof of the theorem.  $\square$

Similarly, by using Theorem 4.1.2 instead of Theorem 4.1.1, we obtain the following result.

**Theorem 4.1.4** Assume that there exist  $p, \alpha_i, \beta_i \in \mathbb{R}^+$ ,  $h_i \in \mathbb{Z}^+$ ,  $i = 1, \dots, r$ , where  $\sum_{i=0}^r \alpha_i = 1$  and  $\prod_{i=0}^r \beta_i < p \leq 1$  such that

$$|f(n, x_n, x_{n-h_1}, \dots, x_{n-h_r})| \leq \prod_{i=0}^r \beta_i |x_{n-h_i}|^{\alpha_i},$$

for all  $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$ . Then there exists  $\lambda_0 \in (0, 1)$  such that every solution  $\{x_n\}$  of (4.5) satisfies

$$|x_n| \leq \left( \max_{-h_r \leq i \leq 0} \{|x_i|\} \right) \lambda_0^n, \quad n \in \mathbb{Z}^0,$$

where  $\lambda_0$  is chosen as in Theorem 4.1.2.

**Remark 4.1.5** In [18], a discrete Halanay-type inequality is given as in Theorem 4.1.1 where the inequality (4.2) is replaced by

$$\Delta x_n \leq -p x_n + q \max\{x_n, x_{n-1}, \dots, x_{n-r}\}, \quad n \in \mathbb{Z}^0, \quad (4.7)$$

where  $0 < q < p \leq 1$ . Note that if a sequence  $\{x_n\}_{n \in \mathbb{Z}^{-r}}$  of positive real numbers satisfies (4.7), then it also satisfies (4.2). On the other hand, let  $r = 1$ ,  $p = \frac{5}{6}$ ,  $q = q_0 = q_1 = \frac{1}{7}$ , then we might easily show that the sequence  $\{\frac{1}{2^n}\}_{n \in \mathbb{Z}^{-1}}$  satisfies (4.2) but not (4.7). Indeed,

$$\Delta x_n = \frac{1}{2^{n+1}} - \frac{1}{2^n} = -\frac{1}{2^{n+1}} \leq -\frac{5}{6} \frac{1}{2^n} + \frac{1}{7} \left( \frac{1}{2^n} + \frac{1}{2^{n-1}} \right) = -\frac{17}{42} \frac{1}{2^n}.$$

On the other hand,

$$\Delta x_n = -\frac{1}{2^{n+1}} > -\frac{5}{6} \frac{1}{2^n} + \frac{1}{7} \max \left\{ \frac{1}{2^n}, \frac{1}{2^{n-1}} \right\} = -\frac{23}{42} \frac{1}{2^n}.$$

Therefore, in the case of positive sequences, the discrete type inequality (4.2) is less conservative than the discrete Halanay-type inequality given by (4.7).

**Remark 4.1.6** In [6], it was shown that if  $p = 1$  and

$$|f(n, x_n, x_{n-1}, \dots, x_{n-r})| \leq \beta \prod_{i=0}^r |x_{n-i}|^{\alpha_i},$$

for all  $(n, x_n, x_{n-h_1}, \dots, x_{n-h_r}) \in \mathbb{Z}^0 \times \mathbb{R}^{r+1}$ , then (4.5) is locally asymptotically stable provided that either  $\sum_{i=0}^r \alpha_i > 1$  and  $\beta > 0$ , or  $\sum_{i=0}^r \alpha_i = 1$  and  $\beta \in (0, 1)$ . On the other hand, when  $p = 1$ , Theorem 4.1.4 ensures the global stability of nonlinear difference equation (4.5) provided that  $\sum_{i=0}^r \alpha_i = 1$  and  $\beta \in (0, 1)$ .

## 4.2 Global exponential stability of nonlinear discrete-time neural networks with time-varying delays

In this section, we introduce new discrete type inequality which will be used to derive global exponential stability conditions in the next theorem.

**Lemma 4.2.1** *Let  $q_i \in \mathbb{R}_0^+$ ,  $k_i \in \mathbb{Z}^+$ ,  $i = 1, \dots, r$ ;  $p, q_r \in \mathbb{R}^+$ , where  $0 = k_0 < k_1 < \dots < k_r$  and  $\sum_{i=0}^r q_i < p \leq 1$ , and let  $\{x_j\}_{j \in \mathbb{Z}^{-k_r}}$  be a sequence of real numbers satisfying the inequality*

$$\Delta x_n \leq -px_n + \sum_{i=0}^r q_i \sum_{j=0}^{k_i} x_{n-j}, \quad n \in \mathbb{Z}^0. \quad (4.8)$$

*Then there exists  $\lambda_0 \in (0, 1)$  such that*

$$x_n \leq \max\{0, x_0, x_{-1}, \dots, x_{-k_r}\} \lambda_0^n, \quad n \in \mathbb{Z}^0.$$

*Moreover,  $\lambda_0$  might be chosen as the smallest root of the polynomial*

$$P(\lambda) = \lambda^{n+1} - (1-p)\lambda^n - q_1 \sum_{j=0}^{k_1} \lambda^{n-j} - q_2 \sum_{j=0}^{k_2} \lambda^{n-j} - \dots - q_M \sum_{j=0}^{k_r} \lambda^{n-j} \quad (4.9)$$

*which lies in the interval  $(0, 1)$ .*

*Proof* This proof is to appear [31], we can following Theorem 4.1.1. Let  $\{y_n\}$  be a solution of the difference equation

$$\Delta y_n = -py_n + \sum_{i=0}^r q_i \sum_{j=0}^{k_i} y_{n-j}, \quad n \in \mathbb{Z}^0. \quad (4.10)$$

Since  $q_i \in \mathbb{R}_0^+$  and  $0 < p < 1$ , we show that if  $x_n$  satisfies (4.8) and  $x_n \leq y_n$  for  $n = -h_r, \dots, 0$ , then  $x_n \leq y_n$  for all  $n \in \mathbb{Z}^0$ . For given  $K > 0$  and  $\lambda \in (0, 1)$ , the sequence  $\{y_n\}$  defined by  $y_n = K\lambda^n$  is a solution of equation (4.10) if and only if  $\lambda$  is a root of the polynomial (4.9). Since  $\lim_{\lambda \rightarrow 0^+} P(\lambda) = -q_r < 0$  and  $P(1) =$

$p - q(1)(k_1 + 1) - \sum_{j=2}^r q(i)(K_i + K_{i-1}) > 0$ , it follows from continuity of  $P$  that exists

a smallest real number  $\lambda_0 \in (0, 1)$  such that  $P(\lambda_0) = 0$ . Thus, for any  $K \in \mathbb{K}_0^+$ , the sequence  $\{K\lambda^n\}$  is a solution of (4.10). Let  $K_0 = \max\{0, x_n, x_{-1}, \dots, x_{-h_r}\}$ . Therefore,  $y_n = \{K_0\lambda_0^n\}$  is a solution of (4.10) and obviously, we have  $x_n \leq y_n$ , for  $n = -h_r, \dots, 0$ . Hence, by using the first part of proof, we conclude that  $x_n \leq y_n = K_0\lambda_0$ ,  $n \in \mathbb{Z}^0$   $\square$

Next, We consider the following nonlinear discrete-time neural networks with time-varying delays:

$$u(k+1) = Cu(k) + Af(u(k)) + \sum_{j=1}^M B_j f(u(k - \tau_j(k))) + b, \quad (4.11)$$

where  $u(k) = [u_1(k), \dots, u_n(k)]^T \in \mathbb{R}^n$  is the neuron state vector,  $b = [b_1, \dots, b_n]^T$  is constant input vector,  $\tau_j(k)$ ,  $j = 1, 2, \dots, M$  are positive integers denote the time-varying delay satisfying

$$0 \leq \tau_j(k) \leq \tau_j, \quad j = 1, 2, \dots, M$$

where  $\tau_j \geq 0$  are known integers,  $C = \text{diag}(c_i)$  ( $c_i \in (0, 1)$ ),  $A$  and  $B_j$  are the interconnection matrices and the activation function  $f_i(\cdot)$   $i = 1, \dots, n$  satisfies the following,

$$|f_i(x) - f_i(y)| \leq L_i |x - y|, \quad \forall x, y \in \mathbb{R}.$$

The initial conditions with system (4.11) are of the form

$$u_i(l) = \phi_i(l), \quad i = 1, 2, \dots, n,$$

where  $l$  is an integer with  $l \in [-\tau, 0]$ . Let  $u^* = [u_1^*, u_2^*, \dots, u_n^*]$  be an equilibrium point of system (4.11). We shift the equilibrium point  $u^*$  to the origin by the transformation  $x(\cdot) = u(\cdot) - u^*$ . Then, we obtain the new system

$$x(k+1) = Cx(k) + Ag(x(k)) + \sum_{j=1}^M B_j g(x(k - \tau_j(k))), \quad (4.12)$$

where  $x(k) = [x_1(k), x_2(k), \dots, x_n(k)]$  is the state vector of the transformed system,  $g_j(x_j(\cdot)) = \{g_1(x_1(\cdot)), \dots, g_n(x_n(\cdot))\}$ ,  $g_j(x_j(\cdot)) = \hat{f}_j(x_j(k) + u_j^*) - \hat{f}_j(u_j^*)$ ,  $j = 1, 2, \dots, n$  and the transformed activation functions satisfy the condition

$$|g_j(x)| \leq L_j |x|,$$

and  $\tau_j(k)$  are the time-varying delays satisfying

$$0 \leq \tau_j(k) \leq \tau_j, \quad j = 1, 2, \dots, M$$

where  $\tau_j$  are known positive integers. The initial conditions with system (4.12) are of the form

$$x_i(l) = \xi_i(l), \quad i = 1, 2, \dots, n,$$

**Theorem 4.2.2** *The equilibrium point of system (4.12) is globally exponentially stable if*

$$c_{\max} + l\|A\| + l \sum_{j=1}^M (\tau_j + 1) \|B_j\| < 1, \quad (4.13)$$

where  $l = \max_j(L_j)$  and  $c_{\max} = \max_j(c_j)$ .

*Proof* Consider  $z(k) = \|x(k)\|$  and the difference of system is given by

$$\begin{aligned}
\Delta z(k) &= \|x(k+1)\| - \|x(k)\| \\
&= \|Cx(k) + Ag(x(k)) + \sum_{j=1}^M B_j g(x(k - \tau_j(k)))\| - \|x(k)\| \\
&\leq \|Cx(k)\| + \|Ag(x(k))\| + \sum_{j=1}^M \|B_j g(x(k - \tau_j(k)))\| - \|x(k)\| \\
&\leq -(1 - c_{\max} - l\|A\|)\|x(k)\| + l \sum_{j=1}^M \|B_j\| \|x(k - \tau_j(k))\| \\
&\leq -(1 - c_{\max} - l\|A\|)\|x(k)\| + l \sum_{j=1}^M \sum_{i=0}^{\tau_j} \|B_j\| \|x(k - i)\| \\
&\leq -(1 - c_{\max} - l\|A\|)\|x(k)\| + l \sum_{j=1}^M (\tau_j + 1) \|B_j\| \sum_{i=0}^{\tau_j} \|x(k - i)\| \\
&= -pz(k) + \sum_{j=0}^M q_j \sum_{i=0}^{\tau_j} z(k - i),
\end{aligned}$$

where

$$p = 1 - c_{\max} - l\|A\|, \quad \sum_{j=0}^M q_j = l \sum_{j=1}^M (\tau_j + 1) \|B_j\|.$$

It follows from assumptions of the Lemma 4.2.1 that there exist  $\lambda_0 \in (0, 1)$  such that

$$\sum_{i=0}^M q_i < p < 1, \quad (4.14)$$

then we obtain

$$\begin{aligned}
z(k) &= \|x(k)\| \\
&\leq \max\{0, z(0), z(-1), \dots, z(-\tau_M)\} \lambda_0^n \\
&= \max\{0, \|x(0)\|, \|x(-1)\|, \dots, \|x(-\tau_M)\|\} \lambda_0^n \\
&\leq \|\xi\| \lambda_0^n.
\end{aligned} \quad (4.15)$$

By Definition 2.4.14, we conclude that the equation (4.12) is globally exponentially stable the proof is complete.  $\square$

**Remark 4.2.3** In our main result, we derived global exponential stability of discrete time neural network condition for multiple time varying delays by use discrete type inequality. When there have one time varying delays, the sufficient condition is in [35], the author studied the global exponential stability of discrete time hopfield neural networks is given by theorem 2.9. Nevertheless, the condition in [35] can not applied to multiple time varying delays.

**Example 4.2.4** Consider the NNs (4.11) with  $M=2$  where

$$C = \begin{bmatrix} 0.25 & 0 \\ 0 & -0.15 \end{bmatrix}, A = \begin{bmatrix} 0.15 & -0.2 \\ -0.35 & 0.4 \end{bmatrix}, B_1 = \begin{bmatrix} 0.25 & -0.1 \\ -0.3 & 0.15 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.4 & -0.25 \\ 0.35 & -0.3 \end{bmatrix},$$

$$\tau_1(k) = 1 + \sin\left(\frac{k\pi}{2}\right), \tau_2(k) = 2 + \sin\left(\frac{k\pi}{2}\right), f_1(s) = \tanh(-0.4s) + 0.2 \sin(s),$$

$$f_2(s) = \tanh(0.2s), \text{ and we choose } l = 1, \tau_1 = 1, \text{ and } \tau_2 = 2, b = [-0.5 \ 0.6].$$

It's easy to check that

$$\|A\|_1 + (\tau_1 + 1)\|B_1\|_1 + (\tau_2 + 1)\|B_2\|_1 = 0.65 < 1 - c_{\max} = 0.75.$$

Therefore, from Theorem 4.2.2, it follows that the solutions of system (4.11) is globally exponentially stable.

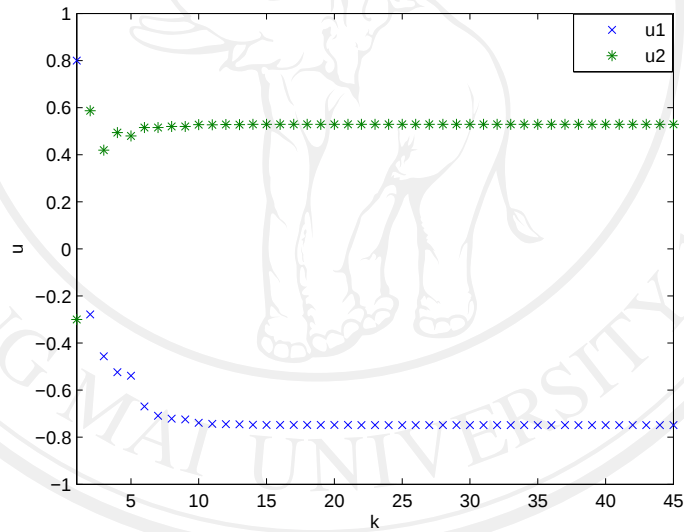


Figure 4.1: The trajectory of solutions of system (4.11) in Example 4.2.4.

**Example 4.2.5** Consider the NNs (4.11) with  $M=2$  where

$$C = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, A = \begin{bmatrix} -0.2 & 0.3 \\ 0.4 & -0.15 \end{bmatrix}, B_1 = \begin{bmatrix} 0.2 & -0.1 \\ -0.3 & 0.2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.25 & 0.3 \\ 0.1 & -0.15 \end{bmatrix},$$

$$\tau_1(k) = 1 + \sin\left(\frac{k\pi}{2}\right), \tau_2(k) = 2 + \sin\left(\frac{k\pi}{2}\right), f_1(s) = \tanh(-0.4s) + 0.2 \sin(s),$$

$$f_2(s) = \tanh(0.2s), \text{ and we chosen } l = 1, \tau_1 = 1 \text{ and } \tau_2 = 2.$$

It's easy to check that

$$\|A\|_{\infty} + (\tau_1 + 1)\|B_1\|_{\infty} + (\tau_2 + 1)\|B_2\|_{\infty} = 0.6 < 1 - c_{\max} = 0.8.$$

Therefore, from Theorem 4.2.2, it follows that the solutions of system (4.11) is globally exponentially stable.

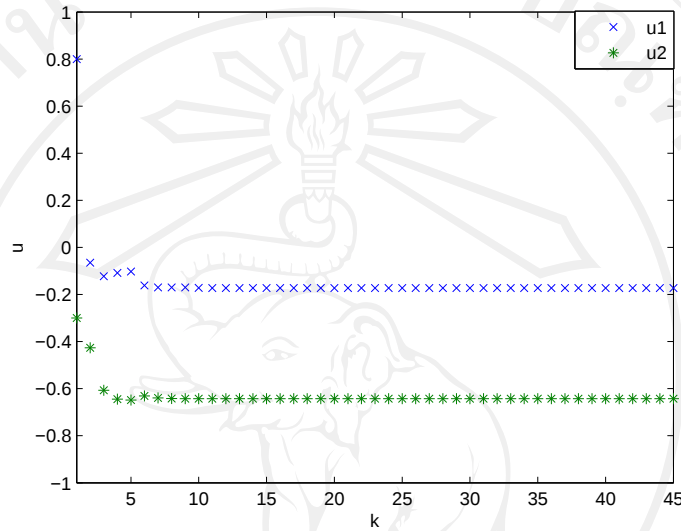


Figure 4.2: The trajectory of solutions of system (4.11) in Example 4.2.5.