

Chapter 5

H_∞ filtering problem for nonlinear discrete-time neural networks with time-varying delay

In this chapter, we study H_∞ filtering problem for discrete-time neural networks with time-varying delay. Based on Lyapunov stability theory and the S-procedure, we derive criteria in terms of LMIs. Numerical examples are given to illustrate the effectiveness of our theoretical results.

We consider the following discrete-time neural network with time-varying delay:

$$\begin{cases} x(k+1) = A_1x(k) + A_2f(x(k)) + A_3g(x(k-\tau(k))) + B\omega(k) \\ y(k) = C_1x(k) + C_2f(x(k)) + C_3g(x(k-\tau(k))) + D\omega(k) \\ z(k) = K_1x(k) + K_2f(x(k)) + K_3g(x(k-\tau(k))) + G\omega(k) \\ x(k) = \phi(k), \quad k = -\tau_2, \tau_2 + 1, \dots, 0, \end{cases} \quad (5.1)$$

where $x(k) \in \mathbb{R}^n$ is the neuron state vector, $y(k) \in \mathbb{R}^m$ is the measurement vector, $\omega(k) \in \mathbb{R}^q$ is the noise signal vector belonging to $l_2[0, +\infty)$, $\tau(k)$ is a time-varying delay satisfying

$$\tau_1 \leq \tau(k) \leq \tau_2, \quad (5.2)$$

where $\tau_1, \tau_2 \geq 0$ are known integers, $A_1, A_2, A_3, B, C_1, C_2, C_3, D, K_1, K_2, K_3$ and G are the constant matrices with appropriate dimensions (the interconnection matrices) and the activation functions $f_i(\cdot)$ $i = 1, \dots, n$ and $g_i(\cdot)$ $i = 1, \dots, n$ satisfy the following conditions

$$l_j^- \leq \frac{f_j(x) - f_j(y)}{x - y} \leq l_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (5.3)$$

$$v_j^- \leq \frac{g_j(x) - g_j(y)}{x - y} \leq v_j^+ \quad \forall x, y \in \mathbb{R}, x \neq y, j = 1, 2, \dots, n, \quad (5.4)$$

where $l_j^-, l_j^+, v_j^-, v_j^+, j = 1, 2, \dots, n$ are known constants. We design a full-order filter for system (5.1) as follows:

$$\begin{aligned}\hat{x}(k+1) &= A_F \hat{x}(k) + B_F y(k), \quad \hat{x}(0) = 0 \\ \hat{z}(k) &= C_F \hat{x}(k) + D_F y(k),\end{aligned}\tag{5.5}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the filter state, $\hat{z}(k)$ is the estimation of $z(k)$, $A_F \in \mathbb{R}^{n \times n}$, $B_F \in \mathbb{R}^{n \times m}$, $C_F \in \mathbb{R}^{p \times n}$, $D_F \in \mathbb{R}^{p \times m}$ are filter parameters to be designed for system (5.1). Let $\bar{x}(k) = x(k) - \hat{x}(k)$ and $e(k) = z(k) - \hat{z}(k)$. Then, we obtain the filter error dynamics can be described by

$$\begin{cases} \bar{x}(k+1) = \bar{A}_1 x(k) - A_F \hat{x}(k) + \bar{A}_2 f(x(k)) + \bar{A}_3 g(x(k - \tau(k))) + \bar{B} \omega(k) \\ e(k) = \bar{K}_1 x(k) - C_F \hat{x}(k) + \bar{K}_2 f(x(k)) + \bar{K}_3 g(x(k - \tau(k))) + \bar{G} \omega(k) \\ \bar{x}(k) = \phi(k), \quad k = -\tau_2, \tau_2 + 1, \dots, 0, \end{cases}\tag{5.6}$$

where $\bar{A}_1 = A_1 - B_F C_1$, $\bar{A}_2 = A_2 - B_F C_2$, $\bar{A}_3 = A_3 - B_F C_3$, $\bar{B} = B - B_F D$, $\bar{K}_1 = K_1 - D_F C_1$, $\bar{K}_2 = K_2 - D_F C_2$, $\bar{K}_3 = K_3 - D_F C_3$, $\bar{G} = G - D_F D$.

Our goal is to design a robust H_∞ filter of the form (5.5) such that the filtering error system (5.6) satisfy the following two conditions hold:

Given integers $\tau_2 \geq \tau_1 > 0$ and scalar $\gamma > 0$

- (i) The filtering error system (5.6) with $\omega(k) = 0$ is asymptotically stable ; and
- (ii) The H_∞ performance

$$\|e\|_2 \leq \gamma \|\omega\|_2\tag{5.7}$$

is guaranteed under zero-initial conditions for all nonzero $\omega(k) \in l_2[0, +\infty)$ and a prescribed $\gamma > 0$.

5.1 Stability analysis

First, we derive asymptotic stability condition of system (5.6) with $\omega(k) = 0$.

Proposition 5.1.1 *Given the filter parameters A_F, B_F, C_F and D_F . Then, the filtering error system (5.6) with time-varying delay $\tau(k)$ satisfying (5.2) with $\omega(k) = 0$ is asymptotically stable if there exist $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q = Q^T > 0$, $T = \text{diag}\{t_1, t_2, \dots, t_n\} \geq 0$, $S = \text{diag}\{s_1, s_2, \dots, s_n\} \geq 0$ satisfying the following conditions*

$$M = \begin{bmatrix} M_0 & M_1^T P_1 & M_2^T P_2 \\ * & -P_1 & 0 \\ * & * & -P_2 \end{bmatrix} < 0,\tag{5.8}$$

where

$$M_0 = \begin{bmatrix} -P_2 - 2L_1T + \hat{\tau}Q & P_2 & 0 & -2L_2T & 0 \\ * & -P_2 & 0 & 0 & 0 \\ * & * & -Q - 2V_1S & 0 & -2V_2S \\ * & * & * & -2T & 0 \\ * & * & * & * & -2S \end{bmatrix},$$

$$M_1 = [A_1 \ 0 \ 0 \ A_2 \ A_3], \ M_2 = [\bar{A}_1 \ -A_F \ 0 \ \bar{A}_2 \ \bar{A}_3], \ \bar{A}_1 = A_1 - B_F C_1, \ \bar{A}_2 = A_2 - B_F C_2, \ \bar{A}_3 = A_3 - B_F C_3, \ \hat{\tau} = \tau_2 - \tau_1 + 1.$$

Proof Consider the Lyapunov function candidate

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k),$$

where

$$V_1(k) = x^T(k)P_1x(k), \ V_2(k) = \bar{x}^T(k)P_2\bar{x}(k), \ V_3(k) = \sum_{l=k-\tau(k)}^{k-1} x^T(l)Qx(l),$$

$$V_4(k) = \sum_{j=k-\tau(k)}^{-\tau_1+1} \sum_{l=k+j-1}^{k-1} x^T(l)Qx(l).$$

The Lyapunov difference along trajectory of solution of system is given by

$$\begin{aligned} \Delta V_1(k) &= x^T(k+1)P_1x(k+1) - x^T(k)P_1x(k) \\ &= [A_1x(k) + A_2f(x(k)) + A_3g(x(k-\tau(k)))]^T P_1 [A_1x(k) + A_2f(x(k)) \\ &\quad + A_3g(x(k-\tau(k)))] - x^T(k)P_1x(k) \\ &= y^T(k)M_1^T P_1 M_1 y(k) - x^T(k)P_1x(k), \end{aligned}$$

where

$$y(k) = [x^T(k) \ \hat{x}^T(k) \ x^T(k-\tau(k)) \ f^T(x(k)) \ g^T(x(k-\tau(k)))]^T,$$

$$M_1 = [A_1 \ 0 \ 0 \ A_2 \ A_3],$$

and

$$\begin{aligned} \Delta V_2(k) &= \bar{x}^T(k+1)P_2\bar{x}(k+1) \\ &= [\bar{A}_1x(k) - A_F\hat{x}(k) + \bar{A}_2f(x(k)) + \bar{A}_3g(x(k-\tau(k)))]^T P_2 [\bar{A}_1x(k) \\ &\quad - A_F\hat{x}(k) + \bar{A}_2f(x(k)) + \bar{A}_3g(x(k-\tau(k)))] - \bar{x}^T(k)P_2\bar{x}(k) \\ &= y^T(k)M_2^T P_2 M_2 y(k) - [x^T(k)P_2x(k) - \hat{x}^T(k)P_2x(k) - x^T(k)P_2\hat{x}(k) \\ &\quad + \hat{x}^T(k)P_2\hat{x}(k)] \\ &= y^T(k)M_2^T P_2 M_2 y(k) - x^T(k)P_2x(k) + 2\hat{x}^T(k)P_2x(k) - \hat{x}^T(k)P_2\hat{x}(k), \end{aligned}$$

where $M_2 = [\bar{A}_1 \ -A_F \ 0 \ \bar{A}_2 \ \bar{A}_3]$.

Similarly, we obtain

$$\begin{aligned}
\Delta V_3(k) &= \sum_{l=k+1-\tau(k+1)}^k x^T(l)Qx(l) - \sum_{l=k-\tau(k)}^{k-1} x^T(l)Qx(l) \\
&= \sum_{l=k-\tau(k+1)}^{k-\tau_1} x^T(l)Qx(l) + x^T(k)Qx(k) - x^T(k-\tau(k))Qx(k-\tau(k)) \\
&\quad + \sum_{l=k+1-\tau_1}^{k-1} x^T(l)Qx(l) - \sum_{l=k+1-\tau(k)}^{k-1} x^T(l)Qx(l). \tag{5.9}
\end{aligned}$$

Since $\tau(k) \geq \tau_1$, we get

$$\sum_{l=k+1-\tau_1}^{k-1} x^T(l)Qx(l) - \sum_{l=k+1-\tau(k)}^{k-1} x^T(l)Qx(l) \leq 0,$$

which implies that

$$\Delta V_3(k) \leq \sum_{l=k-\tau(k+1)}^{k-\tau_1} x^T(l)Qx(l) + x^T(k)Qx(k) - x^T(k-\tau(k))Qx(k-\tau(k)).$$

We have

$$\begin{aligned}
\Delta V_4(k) &= \sum_{l=-\tau_2+2}^{-\tau_1+1} \left[x^T(k)Qx(k) + \sum_{m=k+j}^{k-1} x^T(m)Qx(m) - \sum_{m=k+j-1}^{k-1} x^T(m)Qx(m) \right] \\
&= (\tau_2 - \tau_1)x^T(k)Qx(k) - \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(l)Qx(l).
\end{aligned}$$

From $\Delta V_3(k)$ and $\Delta V_4(k)$, we get

$$\begin{aligned}
\Delta V_3(k) + \Delta V_4(k) &\leq \hat{\tau}x^T(k)Qx(k) - x^T(k-\tau(k))Qx(k-\tau(k)) \\
&\quad + \sum_{l=k+1}^{k-\tau_1} x^T(l)Qx(l) - \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(l)Qx(l),
\end{aligned}$$

where $\hat{\tau} = \tau_2 - \tau_1 + 1$. Since $\tau(k) \leq \tau_2$, we obtain

$$\sum_{l=k+1-\tau(k+1)}^{k-\tau_1} x^T(l)Qx(l) - \sum_{l=k+1-\tau_2}^{k-\tau_1} x^T(l)Qx(l) \leq 0.$$

Thus,

$$\Delta V_3(k) + \Delta V_4(k) \leq \hat{\tau}x^T(k)Qx(k) - x^T(k-\tau(k))Qx(k-\tau(k)).$$

As a result, we have

$$\begin{aligned}\Delta V(k) &= \Delta V_1(k) + \Delta V_2(k) + \Delta V_3(k) + \Delta V_4(k) \\ &\leq y^T(k)M_1^T P_1 M_1 y(k) + y^T(k)M_2^T P_2 M_2 y(k) - x^T(k)P_2 x(k) + 2\hat{x}^T(k)P_2 x(k) \\ &\quad - \hat{x}^T(k)P_2 \hat{x}(k) + \hat{\tau}x^T(k)Qx(k) - x^T(k - \tau(k))Qx(k - \tau(k)).\end{aligned}\quad (5.10)$$

From (5.3) and (5.4), we have

$$(f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \leq 0, \quad j = 1, 2, \dots, n,$$

and

$$(g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) - v_j^- x_j(k - \tau(k))) \leq 0, \\ j = 1, 2, \dots, n.$$

Thus,

$$(f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \leq 0, \quad j = 1, 2, \dots, n, \quad (5.11)$$

and

$$(g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) - v_j^- x_j(k - \tau(k))) \leq 0, \\ j = 1, 2, \dots, n, \quad (5.12)$$

which are equivalent to

$$\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} l_j^+ l_j^- d_j d_j^T & -\frac{l_j^+ + l_j^-}{2} d_j d_j^T \\ -\frac{l_j^+ + l_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \leq 0, \quad j = 1, 2, \dots, n, \quad (5.13)$$

$$\begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} v_j^+ v_j^- d_j d_j^T & -\frac{v_j^+ + v_j^-}{2} d_j d_j^T \\ -\frac{v_j^+ + v_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \leq 0, \quad (5.14) \\ j = 1, 2, \dots, n,$$

where d_k denotes the unit column vector having “1” element on its k th row and zeros elsewhere. By (5.11), (5.12) and the S-procedure, $\Delta V(k) < 0$ if the following inequality holds,

$$\begin{aligned}\Delta V(k) &- 2 \sum_{j=1}^n t_j (f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) - 2 \sum_{j=1}^n s_j (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) - v_j^- x_j(k - \tau(k))) < 0.\end{aligned}\quad (5.15)$$

We will now show that, under assumptions of the theorem, (5.15) holds. From (5.13) and (5.14), we have

$$\begin{aligned}
\Delta V(k) &= 2 \sum_{j=1}^n t_j (f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \\
&\quad - 2 \sum_{j=1}^n s_j (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) - v_j^- x_j(k - \tau(k))) \\
&= \Delta V(k) - 2 \sum_{j=1}^n t_j \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} l_j^+ l_j^- d_j d_j^T & -\frac{l_j^+ + l_j^-}{2} d_j d_j^T \\ -\frac{l_j^+ + l_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\
&\quad - 2 \sum_{j=1}^n s_j \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} v_j^+ v_j^- d_j d_j^T & -\frac{v_j^+ + v_j^-}{2} d_j d_j^T \\ -\frac{v_j^+ + v_j^-}{2} d_j d_j^T & d_j d_j^T \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Delta V(k) &= 2 \sum_{j=1}^n t_j (f_j(x_j(k)) - l_j^+ x_j(k))(f_j(x_j(k)) - l_j^- x_j(k)) \\
&\quad - 2 \sum_{j=1}^n s_j (g_j(x_j(k - \tau(k))) - v_j^+ x_j(k - \tau(k)))(g_j(x_j(k - \tau(k))) - v_j^- x_j(k - \tau(k))) \\
&= \Delta V(k) - 2 \sum_{j=1}^n t_j \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\
&\quad - 2 \sum_{j=1}^n s_j \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}, \tag{5.16}
\end{aligned}$$

where $L_1 = \text{diag}(l_1^+ l_1^-, l_2^+ l_2^-, \dots, l_n^+ l_n^-)$, $L_2 = \text{diag}(-\frac{l_1^+ + l_1^-}{2}, -\frac{l_2^+ + l_2^-}{2}, \dots, -\frac{l_n^+ + l_n^-}{2})$, $V_1 = \text{diag}(v_1^+ v_1^-, v_2^+ v_2^-, \dots, v_n^+ v_n^-)$ and $V_2 = \text{diag}(-\frac{v_1^+ + v_1^-}{2}, -\frac{v_2^+ + v_2^-}{2}, \dots, -\frac{v_n^+ + v_n^-}{2})$. Therefore,

$$\begin{aligned}
\Delta V(k) &= 2 \sum_{j=1}^n t_j \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\
&\quad - 2 \sum_{j=1}^n s_j \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \tag{5.17} \\
&\leq y^T(k) [M_0 + M_1^T P_1 M_1 + M_2^T P_2 M_2] y(k).
\end{aligned}$$

By Lemma 2.4.15, it follows from (5.8) that $M_0 + M_1^T P_1 M_1 + M_2^T P_2 M_2 < 0$ which implies that (5.16) holds. Hence, the error system (5.6) with $\omega(k) = 0$ is asymptotically stable. This completes the proof of the theorem. $\square$

Now, we provide a sufficient condition of H_∞ filtering problem for (5.6).

Theorem 5.1.2 *Given the filter parameters A_F, B_F, C_F, D_F and scalar $\gamma > 0$. Then, the filtering error system (5.6) with time-varying delay $\tau(k)$ satisfying (5.2) is asymptotically stable if there exist $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q = Q^T > 0$, $T = \text{diag}\{t_1, t_2, \dots, t_n\} \geq 0$, $S = \text{diag}\{s_1, s_2, \dots, s_n\} \geq 0$ satisfying the following conditions :*

$$\bar{M} = \begin{bmatrix} \bar{M}_0 & \bar{M}_1^T P_1 & \bar{M}_2^T P_2 & \bar{M}_3^T \\ * & -P_1 & 0 & 0 \\ * & * & -P_2 & 0 \\ * & * & * & -I \end{bmatrix} < 0, \quad (5.18)$$

where

$$\bar{M}_0 = \begin{bmatrix} -P_2 - 2L_1T + \hat{\tau}Q & P_2 & 0 & -2L_2T & 0 & 0 \\ * & -P_2 & 0 & 0 & 0 & 0 \\ * & * & -Q - 2V_1S & 0 & -2V_2S & 0 \\ * & * & * & -2T & 0 & 0 \\ * & * & * & * & -2S & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\begin{aligned} \bar{M}_1 &= [A_1 \ 0 \ 0 \ A_2 \ A_3 \ B], \bar{M}_2 = [\bar{A}_1 \ -A_F \ 0 \ \bar{A}_2 \ \bar{A}_3 \ \bar{B}], \bar{M}_3 = [\bar{K}_1 \ -C_F \ 0 \ \bar{K}_2 \ \bar{K}_3 \ \bar{G}], \\ \bar{A}_1 &= A_1 - B_F C_1, \bar{A}_2 = A_2 - B_F C_2, \bar{A}_3 = A_3 - B_F C_3, \bar{B} = B - B_F D, \\ \bar{K}_1 &= K_1 - D_F C_1, \bar{K}_2 = K_2 - D_F C_2, \bar{K}_3 = K_3 - D_F C_3, \bar{G} = G - D_F D, \\ \hat{\tau} &= \tau_2 - \tau_1 + 1. \end{aligned}$$

Proof Consider the following Lypunov function candidate

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k),$$

where

$$V_1(k) = x^T(k)P_1x(k), \quad V_2(k) = \bar{x}^T(k)P_2\bar{x}(k), \quad V_3(k) = \sum_{l=k-\tau(k)}^{k-1} x^T(l)Qx(l),$$

$$V_4(k) = \sum_{j=k-\tau(k)}^{-\tau_1+1} \sum_{l=k+j-1}^{k-1} x^T(l)Qx(l).$$

One may show, with the same argument as in the proof of Theorem 5.1.1, that the Lyapunov difference along any trajectory of solution of (5.6) satisfies

$$\begin{aligned} \Delta V(k) &- 2 \sum_{j=1}^n t_j \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\ &- 2 \sum_{j=1}^n s_j \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \\ &\leq \bar{y}^T(k) \left[\bar{M}_0 + \bar{M}_1^T P_1 \bar{M}_1 + \bar{M}_2^T P_2 \bar{M}_2 \right] \bar{y}(k), \end{aligned}$$

where

$$M_0 = \begin{bmatrix} -P_2 - 2L_1T + \hat{\tau}Q & P_2 & 0 & -2L_2T & 0 & 0 \\ * & -P_2 & 0 & 0 & 0 & 0 \\ * & * & -Q - 2V_1S & 0 & -2V_2S & 0 \\ * & * & * & -2T & 0 & 0 \\ * & * & * & * & -2S & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\begin{aligned} \bar{M}_1 &= [A_1 \ 0 \ 0 \ A_2 \ A_3 \ B], \bar{M}_2 = [\bar{A}_1 \ -A_F \ 0 \ \bar{A}_2 \ \bar{A}_3 \ \bar{B}], \bar{A}_1 = A_1 - B_F C_1, \\ \bar{A}_2 &= A_2 - B_F C_2, \bar{A}_3 = A_3 - B_F C_3, \bar{B} = B - B_F D, \hat{\tau} = \tau_2 - \tau_1 + 1, \\ \bar{y}(k) &= [x^T(k) \ \hat{x}^T(k) \ x^T(k - \tau(k)) \ f^T(x(k)) \ g^T(x(k - \tau(k))) \ \omega(k)]^T. \end{aligned}$$

We are now ready to deal with the H_∞ performance of the filtering process.

We denote

$$J(n) := \sum_{k=0}^n [e^T(k)e(k) - \gamma^2 \omega^T(k)\omega(k)],$$

where n is any nonnegative integer. We will now show that $J(n) < 0$, for all $n \in \mathbb{Z}^+$. Under the zero initial condition, we get

$$\begin{aligned} J(n) &= \sum_{k=0}^n [e^T(k)e(k) - \gamma^2 \omega^T(k)\omega(k) + \Delta V(k)] - V(n+1) \\ &\leq \sum_{k=0}^n \left[[\bar{K}_1 x(k) - C_F \hat{x}(k) + \bar{K}_2 f(x(k)) + \bar{K}_3 g(x(k - \tau(k))) + \bar{G} \omega(k)]^T [\bar{K}_1 x(k) \right. \\ &\quad \left. - C_F \hat{x}(k) + \bar{K}_2 f(x(k)) + \bar{K}_3 g(x(k - \tau(k))) + \bar{G} \omega(k)] - \gamma^2 \omega^T(k)\omega(k) \right. \\ &\quad \left. + \bar{y}^T(k) [M_0 + \bar{M}_1^T P_1 \bar{M}_1 + \bar{M}_2^T P_2 \bar{M}_2] \bar{y}(k) \right] \\ &= \sum_{k=0}^n \left[\bar{y}^T(k) [\bar{M}_0 + \bar{M}_1^T P_1 \bar{M}_1 + \bar{M}_2^T P_2 \bar{M}_2 + \bar{M}_3^T \bar{M}_3^T] \bar{y}(k) \right], \end{aligned} \quad (5.19)$$

where

$$\bar{M}_0 = \begin{bmatrix} -P_2 - 2L_1T + \hat{\tau}Q & P_2 & 0 & -2L_2T & 0 & 0 \\ * & -P_2 & 0 & 0 & 0 & 0 \\ * & * & -Q - 2V_1S & 0 & -2V_2S & 0 \\ * & * & * & -2T & 0 & 0 \\ * & * & * & * & -2S & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\begin{aligned} \bar{M}_1 &= [A_1 \ 0 \ 0 \ A_2 \ A_3 \ B], \bar{M}_2 = [\bar{A}_1 \ -A_F \ 0 \ \bar{A}_2 \ \bar{A}_3 \ \bar{B}], \bar{M}_3 = [\bar{K}_1 \ -C_F \ 0 \ \bar{K}_2 \ \bar{K}_3 \ \bar{G}], \\ \bar{K}_1 &= K_1 - D_F C_1, \bar{K}_2 = K_2 - D_F C_2, \bar{K}_3 = K_3 - D_F C_3, \bar{G} = G - D_F D, \\ \bar{y}(k) &= [x^T(k) \ \hat{x}^T(k) \ x^T(k - \tau(k)) \ f^T(x(k)) \ g^T(x(k - \tau(k))) \ \omega(k)]^T. \end{aligned}$$

By Lemma 2.4.15 and (5.18), we obtain $\bar{M}_0 + \bar{M}_1^T P_1 \bar{M}_1 + \bar{M}_2^T P_2 \bar{M}_2 + \bar{M}_3^T \bar{M}_3 < 0$, which implies that $J(n) < 0$ from (5.21). Let $n \rightarrow \infty$, we obtain

$$\|e\|_2 \leq \gamma \|\omega\|_2.$$

This completes the proof of the theorem. $\square$

5.2 H_∞ filter design

In this section, we consider the H_∞ filter design problem which is based on Theorem 5.1.2.

Theorem 5.2.1 *Given the scalar $\gamma > 0$. Then, the nonlinear discrete-time neural network (5.1) an H_∞ filter (5.5) can be designed such that the filtering error system (5.6) with time-varying delay $\tau(k)$ satisfying (5.2) if there exist $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $Q = Q^T > 0$, $\bar{A}_F$, $\bar{B}_F$, $\bar{C}_F$, $\bar{D}_F$, $T = \text{diag}\{t_1, t_2, \dots, t_n\} \geq 0$, $S = \text{diag}\{s_1, s_2, \dots, s_n\} \geq 0$ satisfying the following conditions*

$$\bar{M} = \begin{bmatrix} \bar{M}_0 & \bar{M}_1^T P_1 & \bar{M}_2^T P_2 & \bar{M}_3^T \\ * & -P_1 & 0 & 0 \\ * & * & -P_2 & 0 \\ * & * & * & -I \end{bmatrix} < 0 \quad (5.20)$$

where $\bar{M}_0, \bar{M}_1, \bar{M}_2, \bar{M}_3$ are defined in Theorem 5.1.2. Moreover, the filter parameters of the form (5.5) is given as follows

$$A_F = -P_2^{-1} \bar{A}_F, \quad B_F = -P_2^{-1} \bar{B}_F, \quad C_F = -\bar{C}_F, \quad D_F = \bar{D}_F.$$

Proof As in Theorem 5.1.2, we consider the following Lypunov function candidate

$$V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k),$$

where

$$\begin{aligned} V_1(k) &= x^T(k) P_1 x(k), \quad V_2(k) = \bar{x}^T(k) P_2 \bar{x}(k), \quad V_3(k) = \sum_{l=k-\tau(k)}^{k-1} x^T(l) Q x(l), \\ V_4(k) &= \sum_{j=k-\tau(k)}^{-\tau_1+1} \sum_{l=k+j-1}^{k-1} x^T(l) Q x(l). \end{aligned}$$

One may show, with the same argument as in the proof of Theorem 5.1.1, that the Lyapunov difference along any trajectory of solution of (5.6) satisfies

$$\begin{aligned} \Delta V(k) &- 2 \sum_{j=1}^n t_j \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} L_1 & L_2 \\ L_2 & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \\ &- 2 \sum_{j=1}^n s_j \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix}^T \begin{bmatrix} V_1 & V_2 \\ V_2 & I \end{bmatrix} \begin{bmatrix} x(k - \tau(k)) \\ g(x(k - \tau(k))) \end{bmatrix} \\ &\leq \bar{y}^T(k) [\bar{M}_0 + \bar{M}_1^T P_1 \bar{M}_1 + \bar{M}_2^T P_2 \bar{M}_2] \bar{y}(k), \end{aligned}$$

where

$$\bar{M}_0 = \begin{bmatrix} -P_2 - 2L_1T + \hat{\tau}Q & P_2 & 0 & -2L_2T & 0 & 0 \\ * & -P_2 & 0 & 0 & 0 & 0 \\ * & * & -Q - 2V_1S & 0 & -2V_2S & 0 \\ * & * & * & -2T & 0 & 0 \\ * & * & * & * & -2S & 0 \\ * & * & * & * & * & 0 \end{bmatrix},$$

$$\begin{aligned} \bar{M}_1 &= [A_1 \ 0 \ 0 \ A_2 \ A_3 \ B], \bar{M}_2 = [\bar{A}_1 \ -A_F \ 0 \ \bar{A}_2 \ \bar{A}_3 \ \bar{B}], \bar{A}_1 = A_1 - B_F C_1, \\ \bar{A}_2 &= A_2 - B_F C_2, \bar{A}_3 = A_3 - B_F C_3, \bar{B} = B - B_F D, \hat{\tau} = \tau_2 - \tau_1 + 1, \\ \bar{y}(k) &= [x^T(k) \ \hat{x}^T(k) \ x^T(k - \tau(k)) \ f^T(x(k)) \ g^T(x(k - \tau(k))) \ \omega(k)]^T. \end{aligned}$$

We are now ready to deal with the H_∞ performance of the filtering process. Denote

$$J(n) := \sum_{k=0}^n [e^T(k)e(k) - \gamma^2 \omega^T(k)\omega(k)],$$

where n is any nonnegative integer. We will now show that $J(n) < 0$. Under the zero initial condition, we get

$$J(n) \leq \sum_{k=0}^n [\bar{y}^T(k) [\bar{M}_0 + \bar{M}_1^T P_1 \bar{M}_1 + \bar{M}_2^T P_2 \bar{M}_2 + \bar{M}_3^T \bar{M}_3] \bar{y}(k)], \quad (5.21)$$

where

$$\bar{M}_0 = \begin{bmatrix} -P_2 - 2L_1T + \hat{\tau}Q & P_2 & 0 & -2L_2T & 0 & 0 \\ * & -P_2 & 0 & 0 & 0 & 0 \\ * & * & -Q - 2V_1S & 0 & -2V_2S & 0 \\ * & * & * & -2T & 0 & 0 \\ * & * & * & * & -2S & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\begin{aligned} \bar{M}_1 &= [A_1 \ 0 \ 0 \ A_2 \ A_3 \ B], \bar{M}_2 = [\bar{A}_1 \ -A_F \ 0 \ \bar{A}_2 \ \bar{A}_3 \ \bar{B}], \bar{M}_3 = [\bar{K}_1 \ -C_F \ 0 \ \bar{K}_2 \ \bar{K}_3 \ \bar{G}], \\ \bar{K}_1 &= K_1 - D_F C_1, \bar{K}_2 = K_2 - D_F C_2, \bar{K}_3 = K_3 - D_F C_3, \bar{G} = G - D_F D, \\ \bar{y}(k) &= [x^T(k) \ \hat{x}^T(k) \ x^T(k - \tau(k)) \ f^T(x(k)) \ g^T(x(k - \tau(k))) \ \omega(k)]^T. \end{aligned}$$

As, the filter parameters of the form (5.5) is designed as follows

$$A_F = -P_2^{-1}\bar{A}_F, B_F = -P_2^{-1}\bar{B}_F, C_F = -\bar{C}_F, D_F = \bar{D}_F.$$

Thus, by Lemma 2.4.15 and (5.18), we obtain $\bar{M}_0 + \bar{M}_1^T P_1 \bar{M}_1 + \bar{M}_2^T P_2 \bar{M}_2 + \bar{M}_3^T \bar{M}_3 < 0$, which implies that $J(n) < 0$ from (5.21). Let $n \rightarrow \infty$, we obtain

$$\|e\|_2 \leq \gamma \|\omega\|_2.$$

This completes the proof of the theorem. $\square$

Example 5.2.2 Consider the nonlinear NNs (5.1) where

$$\begin{aligned} A_1 &= \begin{bmatrix} 0.1 & 0 & -0.1 \\ 0.1 & 0.4 & 0.1 \\ 0.1 & 0 & 0.4 \end{bmatrix}, A_2 = \begin{bmatrix} 0.1 & -0.1 & 0 \\ 0.1 & -0.25 & 0 \\ 0 & -0.2 & -0.1 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -0.2 & 0.1 & 0 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix}, C_1 = \begin{bmatrix} 0.2 & -0.4 & 0.4 \\ -0.2 & 0.2 & 0.1 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0.04 & -0.02 & 0.04 \\ 0.02 & 0.04 & 0.01 \end{bmatrix}, C_3 = \begin{bmatrix} -0.26 & 0.23 & -0.12 \\ -0.29 & 0.04 & -0.09 \end{bmatrix}, \\ K_1 &= \begin{bmatrix} 0.2 & -0.3 & 0 \\ -0.2 & 0 & 0.3 \end{bmatrix}, K_2 = \begin{bmatrix} -0.3 & 0 & 0.1 \\ 0.2 & -0.2 & 0 \end{bmatrix}, \\ K_3 &= \begin{bmatrix} 0.1 & -0.2 & -0.1 \\ 0.2 & 0.1 & 0.1 \end{bmatrix}, L_1 = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}, L_2 = \begin{bmatrix} -0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \\ V_1 &= \begin{bmatrix} -0.12 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, V_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}, B = \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \\ D &= \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}, G = \begin{bmatrix} 0.1 & -0.1 \\ 0 & 0.3 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \\ \gamma &= 1.3999, \tau(k) = 4 + \sin\left(\frac{k\pi}{2}\right), \tau_1 = 3, \tau_2 = 5, \hat{f}_1(s) = \tanh(0.6s) - 0.2 \sin s, \\ \hat{f}_2(s) &= \tanh(-0.4s), \hat{f}_3(s) = \tanh(-0.2s), \hat{g}_1(s) = \tanh(-0.4s) + 0.2 \sin s, \\ \hat{g}_2(s) &= \tanh(0.2s), \hat{g}_3(s) = \tanh(0.4s), \omega(k) = \frac{1}{k+1}. \end{aligned}$$

By using the Matlab LMI toolbox, we can solve for P_1, P_2, Q, S and T , which satisfy the conditions (5.18) in Theorem 5.1.2. Thus, the error system (5.6) is

asymptotically stable. A set of solutions of (5.6) are given below

$$P_1 = \begin{bmatrix} 2.8402 & -0.3187 & -0.0675 \\ -0.3187 & 2.1536 & 0.1289 \\ -0.0675 & 0.1289 & 2.2039 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.9960 & -0.3677 & -0.1774 \\ -0.3677 & 1.7160 & -0.2612 \\ -0.1774 & -0.2612 & 1.7863 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.4326 & -0.0158 & 0.0191 \\ -0.0158 & 1.3882 & -0.0826 \\ 0.0191 & -0.0826 & 1.2292 \end{bmatrix}, \quad S = \begin{bmatrix} 1.7245 & 0 & 0 \\ 0 & 1.7245 & 0 \\ 0 & 0 & 1.7245 \end{bmatrix},$$

$$T = \begin{bmatrix} 4.1849 & 0 & 0 \\ 0 & 4.1849 & 0 \\ 0 & 0 & 4.1849 \end{bmatrix}.$$

Then, the corresponding filter parameters are

$$A_F = \begin{bmatrix} -0.1156 & 0.1235 & -0.4062 \\ -0.1067 & 0.3007 & -0.2127 \\ -0.1206 & 0.2765 & -0.4026 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0.4130 & -0.3012 \\ -0.1202 & -0.3121 \\ -0.1303 & 0.3021 \end{bmatrix}$$

$$C_F = \begin{bmatrix} -0.1056 & -0.2104 & 0.3014 \\ -0.2026 & 0.3215 & -0.1905 \end{bmatrix}, \quad D_F = \begin{bmatrix} 0.2515 & -0.1051 \\ 0.3451 & 0.4062 \end{bmatrix},$$

$$\bar{A}_F = \begin{bmatrix} 0.1701 & -0.0869 & 0.6611 \\ 0.1091 & -0.3984 & 0.1105 \\ 0.1671 & -0.3935 & 0.5915 \end{bmatrix}, \quad \bar{B}_F = \begin{bmatrix} -0.8917 & 0.5400 \\ 0.3241 & 0.5037 \\ 0.2746 & -0.6746 \end{bmatrix},$$

$$\bar{C}_F = C_F, \quad \bar{D}_F = D_F,$$

and by solving the condition (5.6), the minimum value γ may be calculated as $\gamma_{\min} = 0.4565$ in Example 5.2.2 is feasible.

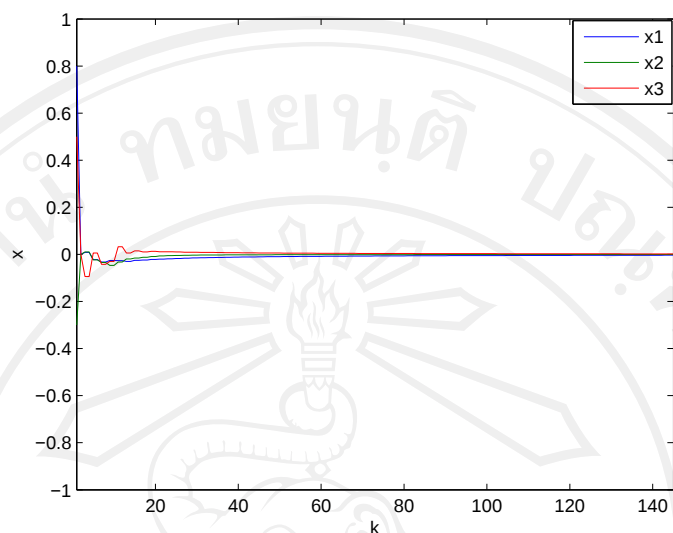


Figure 5.1: The trajectory of solutions $x(k)$ of system (5.1) in Example 5.2.2.

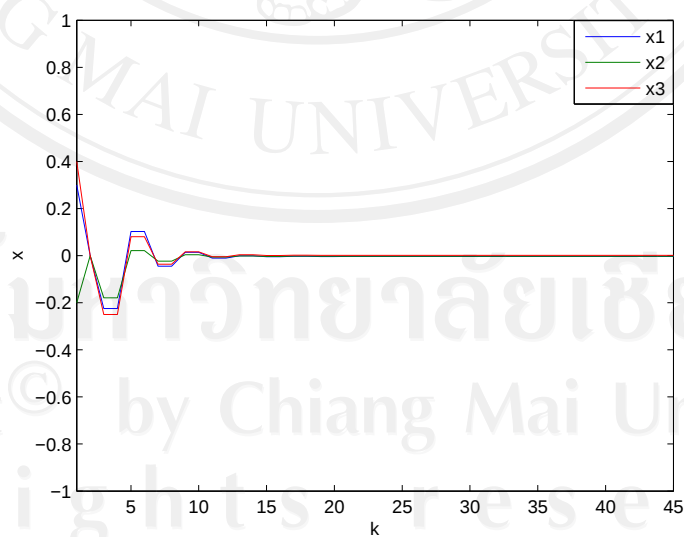


Figure 5.2: The trajectory of solutions $\hat{x}(k)$ of system (5.5) in Example 5.2.2.

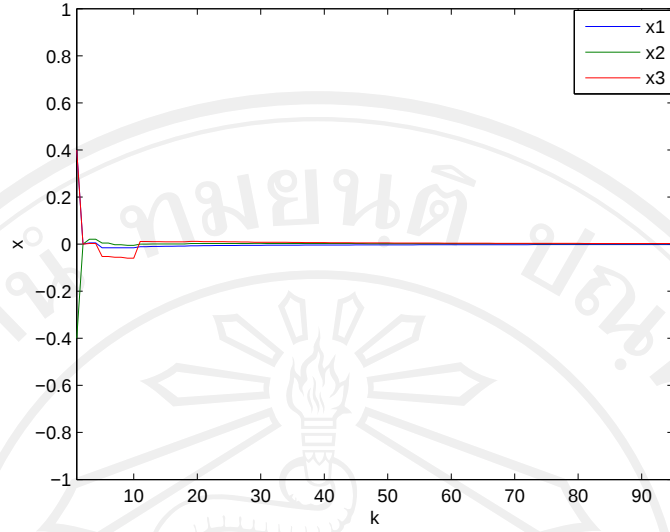


Figure 5.3: The trajectory of solutions $\bar{x}(k)$ of system (5.6) in Example 5.2.2.

Example 5.2.3 Consider the nonlinear NNs (5.1) where

$$A_1 = \begin{bmatrix} 0.5 & 0 & 0.1 \\ 0.1 & -0.4 & 0.1 \\ 0.1 & 0 & 0.4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0.2 & 0.1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0.8 & 0.7 \\ -0.6 & 0.9 & 0.6 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -0.1 & 0 & 0.1 \\ -0.1 & -0.1 & 0 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.8 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 0.7 \end{bmatrix},$$

$$L_2 = \begin{bmatrix} -0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -0.12 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$V_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}, \quad B = \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \quad D = \begin{bmatrix} 0.9 & -0.6 \\ 0.5 & 0.8 \end{bmatrix},$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \gamma = 0.8, \tau(k) = 4 + \sin\left(\frac{k\pi}{2}\right), \quad \tau_1 = 1, \quad \tau_2 = 2.$$

By using the Matlab LMI toolbox, we can solve for P_1 , P_2 , Q , S and T , which satisfy the conditions (5.18) in Theorem 5.1.2. Thus, the error system (5.6) is

asymptotically stable. A set of solutions of (5.6) are given below

$$P_1 = \begin{bmatrix} 0.9971 & -0.1660 & -0.0475 \\ -0.1660 & 1.2169 & -0.0709 \\ -0.0475 & -0.0709 & 1.1162 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.0179 & -0.1341 & -0.1319 \\ -0.1341 & 0.6369 & -0.2239 \\ -0.1319 & -0.2239 & 0.9074 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.6446 & 0.0169 & -0.0195 \\ 0.0169 & 0.7178 & 0.0159 \\ -0.0195 & 0.0159 & 0.6222 \end{bmatrix}, \quad S = \begin{bmatrix} 0.7035 & 0 & 0 \\ 0 & 0.7035 & 0 \\ 0 & 0 & 0.7035 \end{bmatrix},$$

$$T = \begin{bmatrix} 2.0623 & 0 & 0 \\ 0 & 2.0623 & 0 \\ 0 & 0 & 2.0623 \end{bmatrix}.$$

The corresponding filter parameters are

$$A_F = \begin{bmatrix} 0.2173 & -0.1319 & 0.0590 \\ -0.2965 & -0.2322 & 0.4101 \\ 0.1903 & -0.1228 & -0.1073 \end{bmatrix}, \quad B_F = \begin{bmatrix} 0.2057 & -0.1876 \\ 0.3118 & -0.4662 \\ 0.0516 & -0.2525 \end{bmatrix}.$$

Remark 5.2.4 We now compare our obtained results with those obtained in [21]. In our main results, we derived asymptotically stable conditions for discrete-time nonlinear system, our system is in the following form

$$\begin{cases} x(k+1) = A_1x(k) + A_2f(x(k)) + A_3g(x(k-\tau(k))) + B\omega(k) \\ y(k) = C_1x(k) + D\omega(k) \\ z(k) = K_1x(k) \\ x(k) = \phi(k), \quad k = -\tau_2, \tau_2 + 1, \dots, 0, \end{cases} \quad (5.22)$$

We design a full-order filter for system (5.22) as follows:

$$\begin{aligned} \hat{x}(k+1) &= A_F\hat{x}(k) + B_Fy(k), \quad \hat{x}(0) = 0 \\ \hat{z}(k) &= K_1\hat{x}(k). \end{aligned} \quad (5.23)$$

Then, we obtained the filter error dynamics as

$$\begin{cases} \bar{x}(k+1) = \bar{A}_1x(k) - A_F\hat{x}(k) + \bar{A}_2f(x(k)) + \bar{A}_3g(x(k-\tau(k))) + \bar{B}\omega(k) \\ e(k) = K_1x(k) - K_1\hat{x}(k) \\ \bar{x}(k) = \phi(k), \quad k = -\tau_2, \tau_2 + 1, \dots, 0, \end{cases} \quad (5.24)$$

where $\bar{A}_1 = A_1 - B_FC_1$, $\bar{A}_2 = A_2 - B_FC_2$, $\bar{A}_3 = A_3 - B_FC_3$, $\bar{B} = B - B_FD$.

If we consider system without stochastic disturbance and without uncertainties, the system considered in [21] and in this thesis are the same. Moreover, in Example 5.2.3, if we let $\tau_1 = 1$ and $\tau_2 = 2$, then by using our main results (Theorem 5.1.2), the minimum value γ may be calculated as $\gamma_{\min} = 0.1221$. However, by using LMI in the main results of [21], the H_∞ filter in Example 5.2.3 is infeasible.