Chapter 1
Introduction

This chapter is organized as follows: Section 1.1 presents elementary concepts, notations and a brief history in fixed point theory of nonexpansive mappings. Section 1.2 is devoted to definition, properties and some known researches concerning strict pseudocontractions. In Section 1.3 we give definition and properties of relatively quasi-nonexpansive mappings in Banach spaces. In Section 1.4 we recall some well-known results and applications of equilibrium problems.

1.1 The Background of Fixed Point Theory and Nonexpansive Mappings

Let X be a nonempty set and let $T : X \to X$ be a nonlinear mapping. We say that $x \in X$ is a fixed point of T if

$$Tx = x$$

and denote by $F(T)$ the fixed points set of T.

Example 1.1.1. (1) If $X = \mathbb{R}$ and $Tx = x^2 - 5x + 9$, then $F(T) = \{3\}$;

(2) If $X = \mathbb{R}$ and $Tx = x^2 - 2$, then $F(T) = \{-1, 2\}$;

(3) If $X = \mathbb{R}$ and $Tx = x + 5$, then $F(T) = \emptyset$;

(4) If $X = \mathbb{R}$ and $Tx = x$, then $F(T) = \mathbb{R}$.

Fixed point theory plays an important role in nonlinear analysis. This is because many practical problems in applied science, economics, physics and engineering can be reformulated as a problem of finding fixed points of nonlinear mappings.

The study of fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator $T : X \to X$, in order to obtain results on:

- the existence and the uniqueness of fixed points;
- the structure of fixed point sets;
- the approximation of fixed points.

Let X be a nonempty set and let $T : X \to X$ be a nonlinear mapping. For any given $x \in X$, we define $T^n x$ inductively by $T^0 x = x$ and $T^{n+1}x = TT^n x$; we call $T^n x$ the *iterate of x under T*. The mapping $T^n (n \geq 1)$ is called the n^{th} *iterate* of T. For any $x_0 \in X$, the sequence $\{x_n\}$ given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \cdots$$

is called *the sequence of successive approximations with the initial value* x_0. It is known as the *Picard iteration* starting at x_0.

Iteration procedures are used in nearly every branch of applied mathematics, and convergence proofs and error estimates are very often obtained by an application of Banach fixed point theorem (or more difficult fixed point theorems).

In 1922, Banach proved the following famous theorem in fixed point theory for a contraction.

Theorem 1.1.2. (**The Banach contraction principle**) *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a contraction, that is, there exists* $\alpha \in [0, 1)$ *such that, for all* $x, y \in X$,

$$d(Tx, Ty) \le \alpha d(x, y).$$

Then T *has a unique fixed point. Moreover, for each* $x \in X$, *the sequence* $\{T^n x\}$ *converges strongly to this fixed point.*

Let C be a nonempty subset of a Banach space X and let $T : C \to C$ be a nonlinear mapping. Then T is called *nonexpansive* if, for all $x, y \in C$,

$$\|Tx - Ty\| \le \|x - y\|.$$

Many researchers concentrate in obtaining (additional) condition on T and X as general as possible, and which should guarantee the (strong) convergence of the Picard iteration to a fixed point of T. Moreover, if the Picard iteration converges to a fixed point of T, they will be interested in evaluating the error estimate (or alternatively, the rate of convergence) of the method, that is, in obtaining a stopping criterion for the sequence of successive approximations. However, the Picard iteration may not converge even in the weak topology.

To overcome this difficulty, in 1953, Mann [60] introduced the iteration as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \ge 0, \tag{1.1.1}$$

where $\{\alpha_n\} \subset (0, 1)$. If T is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n (1 - \alpha_n) = +\infty$, then the sequence $\{x_n\}$ defined by (1.1.1) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [87]). See also [10, 38].

Due to the weak convergence of Mann's iteration, in 1967, Halpern [43] introduced the modified Mann's iteration (1.1.1) as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad \forall n \ge 0, \tag{1.1.2}$$

where $\{\alpha_n\} \subset (0, 1)$ and $u \in C$. He proved, in a real Hilbert space, the convergence of $\{x_n\}$ to a fixed point of T where $\alpha_n := n^{-a}, \ a \in (0, 1)$.

In 1977, Lions [58] obtained a strong convergence provide the real sequence $\{\alpha_n\}$ satisfies the following conditions:

L1: $\lim_{n\to\infty} \alpha_n = 0$;

L2: $\sum_{n=0}^{\infty} \alpha_n = +\infty$;
L3: $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0$.

Reich [92] also extended the result of Halpern from Hilbert spaces to uniformly smooth Banach spaces. However, both Halpern's and Lions' conditions imposed on the real sequence $\{\alpha_n\}$ exclude the canonical choice $\alpha_n = \frac{1}{n+1}$.

In 1992, Wittmann [111] proved that the sequence $\{x_n\}$ converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies the following conditions:

W1: $\lim_{n\to\infty} \alpha_n = 0$;
W2: $\sum_{n=0}^{\infty} \alpha_n = +\infty$;
W3: $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$.

Shioji-Takahshi [95] extended Wittmann's result to real Banach spaces with uniformly Gâteaux differentiable norms and in which each nonempty, closed, convex and bounded subset has the fixed point property for nonexpansive mappings. The concept of Halpern's iteration has been widely used to approximate the fixed points of nonexpansive mappings (see, for examples, [6, 32, 51, 91, 112, 113]).

In 2003, Nakajo-Takahashi [71] (see also [9]) introduced another modified Mann's iteration for a nonexpansive mapping T in a Hilbert space H as follows: $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad n \ge 1, \end{cases} \tag{1.1.3}$$

where $\{\alpha_n\} \subset [0,1]$ and P_K is a metric projection from H into a nonempty, closed and convex subset K. Such an algorithm is called the *CQ method*. They proved that the sequence $\{x_n\}$ generated by (1.1.3) converges strongly to a fixed point of T if $\limsup_{n\to\infty} \alpha_n < 1$.

Recently, Takahashi et al. [104] introduced a new modification of Mann's iteration called the *shrinking projection method* for a nonexpansive mapping T in a Hilbert space H as follows: for $x_0 \in H$, $C_1 = C$ and $x_1 = P_{C_1} x_0$, define the sequence $\{x_n\}$ by

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad n \ge 1, \end{cases} \tag{1.1.4}$$

where $\{\alpha_n\} \subset [0,1]$. They proved that the sequence $\{x_n\}$ generated by (1.1.4) converges strongly to a fixed point of T if $\limsup_{n\to\infty} \alpha_n < 1$.

The CQ method and the shrinking projection method are now become a hot topics in nonlinear analysis. This is because one can get strong convergence of the generated sequence from its construction.

On the other hand, the problem of finding common fixed points is now has been extensively studied by mathematicians. To deal with a fixed point problem of a family of nonlinear mappings, there have been several ways appeared in the literature.

Let C be a nonempty, closed and convex of a Banach space X. Let $\{T_i\}_{i=1}^N : C \to C$ be a finite family of nonexpansive mappings.

In 1973, Bruck [22] investigated the mapping defined via the convex combination of a sequence of nonexpansive mappings $\{T_i\}_{i=1}^N$ as follows:

$$V = \sum_{i=1}^{N} \beta_i T_i,$$

where $\{\beta_i\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N \beta_i = 1$. If X is strictly convex, then $F(V) = \bigcap_{i=1}^N F(T_i)$.

In 1999, Atsushiba-Takahashi [8] defined the mapping W as follows:

$$\begin{aligned}
U_1 &= \beta_1 T_1 + (1-\beta_1)I, \\
U_2 &= \beta_2 T_2 U_1 + (1-\beta_2)I, \\
U_3 &= \beta_3 T_3 U_2 + (1-\beta_3)I, \\
&\vdots \\
U_{N-1} &= \beta_{N-1} T_{N-1} U_{N-2} + (1-\beta_{N-1})I, \\
W &= U_N = \beta_N T_N U_{N-1} + (1-\beta_N)I,
\end{aligned}$$

where $\{\beta_i\}_{i=1}^N \subset (0,1)$. This mapping is called the $W-$*mapping* generated by $T_1, T_2, \cdots, T_N$ and $\beta_1, \beta_2, \cdots, \beta_N$. In 2000 Takahashi-Shimoji [107] proved that if X is a strictly convex Banach space then $F(W) = \bigcap_{i=1}^N F(T_i)$.

Recently, Kangtunyakarn-Suantai [49] introduced another mapping as follows:

$$\begin{aligned}
U_1 &= \beta_1 T_1 + (1-\beta_1)I, \\
U_2 &= \beta_2 T_2 U_1 + (1-\beta_2)U_1, \\
U_3 &= \beta_3 T_3 U_2 + (1-\beta_3)U_2, \\
&\vdots \\
U_{N-1} &= \beta_{N-1} T_{N-1} U_{N-2} + (1-\beta_{N-1})U_{N-2}, \\
K &= U_N = \beta_N T_N U_{N-1} + (1-\beta_N)U_{N-1},
\end{aligned}$$

where $\{\beta_i\}_{i=1}^N \subset (0,1)$. This mapping is called the K-*mapping* generated by $T_1, T_2, \cdots, T_N$ and $\beta_1, \beta_2, \cdots, \beta_N$. Also, $F(K) = \bigcap_{i=1}^N F(T_i)$ in a strictly convex Banach space.

Following the idea of Bruck [22], in recent years, there have been considerable interests in the study of a common fixed point problem for some certain classes of nonlinear mappings (see [7, 8, 49, 50, 107]).

1.2 Fixed Points of Strict Pseudocontractions

We begin this section by recalling definition and properties of strict pseudocontractions.

Definition 1.2.1. A mapping T with domain $D(T)$ and range $R(T)$ in X is called λ-*strictly pseudocontractive* [75] if for all $x, y \in D(T)$, there exist $0 < \lambda < 1$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2$$

equivalently

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \lambda\|(I-T)x - (I-T)y\|^2.$$

Remark 1.2.2. Every λ-strict pseudocontraction is $\frac{\lambda+1}{\lambda}$-Lipschitzian [30].

Remark 1.2.3. Let C be a nonempty subset of a Hilbert space and let $T : C \to C$ be a mapping. Then T is called *κ-strictly pseudocontractive* [19], if for all $x, y \in D(T)$, there exists $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2. \tag{1.2.1}$$

It is known that (1.2.1) is equivalent to the following

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1 - \kappa}{2}\|(I - T)x - (I - T)y\|^2.$$

The following examples are strict pseudocontractions but not nonexpansive.

Example 1.2.4. [120] Let $C = (0, \infty)$ and define $T : C \to C$ by

$$Tx = \frac{x^2}{1 + x}.$$

Example 1.2.5. [34] Let $C = [-1, 2]$ and define $T : C \to \mathbb{R}$ by

$$Tx = \begin{cases} x, & x \in [-1, 0), \\ x - 4x^2, & x \in [0, 2). \end{cases}$$

In 1967, Browder-Petryshyn first introduced in [19] the class of strict pseudocontractions and proved the existence and the weak convergence in a Hilbert space of Mann's iteration (1.1.1) with a constant sequence $\alpha_n = \alpha$ for all $n \geq 0$.

In 2006, Marino-Xu [61] proved that the sequence generated by Mann's iteration weakly converges to a fixed point of a κ-strict pseudocontraction on a closed and convex subset of a Hilbert space if the control sequence $\{\alpha_n\}$ is chosen such that $\kappa < \alpha_n < 1$ and $\sum_{n=0}^{\infty}(\alpha_n - \kappa)(1 - \alpha_n) = +\infty$. Zhou [120] extended, still in Hilbert spaces, the results of Browder-Petryshyn [19] to Mann's iteration. Zhou [121] also investigated weak and strong convergence in 2-uniformly smooth Banach spaces. In a much more general setting, Zhang-Guo [118] and Zhang-Su [119] studied the convergence in q-uniformly smooth Banach spaces. In 2010, Chidume-Shahzad [34] also studied the weak convergence of Mann's iteration in a uniformly smooth Banach space. However, it should be noted that Mann's iteration fails to converge for a class of Lipschitz pseudocontractions in Hilbert spaces [33]. Since 1967, the constructions of fixed points for pseudocontractions via iteration processes have been extensively studied by many authors (see, for examples, [1, 26, 27, 28, 75, 82]).

Very recently, Zhou [122] proved the following theorem in the framework of q-uniformly smooth Banach spaces.

Theorem 1.2.6. *Let C be a nonempty, closed and convex subset of a real q uniformly smooth Banach space X that either is uniformly convex or satisfies Opial's condition. Let $T : C \to C$ be a λ-strict pseudocontraction for some $0 < \lambda < 1$. Let $\mu = \min\{1, \left(\frac{q\lambda}{D}\right)^{\frac{1}{q-1}}\}$ for some $D > 0$. Let $\{x_n\}$ be the sequence generated by the*

Mann's iteration algorithm (1.1.1). *Assume that the control sequence* $\{\alpha_n\}$ *is chosen such that* $\alpha_n \in [0, \mu]$ *for all* $n \geq 0$ *and*

$$\sum_{n=0}^{\infty} \left(\mu^{q-1} - \alpha_n^{q-1}\right)\alpha_n = +\infty.$$

Then the following statements are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{x_n\}$ *is bounded;*
(c) $\{x_n\}$ *converges weakly to a fixed point of* T.

Further, in order to get strong convergence results, the modification of Mann's iteration is introduced. To be more precise, he proved the following theorem.

Theorem 1.2.7. *Let* C *be a closed and convex subset of a real* q*-uniformly smooth Banach space* X *and let* $T : C \to C$ *be a* λ*-strict pseudocontraction such that* $F(T) \neq \emptyset$. *Let* $\mu = \min\{1, \left(\frac{q\lambda}{D}\right)^{\frac{1}{q-1}}\}$ *for some* $D > 0$. *Given* $u, x_0 \in C$ *and sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *in* $(0, 1)$, *the following control conditions are satisfied:*

(a) $a \leq \alpha_n \leq \mu$ *for some* $a > 0$ *and for all* $n \geq 0$;
(b) $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 0$;
(c) $\beta_n \to 0$ *as* $n \to \infty$, $\sum_{n=0}^{\infty} \beta_n = +\infty$;
(d) $\alpha_{n+1} - \alpha_n \to 0$ *as* $n \to \infty$;
(e) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Let the sequence $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \quad n \geq 0, \end{cases}$$

Then $\{x_n\}$ *converges strongly to a fixed point* z *of* T, *where* $z = Q_{F(T)}(u)$ *and* $Q_{F(T)} : C \to F(T)$ *is the unique sunny nonexpansive retraction from* C *onto* $F(T)$.

Meanwhile, in the end of his article, he put an interesting question as follows:

Can one extend the main results of this paper to both uniformly smooth and uniformly convex Banach spaces?

In recent years, there have been several types of explicit algorithms for a class of strict pseudocontractions proposed and studied in Hilbert spaces, 2-uniformly smooth Banach spaces and q-uniformly smooth Banach spaces (see [118, 119, 120, 121]).

1.3 Fixed Points of Relatively Quasi-nonexpansive Mappings

We begin this section by recalling definition and properties of relatively quasi-nonexpansive mappings.

Let C be a nonempty subset of a smooth Banach space X and define the *Lyapunov* function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in X$.

Let $T : C \to C$ be a nonlinear mapping. A point p in C is said to be an *asymptotic fixed point* of T [29, 86] if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

We say that the mapping T is *relatively nonexpansive* [62, 64] if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$;
(R2) $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;
(R3) $F(T) = \widehat{F}(T)$.

The mapping T will be called *relatively quasi-nonexpansive* if it satisfies (R1) and (R2).

The following examples are relatively quasi-nonexpansive mappings.

Example 1.3.1. [64] Let X be a uniformly smooth and strictly convex Banach space and $A \subset X \times X^*$ be maximal monotone such that $A^{-1}(0) \neq \emptyset$. Then $J_r = (J + rA)^{-1}J$ is a relatively quasi-nonexpansive mapping from X onto $D(A)$ and $F(J_r) = A^{-1}(0)$.

Example 1.3.2. [3, 48] Let Π_C be the generalized projection from a smooth, strictly convex and reflexive Banach space X onto a nonempty, closed and convex subset C of X. Then Π_C is a relatively quasi-nonexpansive mapping and $F(\Pi_C) = C$.

Example 1.3.3. [105] Let C be a nonempty, closed and convex subset of a smooth, reflexive and strictly convex Banach space X. Let f be a bifunction from $C \times C \to \mathbb{R}$. For each $r > 0$, define the mapping $T_r : X \to C$ as in Lemma 2.8 of [105]. Then T_r is a relatively quasi-nonexpansive mapping and $F(T_r) = EP(f)$ (see Lemma 2.9 of [105]).

Example 1.3.4. [117] Let C be a nonempty, closed and convex subset of a smooth, reflexive and strictly convex Banach space X. Let f be a bifunction from $C \times C \to \mathbb{R}$, $A : C \to X^*$ a continuous and monotone mapping, and $\varphi : C \to \mathbb{R}$ a convex and lower semi-continuous function. For each $r > 0$, define the mapping $S_r : X \to C$ as in Lemma 1.5 of [117]. Then S_r is a relatively quasi-nonexpansive mapping and $F(S_r) = GMEP(f, A, \varphi)$.

Matsushita-Takahashi [62] introduced the modified Mann's iteration for a relatively nonexpansive T as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \Pi_C J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTx_n\big) \tag{1.3.1}$$

where $\{\alpha_n\} \subset (0,1)$, Π_C is a generalized projection of X onto C and J is the normalized duality mapping on X. They proved a weak convergence of the sequence $\{x_n\}$ generated by (1.3.1).

In 2005, Matsushita-Takahashi [64] extended the result of Nakajo-Takahashi [71] for a relatively nonexpansive mapping T in uniformly smooth and uniformly convex

Banach spaces and defined the following iteration:

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTx_n\big), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad n \ge 0, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

In 2007, Plubtieng-Ungchittrakool [83] introduced new general processes of two relatively nonexpansive mappings S and T in uniformly smooth and uniformly convex Banach spaces as follows:

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTz_n\big), \\ z_n = J^{-1}\big(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n\big), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad n \ge 0, \end{cases}$$

and

$$\begin{cases} x_0 = x \in C \quad \textit{chosen arbitrarily}, \\ y_n = J^{-1}\big(\alpha_n Jx_0 + (1-\alpha_n)JTx_n\big), \\ z_n = J^{-1}\big(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n\big), \\ C_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx\rangle)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad n \ge 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$ and $\{\beta_n^{(3)}\}$ are sequences in $[0,1]$ with $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$.

Su et al. [97] proposed the following monotone hybrid algorithm for a relatively quasi-nonexpansive mapping T in uniformly smooth and uniformly convex Banach spaces:

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTx_n\big), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\}, \\ C_0 = \{z \in C : \phi(z, y_0) \le \phi(z, x_0)\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \ge 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x, \quad n \ge 0, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$.

Qin et al. [85] introduced the following shrinking projection method for a relatively quasi-nonexpansive mapping T in a uniformly smooth and uniformly convex Banach space:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \quad x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}\big(\alpha_n Jx_1 + (1-\alpha_n)JTx_n\big), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \ge 1, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$.

The problem of finding fixed points of relatively nonexpansive mappings and relatively quasi-nonexpansive mappings has been investigated by many authors (see, for examples, [5, 7, 16, 53, 54, 62, 93, 110]).

1.4 Equilibrium Problems

In 1972, Ky Fan [37] established the famous Ky Fan inequality in a Hausdorff topological vector space. It is well-known that Fan's minimax inequalities have played very important roles in the study of modern nonlinear analysis. By a powerful tool of this inequality, since 1972, a number of generalized versions are continuously established in the literature (see, for instance, [4, 24, 59, 109]). In this connection, utilizing Ky Fan inequality, Blum-Oettli [13] proved the existence under the restriction on a bifunction of the equilibrium model which closely relates to optimal theory, fixed point theory, saddle points, variational inequalities, complementarity problems, Nash equilibrium in game theory and so on (see [23, 39, 40, 68, 76, 96]).

Let C be a closed and convex subset of a Banach space X and let $f : C\times C \to \mathbb{R}$ be a bifunction which satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f\big(tz + (1-t)x, y\big) \leq f(x, y)$;
(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semi-continuous.

We next give examples of bifunctions which satisfy the conditions (A1)-(A4).

Example 1.4.1. Let $X = \mathbb{R}$ and $C = [-1, 1]$. Define $f : C \times C \to \mathbb{R}$ by

$$f(x, y) = -9x^2 + xy + 8y^2.$$

Then f is a bifunction satisfying (A1)-(A4).

Example 1.4.2. Let $X = \mathbb{R}$ and $C = [-5, 5]$. Define $f : C \times C \to \mathbb{R}$ by

$$f(x, y) = -4x^2 + 4y^2.$$

Then f is a bifunction satisfying (A1)-(A4).

Blum-Oettli [13] proved the following existence result:

Lemma 1.4.3. [13] *Let C be a closed and convex subset of a smooth, strictly convex, and reflexive Banach space X, let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions* (A1)-(A4)*, and let $r > 0$ and $x \in X$. Then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\big\langle Jz - Jx, y - z\big\rangle \geq 0, \quad \forall y \in C.$$

It is known that the above inequality relates to the equilibrium problem which is the problem of finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \quad \forall y \in C. \tag{1.4.1}$$

The solutions set of (1.4.1) is denoted by $EP(f)$.

Employing the ideas of Blum-Oettli [13] and Combettes-Hirstoaga [36], Takahashi-Zembayashi first gave in [105] some nice properties of the auxiliary operator of the equilibrium problem via fixed point theory. They also studied, in uniformly smooth and uniformly convex Banach spaces, the strong convergence of the sequence generated by the following algorithm:

$$\begin{cases} x_0 = x \in C, \quad C_0 = C, \\ y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JTx_n\big), \\ u_n \in C \ \ \textit{such that} \ \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \ \ \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and T is a relatively nonexpansive mapping on C. The results obtained by Takahashi-Zembayashi [105] mainly improve and extend those of Tada-Takahashi [99] from Hilbert spaces to Banach spaces.

As a generalization, the generalized mixed equilibrium problem is the problem of finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle + \varphi(y) \geq \varphi(\hat{x}), \ \forall y \in C \tag{1.4.2}$$

where φ is a real-valued function on C and A is a mapping from C to X^*, the dual of X. The solutions set of (1.4.2) is denoted by $GMEP(f, A, \varphi)$ (see Peng-Yao [79]).

If $A \equiv 0$, then the generalized mixed equilibrium problem (1.4.2) reduces to the following mixed equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \varphi(y) \geq \varphi(\hat{x}), \ \forall y \in C. \tag{1.4.3}$$

The solutions set of (1.4.3) is denoted by $MEP(f, \varphi)$ (see Ceng-Yao [25]).

If $f \equiv 0$, then the generalized mixed equilibrium problem (1.4.2) reduces to the following mixed variational inequality problem: finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x}\rangle + \varphi(y) \geq \varphi(\hat{x}), \ \forall y \in C. \tag{1.4.4}$$

The solutions set of (1.4.4) is denoted by $VI(C, A, \varphi)$ (see Noor [73]).

If $\varphi \equiv 0$, then the generalized mixed equilibrium problem (1.4.2) reduces to the following generalized equilibrium problem: finding $\hat{x} \in C$ such that

$$f(\hat{x}, y) + \langle A\hat{x}, y - \hat{x}\rangle \geq 0, \ \forall y \in C. \tag{1.4.5}$$

The solutions set of (1.4.5) is denoted by $GEP(f, A)$ (see Moudafi [67] and Takahashi-Takahashi [108]).

If $f \equiv 0$, then the mixed equilibrium problem (1.4.3) reduces to the following convex minimization problem: finding $\hat{x} \in C$ such that

$$\varphi(y) \geq \varphi(\hat{x}), \ \forall y \in C. \tag{1.4.6}$$

The solutions set of (1.4.6) is denoted by $CMP(\varphi)$.

If $\varphi \equiv 0$, then the mixed variational inequality problem (1.4.4) reduces to the following variational inequality problem: finding $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x} \rangle \geq 0, \ \forall y \in C. \tag{1.4.7}$$

The solutions set of (1.4.7) is denoted by $VI(C, A)$ (see Stampacchia [96]).

The problem of finding solutions of the equilibrium problem under constraint the fixed point problem of nonlinear mappings has been studied by many authors (see, for instance, [28, 35, 46, 47, 50, 57, 67, 77, 78, 79, 80, 81, 82]).

The purpose of this thesis are three folds. Firstly, we construct and study new methods for a common fixed point of a countable family of nonexpansive mappings, quasi-nonexpansive mappings, strict pseudocontractions and relatively quasi-nonexpansive mappings. Secondly, we construct and study new methods for finding solutions of equilibrium problems, variational inequality problems and mixed equilibrium problems. Finally, we find sufficient conditions for weak and strong convergence theorems of the iterative methods defined in the first and the second purposes.

This thesis is divided into 6 chapters. Chapter 1 is an introduction of this thesis. Chapter 2 is devoted to preliminaries, lemmas and propositions which will be used in this thesis. Chapter 3-Chapter 5 are the main results of this thesis and the conclusion is in Chapter 6. To be more precise, Chapter 3 is organized as follows: in Section 3.1, we investigate the convergence of Mann-type iterative scheme for a countable family of strict pseudocontractions in a uniformly convex Banach space with the Fréchet differentiable norm. In Section 3.2, we give some affirmative answer raised by Zhou [122]. We study the weak convergence of Mann-type iteration for a countable family of strict pseudocontractions in a uniformly smooth Banach space which is uniformly convex or satisfies Opial's condition. The strong convergence theorems are also established in this section.

Chapter 4 is organized as follows: in Section 4.1, employing idea of Takahashi [8], we first investigate the W-mapping generated by a sequence of Lipschitz and quasi-nonexpansive mappings. We construct hybrid algorithms for solving the fixed point problem, the variational inclusion, and the generalized equilibrium problem. In Section 4.2, we present hybrid algorithms for solving a system of generalized equilibrium problems under the constraint fixed point problem of strict pseudocontractions. The strong convergence is also discussed in this section. In Section 4.3, using the KKM mapping, we prove the existence of solutions of the mixed equilibrium problem (MEP) in the framework of Banach spaces. Then, by virtue of this result, we construct hybrid algorithms to solve the mixed equilibrium problem and the fixed point problem of nonexpansive mappings.

Finally, Chapter 5 is organized as follows: in Section 5.1, we discuss strong convergence for a system of equilibrium problems and common fixed points set of relatively quasi-nonexpansive mappings in a Banach space. In Section 5.2, we introduce hybrid methods for finding common elements in the solutions set of the equilibrium problem and the common fixed points set of a countable family of relatively quasi-nonexpansive mappings in a Banach space. The Mosco convergence ensures the strong convergence when the sequence of mappings satisfies the $(*)$-condition. The examples of mappings which satisfy the $(*)$-condition are also shown.