Chapter 2
Basic Concepts and Preliminaries

The purpose of this chapter is to collect notations, terminologies and elementary results used throughout the thesis.

2.1 Metric Spaces

Definition 2.1.1. [56] A *metric space* is a pair (X, d), where X is a set and d is a metric on X(or distance function on X), that is, a real- valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$;

(2) $d(x, y) = 0$ if and only if $x = y$;

(3) $d(x, y) = d(y, x)$ (symmetry);

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Example 2.1.2. [12]

(1) $X = \mathbb{R}$; $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$ is a metric on $\mathbb{R}$;

(2) $X = \mathbb{R}^n$; $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2}$, $x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$, is a metric on $\mathbb{R}^n$ (euclidian metric). The following mappings:

$$\delta(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

$$\rho(x, y) = \max_{1 \leq 1 \leq n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

are also metrics on $\mathbb{R}^n$;

(3) Let $X = \{f : [a, b] \to \mathbb{R} : f \text{ is continuous}\}$. We define $d : X \times X \to \mathbb{R}^+$ by

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|, \quad \text{for all } f, g \in X.$$

Then d is a metric on X (Chebyshev metric); the metric space (X, d) is denoted by $C[a, b]$;

(4) Let X be as (3) and let $\delta : X \times X \to \mathbb{R}^+$ be defined by

$$\delta(f, g) = \max_{x \in [a,b]} (|f(x) - g(x)|e^{-\tau|x - x_0|}),$$

for all $f, g \in X$ where $\tau > 0$ is a constant and x_0 is fixed in $[a, b]$.

Then δ is a metric on X (Bielecki metric), and the metric space (X, δ) is denoted by $B[a, b]$.

Definition 2.1.3. [12] Let (X, d) be a metric space. The topology having as basis the family of all open balls, $B(x, r)$, $x \in X$, $r > 0$, is called the *topology induced by the metric d.*

Definition 2.1.4. [12] Two metrics d_1 and d_2 defined on the set X are called *equivalent* if they induce the same topology on X.

Remark 2.1.5. [12]

(1) Two metrics d_1 and d_2 are metrically equivalent if there exist two constants $m > 0$ and $M > 0$ such that

$$md_1(x, y) \leq d_2(x, y) \leq Md_1(x, y), \quad \text{for all } x, y \in X.$$

(2) In Example 2.1.2, the metrics d, δ and ρ from (2) are equivalent; the metrics d from (3) and ρ from (4) are equivalent as well.

Definition 2.1.6. [56] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *convergent* if there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.

Definition 2.1.7. [56] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *Cauchy* if for every $\epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \geq N(\epsilon)$.

Definition 2.1.8. [56] A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Theorem 2.1.9. [56] *Every convergent sequence in a metric space is a Cauchy sequence.*

Theorem 2.1.10. [65] *Let $\{x_n\}$ be a sequence in $\mathbb{R}$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ is convergent.*

Definition 2.1.11. [65] Let (X, d) be a metric space. A subset F of X is *closed* if $\{x_n\} \subset F$ and $x_n \to x$ imply $x \in F$.

Theorem 2.1.12. [103] (**The fundamental properties of closed sets**) *Let (X, d) be a metric space. Then the following conclusions hold:*

(1) *X and $\emptyset$ are closed sets;*

(2) *any intersection of closed sets in X is closed, that is,*

$$F_\mu \ (\mu \in M) \text{ are closed } \Rightarrow \bigcap_{\mu \in M} F_\mu \text{ is closed;}$$

(3) *any finite union of closed sets in X is closed, that is,*

$$F_i \ (i = 1, 2, \cdots, m) \text{ are closed } \Rightarrow \bigcup_{i=1}^{m} F_i \text{ is closed.}$$

Definition 2.1.13. [103] Let X and Y be metric spaces and let f be a mapping of X into Y. Then f is said to be *continuous* at x_0 in X if

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

A mapping f of X into Y is continuous if it is continuous at each x in X.

2.2 Banach Spaces and Hilbert spaces

In this section, we give definition and geometric properties in Banach spaces and Hilbert spaces.

Definition 2.2.1. [65] Let X be a linear space (or vector space). A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\alpha x\| = |\alpha|\|x\|$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

The ordered pair $(X, \|\cdot\|)$ is called a *normed space* .

Definition 2.2.2. [65] Let X be a normed space. The *metric induced by the norm* of X is the metric d on X defined by $d(x, y) = \|x - y\|$ for all $x, y \in X$. The *norm topology* of X is the topology obtained from this metric.

Definition 2.2.3. [56] Let x be an element and $\{x_n\}$ a sequence in a normed space X. Then $\{x_n\}$ *converges strongly* to x written by $x_n \to x$, if $\lim_{n\to\infty} \|x_n - x\| = 0$.

Definition 2.2.4. [65] A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a *Banach space* or *B-space* or *complete normed space* if its norm is a Banach norm.

Definition 2.2.5. Let X be a normed space and $B(X, \mathbb{R})$ the set of all continuous linear functionals of X into $\mathbb{R}$. Then $X^* = B(X, \mathbb{R})$ is called the *dual space* of X.

Definition 2.2.6. [65] Let x be an element and $\{x_n\}$ a sequence in a normed space X. Then $\{x_n\}$ *converges weakly* to x written by $x_n \rightharpoonup x$, if $f(x_n) \to f(x)$ for all $f \in X^*$.

Define the mapping $\varphi : X \to X^{**}$ by $\varphi(x) = f_x$, $x \in X$. Then φ is called the *natural embedding mapping* from X into X^{**} and has the following properties:

(1) φ is linear: $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$ for all $x, y \in X$ and for all $\alpha, \beta \in \mathbb{F}$;

(2) $\varphi(x)$ is isometry: $\|\varphi(x)\| = \|x\|$ for all $x \in X$.

Definition 2.2.7. [2] A normed space X is said to be *reflexive* if the natural embedding mapping $\varphi : X \to X^{**}$ is onto. In this case, we write $X \cong X^{**}$ or $X = X^{**}$.

Lemma 2.2.8. [65] *Let X be a normed space. Then* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *in the following collection of statements.*

(a) *The space X is reflexive.*

(b) *Every bounded sequence in X has a weakly convergent subsequence.*

(c) *Whenever (C_n) is a sequence of nonempty closed bounded convex sets in X such that $C_{n+1} \subset C_n$ for each n, it follows that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$.*

Remark 2.2.9. [2]

(1) Every finite-dimensional Banach space is reflexive.

(2) ℓ_p and L_p for $1 < p < \infty$ are reflexive Banach spaces.

(3) Every Hilbert space is reflexive.

(4) ℓ_1, ℓ_∞, L_1 and L_∞ are not reflexive.

(5) c and c_0 are not reflexive.

Definition 2.2.10. [2, 102] A Banach space X is said to be *strictly convex* if

$$\|x\| = \|y\| = 1 \text{ and } x \neq y \text{ imply } \left\|\frac{x+y}{2}\right\| < 1.$$

Lemma 2.2.11. [2, 102] *A Banach space X is strictly convex if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|(1-\lambda)x + \lambda y\| = 1$ for all $\lambda \in (0,1)$ holds if and only if $x = y$.*

Theorem 2.2.12. [2] *Let C be a nonempty, closed and convex subset of a reflexive and strictly convex Banach space X. Then for $x \in X$, there exists a unique point $z_x \in C$ such that $\|x - z_x\| = D(x, C) = \inf\{\|x - y\| : y \in C\}$.*

Example 2.2.13. [2] Let $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|\cdot\|_2$ defined by $\|x\|_2 = \Big(\sum_{i=1}^n x_i^2\Big)^{1/2}$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then X is strictly convex.

Example 2.2.14. [2] Let $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then X is not strictly convex.

Example 2.2.15. [2] Let $X = \mathbb{R}^n$, $n \geq 2$ with norm $\|\cdot\|_\infty$ defined by $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$, $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. Then X is not strictly convex.

Definition 2.2.16. [2, 102] A Banach space X is called *uniformly convex* if for any $\varepsilon \in (0, 2]$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \varepsilon$, then $\|\frac{x+y}{2}\| \leq 1 - \delta$.

Let X be a Banach space. The *modulus of convexity* of X is the function $\delta_X : (0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\Big\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1;\ \|x - y\| \geq \varepsilon\Big\}.$$

Lemma 2.2.17. [2] *A Banach space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$.*

Remark 2.2.18. [2]

(1) $\delta_X(\varepsilon)/\varepsilon$ is a nondecreasing function on $(0, 2]$.

(2) δ_X is a convex and continuous function.

Example 2.2.19. [2]

(1) $\delta_H(\varepsilon) = 1 - \sqrt{1 - (\varepsilon/2)^2}$.

(2) $\delta_{\ell_p}(\varepsilon) = 1 - \big(1 - (\varepsilon/2)^p\big)^{p/2}$ $(2 \leq p < \infty)$.

(3) If X is uniformly convex, then $\delta_X(\varepsilon) \leq 1 - \sqrt{1-(\varepsilon/2)^2}$.

Remark 2.2.20. [2]

(1) Every Hilbert space is uniformly convex.

(2) The Banach spaces ℓ_p and L_p with $(1 < p < \infty)$ are uniformly convex.

(3) The Banach spaces $\ell_1, \ell_\infty, c, c_0, L_1$ and L_∞ are not uniformly convex.

Theorem 2.2.21. [102] *Let X be a Banach space. Then the following conditions are equivalent:*

(1) *X is uniformly convex;*

(2) *if for any two sequences $\{x_n\}$, $\{y_n\}$ in X,*

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$;

(3) *for any ϵ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ depending only on $\epsilon > 0$ such that*

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$$

for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$.

Theorem 2.2.22. [2, 102] *Every uniformly convex Banach space is strictly convex and reflexive.*

Example 2.2.23. [2] Let $X = c_0$ and let $\beta > 0$ with the norm $\|\cdot\|_\beta$ defined by

$$\|x\|_\beta = \|x\|_{c_0} + \beta\left(\sum_{i=1}^{n}\left(\frac{x_i}{i}\right)^2\right), \; x = \{x_i\} \in c_0.$$

The space $(c_0, \|\cdot\|_\beta)$ for $\beta > 0$ are strictly convex, but not uniformly convex.

Example 2.2.24. [2] Let $X = \mathbb{R}^n$, $n \geq 2$ with the norm $\|\cdot\|_1$ defined by $\|x\|_1 = \sum_{i=1}^n |x_i|$. Then X is reflexive, but not uniformly convex.

Definition 2.2.25. [102] Let X be a Banach space. The multi-valued mapping $J : X \to 2^{X^*}$ is called the *duality mapping* if

$$J(x) = \left\{x^* \in X^* : \; \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\right\}$$

for all $x \in X$, where $\langle\cdot,\cdot\rangle$ denotes the pairing between X and X^*.

Definition 2.2.26. [102] Let X be a Banach space and let $S(X) = \{x \in X : \; \|x\| = 1\}$. Then X is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2.1}$$

exists for each $x, y \in S(X)$. In this case the norm of X is said to be *Gâteaux differentiable*. The space X is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S(X)$, the limit (2.2.1) is attained uniformly for $x \in S(X)$. The norm of X is said to be *Fréchet differentiable* if for each $x \in X$, the limit (2.2.1) is attained uniformly for $y \in S(X)$. The norm of X is said to be *uniformly Fréchet differentiable* (and X is said to be *uniformly smooth*) if the limit (2.2.1) is attained uniformly for $x, y \in S(X)$.

Let X be a Banach space. The *modulus of smoothness* of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(t) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| \le t\Big\}.$$

Example 2.2.27. [2] Let $X = H$ a Hilbert space. Then $\rho_H(t) = \sqrt{1+t^2} - 1$ for all $t \ge 0$.

Lemma 2.2.28. [2] *A Banach space X is uniformly smooth if and only if* $\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$.

Definition 2.2.29. Let $p, q > 1$. A Banach space X is said to be *p-uniformly convex* (resp. *q-uniformly smooth*) if there exists a constant $c > 0$ such that $\delta_X(\varepsilon) \ge c\varepsilon^p$ (resp. $\rho_X(t) \le ct^q$).

Remark 2.2.30. Every p-uniformly convex Banach space (resp. q-uniformly smooth Banach space) is uniformly convex (resp. uniformly smooth).

Theorem 2.2.31. [2] *Every uniformly smooth Banach space is reflexive.*

We next give examples of the duality mapping J in the uniformly smooth and uniformly convex Banach spaces ℓ_p, L_p and W^p_m, $1 < p < +\infty$.

Example 2.2.32. [3]

(1) For ℓ_p, $J(x) = \|x\|^{2-p}_{\ell_p} y \in \ell_q$, where $x = \{x_1, x_2, ...\}$ and $y = \{x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, ...\}$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(2) For L_p, $J(x) = \|x\|^{2-p}_{L_p} |x|^{p-2}x \in L_q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

(3) For W^p_m, $J(x) = \|x\|^{2-p}_{W^p_m} \sum (-1)^{|t|} D^t\big(|D^t x|^{p-2} D^t x\big) \in W^q_{-m}$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 2.2.33. [102] *Let X be a Banach space and let $J : X \to 2^{X^*}$ be the duality mapping. Then*

(1) *for each $x \in X$, $J(x)$ is nonempty, bounded, closed and convex;*

(2) $J(0) = \{0\}$*;*

(3) *for each $x \in X$ and a real α, $J(\alpha x) = \alpha J(x)$;*

(4) *for $x, y \in X$, $x^* \in J(x)$ and $y^* \in J(y)$, $\langle x - y, x^* - y^*\rangle \ge 0$;*

(5) *for $x, y \in X$, $y^* \in J(y)$, $\|x\|^2 - \|y\|^2 \ge 2\langle x - y, y^*\rangle$.*

Proposition 2.2.34. [102] *Let X be a Banach space. Then*

(1) *if X is smooth, then J is single-valued;*

(2) *if X is strictly convex, then J is one-to-one, that is, $x \ne y$ implies $J(x) \cap J(y) = \emptyset$;*

(3) *if X is reflexive, then J is onto, that is, for each $x^* \in X^*$, there exists $x \in X$ such that $x^* \in Jx$;*

(4) *if X has a Fréchet differentiable norm, then J is norm-to-norm continuous;*

(5) *if X is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subsets of X.*

Theorem 2.2.35. [102] *Let X be a Banach space. Then we have the following:*

(1) *if X^* is strictly convex, then X is smooth;*

(2) *if X^* is smooth, then X is strictly convex.*

Remark 2.2.36. [102] The above statements are equivalent if X is a reflexive Banach space.

Theorem 2.2.37. [102] *Let X be a Banach space. Then X is uniformly smooth if and only if X^* is uniformly convex.*

Definition 2.2.38. [2] Let C be a subset of a Banach space X. Then C is said to be *convex* if $(1-\lambda)x + (1-\lambda)y \in C$ for all $x, y \in C$ and for all $\lambda \in (0, 1)$.

Definition 2.2.39. [2] Let X be a Banach space and let $f : X \to (-\infty, \infty)$ be a function. Then $D(f) = \big\{x \in X : \ f(x) < +\infty\big\}$ is called the *effective domain* of f. The function f is called *proper* if $D(f) \neq \emptyset$.

Definition 2.2.40. [2] Let X be a Banach space. A function $f : X \to (-\infty, \infty)$ is *convex* if for any $x, y \in X$ and $t \in (0, 1)$, then $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$.

Definition 2.2.41. [102] Let X be a Banach space. Then $f : X \to (-\infty, \infty)$ is said to be *lower semi-continuous* at $x_0 \in X$, if $\{x_n\}$ is a sequence in X such that $x_n \to x_0$ and $f(x_n) \to y$, then $f(x_0) \leq y$.

Lemma 2.2.42. [102] *Let X be a Banach space, let $\{x_n\}$ be a bounded sequence of X such that $x_n \rightharpoonup x$. Then following inequality holds:*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Definition 2.2.43. [74] A Banach space X is said to satisfy *Opial's condition* if $x_n \rightharpoonup x$ as $n \to \infty$ and $x \neq y$ imply that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$.

It is well known [74] that all Hilbert spaces and ℓ_p spaces where $1 \leq p < \infty$, have this property, while all L_p spaces do not unless $p = 2$.

Definition 2.2.44. A Banach space X is said to have the *Kadec-Klee property* if for every sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$.

Theorem 2.2.45. [65] *Every uniformly convex Banach space has the Kadec-Klee property.*

Definition 2.2.46. [56, 103] An *inner product space* is a vector space X with an inner product defined on X. A *Hilbert space* is a complete inner product space. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; that is, with every pair of vector x and y there is associated a scalar which is written and is called the inner product of x and y, such that for all vectors x, y, z and scalar $\alpha \in \mathbb{F}$ we have:

(1) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$;

(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$.

Theorem 2.2.47. [103](**The Schwarz inequality**) If x and y are any two vectors in an inner product space X, then $|\langle x, y\rangle| \leq \|x\|\|y\|$.

Theorem 2.2.48. [103] (**The parallelogram law**) If x and y are any two vectors in an inner product space X, then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Theorem 2.2.49. [103] Let x and y be elements in an inner product space X and let $\lambda \in (0, 1)$. Then the followings hold:

(1) $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$;

(2) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$;

(3) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$.

Lemma 2.2.50. [103] Let X be an inner product space, let $\{x_n\}$ be a sequence of X and let x be an element of X. Then $x_n \rightharpoonup x$, if for any $y \in X$, $\langle x_n, y\rangle \to \langle x, y\rangle$.

Let C be a nonempty, closed and convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C.

Lemma 2.2.51. [42, 103] *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $x \in H$. Then, for each $y \in C$, the following are equivalent:*

(1) $\langle x - P_C x, y - P_C x\rangle \leq 0$;

(2) $\|x - P_C x\|^2 + \|P_C x - y\|^2 \leq \|x - y\|^2$.

2.3 Inequalities in Banach Spaces

In this section, we collect some useful inequalities in Banach spaces.

Lemma 2.3.1. [2] *Let X be a Banach space and $J : X \to 2^{X^*}$ the duality mapping. Then we have the following:*

(1) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, j(x)\rangle$ *for all* $x, y \in X$, *where* $j(x) \in J(x)$;

(2) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle$ *for all* $x, y \in X$, *where* $j(x + y) \in J(x + y)$.

Remark 2.3.2. In a real Hilbert space H, we have the following inequality:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

Lemma 2.3.3. [114] *Let X be a real q-uniformly smooth Banach space. Then the following inequality holds:*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q,$$

for all $x, y \in X$ and for some $C_q > 0$.

Lemma 2.3.4. [31, 88, 89] *Let X be a uniformly smooth Banach space. Then there exists a nondecreasing continuous function $\beta : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0^+} \beta(t) = 0$ and $\beta(ct) \leq c\beta(t)$ for $c \geq 1$ such that for all $x, y \in X$, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x)\rangle + \max\{\|x\|, 1\}\|y\|\beta(\|y\|).$$

Lemma 2.3.5. [114] *Let X be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x-y\|),$$

where $x, y \in B_r = \{z \in X : \ \|z\| \leq r\}$ and $t \in [0,1]$.

Lemma 2.3.6. [31, 114] *Let $q > 1$ and $r > 0$ be two fixed real numbers and X be a uniformly smooth Banach space. Then there exists a continuous, strictly increasing and convex function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$, $\Phi(0) = 0$ such that for every $x, y \in B_r = \{z \in X : \|z\| \leq r\}$ we get*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + \Phi(\|y\|). \tag{2.3.1}$$

Lemma 2.3.7. [20] *Let C be a bounded, closed and convex subset of a uniformly convex Banach space X. Then there exists a strictly increasing, convex and continuous function $\gamma : [0,\infty) \to [0,\infty)$ such that $\gamma(0) = 0$ and*

$$\gamma\left(\left\|T\left(\sum_{i=1}^{n}\lambda_i x_i\right) - \sum_{i=1}^{n}\lambda_i T x_i\right\|\right) \leq \max_{1\leq j\leq k\leq n}\left(\|x_j - x_k\| - \|Tx_j - Tx_k\|\right)$$

for all $n \in \mathbb{N}$, $\{x_1, x_2, ..., x_n\} \subset C$, $\{\lambda_1, \lambda_2, ..., \lambda_n\} \subset [0,1]$ with $\sum_{i=1}^{n}\lambda_i = 1$ and a nonexpansive mapping T of C into X.

2.4 Some Useful Lemmas, Propositions and Theorems

In this section, we give some useful lemmas, propositions and theorems in order to prove the main results.

Definition 2.4.1. Let B be a subset of topological vector space X. A mapping $G : B \to 2^X$ is called a *KKM mapping* if $co\{x_1, x_2, \cdots, x_m\} \subset \bigcup_{i=1}^{m} G(x_i)$ for $x_i \in B$ and $i = 1, 2, \cdots, m$, where coA denotes the *convex hull* of the set A.

Lemma 2.4.2. [37] *Let B be a nonempty subset of a Hausdorff topological vector space X and let $G : B \to 2^X$ be a KKM mapping. If $G(x)$ is closed for all $x \in B$ and is compact for at least one $x \in B$, then $\bigcap_{x\in B} G(x) \neq \emptyset$.*

Lemma 2.4.3. [18, 42] **Demi-closedness principle**. *Assume that T is a nonexpansive self-mapping of a nonempty, closed and convex subset C of a real Banach space X. If T has a fixed point, then $I - T$ is demi-closed, that is, whenever $x_n \rightharpoonup x \in C$, and the sequence $(I - T)x_n \to y$ for some $y \in C$, it follows that $(I - T)x = y$. Here I is the identity operator of X.*

Lemma 2.4.4. [123] *Let X be a uniformly convex Banach space, C a nonempty, closed and convex subset of X, and $T : C \to C$ be a continuous pseudocontraction. Then $I-T$ is demi-closed at zero.*

Lemma 2.4.5. [123] *Let X be a reflexive Banach space which satisfies Opial's condition, C a nonempty, closed and convex subset of X, and $T : C \to C$ be a continuous pseudocontraction. Then $I - T$ is demi-closed at zero.*

Lemma 2.4.6. [116] *Let X be a uniformly convex Banach space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < b \leq \alpha_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq d$, $\limsup_{n\to\infty} \|y_n\| \leq d$ and $\lim_{n\to\infty} \|\alpha_n x_n + (1 - \alpha_n)y_n\| = d$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.4.7. [98] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be the sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$. Suppose $x_{n+1} = (1 - b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.4.8. [113] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = +\infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4.9. [101] *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n, \quad n \geq 0.$$

If $\sum_{n=0}^{\infty} \delta_n < +\infty$ and $\sum_{n=0}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty} a_n = 0$.

Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $A : H \to H$ be a single-valued nonlinear mapping and let $M : H \to 2^H$ be a set-valued mapping. The variational inclusion is to find $\hat{x} \in H$ such that

$$\theta \in A(\hat{x}) + M(\hat{x}), \tag{2.4.1}$$

where θ is the zero vector in H. The solutions set of problem (2.4.1) is denoted by $I(A, M)$. Recall that a mapping $A : H \to H$ is called *α-inverse strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in H.$$

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H, f \in M(x)$, and $g \in M(y)$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping M is *maximal* if its graph $G(M) := \{(f, x) \in H \times H : f \in M(x)\}$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(M)$ imply $f \in M(x)$. We define the resolvent operator $J_{M,\lambda}$ associated with M and λ as follows:

$$J_{M,\lambda}(x) = (I + \lambda M)^{-1}(x), \quad x \in H, \lambda > 0. \tag{2.4.2}$$

Lemma 2.4.10. [17] *Let $M : H \to 2^H$ be a maximal monotone mapping and $A : H \to H$ be a Lipschitz continuous mapping. Then the mapping $S = M + A : H \to 2^H$ is a maximal monotone mapping.*

Let X be a Banach space, D a nonempty, closed and convex subset of X, and C a nonempty subset of D. Let $Q : D \to C$. Then Q is said to be

(1) *sunny* if for each $x \in D$ and $t \in [0, 1]$, we have

$$Q\big(tx + (1 - t)Qx\big) = Qx;$$

(2) a *retraction* of D onto C if $Qx = x$ for all $x \in C$;

(3) a *sunny nonexpansive retraction* if Q is sunny, nonexpansive and retract onto C.

See Bruck [21], Goebel-Reich [41] and Reich [90].

Lemma 2.4.11. [91] *Let X be a uniformly smooth Banach space and C a nonempty, closed and convex subset of X. Let $T : C \to C$ be a nonexpansive mapping with a fixed point and let $z \in C$. For each $t \in (0, 1)$, let z_t be the unique solution of the equation $x = tz + (1 - t)Tx$. Then $\{z_t\}$ converges to a fixed point of T as $t \to 0$ and*

$$Qz = s - \lim_{t\to 0} z_t$$

defines the unique sunny nonexpansive retraction from C onto $F(T)$.

Lemma 2.4.12. [123] *Let C be a closed and convex subset of a uniformly smooth Banach space X, and $T : C \to C$ a nonexpansive mapping with a nonempty fixed point set $F(T)$. Then there exists a unique sunny nonexpansive retraction $Q_{F(T)} : C \to F(T)$ such that*

$$\limsup_{n\to\infty}\langle u - Q_{F(T)}u, J(x_n - Q_{F(T)}u)\rangle \leq 0,$$

for any given $u \in C$ and for any bounded approximate fixed point sequence $\{x_n\} \subset C$ of T.

Lemma 2.4.13. [100] *Let X be a uniformly convex Banach space with a Fréchet differentiable norm. Let C be a closed and convex subset of X, and $\{S_n\}_{n=1}^{\infty}$ be a family of L_n-Lipschitzian self-mappings on C such that $\sum_{n=1}^{\infty}(L_n-1)<\infty$ and $F=\bigcap_{n=1}^{\infty}F(S_n)\neq\emptyset$. For arbitrary $x_1\in C$, define $x_{n+1}=S_nx_n$ for all $n\geq 1$. Then for every $p,q\in F$, $\lim_{n\to\infty}\langle x_n, j(p-q)\rangle$ exists, in particular, for all $u,v\in\omega_\omega(x_n)$, and $p,q\in F$, $\langle u-v, j(p-q)\rangle=0$.*

Lemma 2.4.14. *Let X be a Banach space with the Fréchet differentiable norm. For $x\in X$, let $\beta^*(t)$ be defined for $0<t<\infty$ by*

$$\beta^*(t)=\sup_{y\in S(X)}\left|\frac{\|x+ty\|^2-\|x\|^2}{t}-2\langle y, j(x)\rangle\right|.$$

Then $\lim_{t\to 0^+}\beta^*(t)=0$, and

$$\|x+h\|^2\leq\|x\|^2+2\langle h, j(x)\rangle+\|h\|\beta^*(\|h\|) \tag{2.4.3}$$

for all $h\in X\setminus\{0\}$.

Proof. Let $x\in X$. Since X has the Fréchet differentiable norm, it follows that

$$\lim_{t\to 0}\sup_{y\in S(X)}\left|\frac{\frac{1}{2}\|x+ty\|^2-\frac{1}{2}\|x\|^2}{t}-\langle y, j(x)\rangle\right|=0.$$

Then $\lim_{t\to 0^+}\beta^*(t)=0$ and hence

$$\left|\frac{\|x+ty\|^2-\|x\|^2}{t}-2\langle y, j(x)\rangle\right|\leq\beta^*(t)\quad\forall y\in S(X)$$

which implies

$$\|x+ty\|^2\leq\|x\|^2+2t\langle y, j(x)\rangle+t\beta^*(t)\quad\forall y\in S(X). \tag{2.4.4}$$

Suppose $h\neq 0$. Put $y=\frac{h}{\|h\|}$ and $t=\|h\|$. By (2.4.4), we have

$$\|x+h\|^2\leq\|x\|^2+2\langle h, j(x)\rangle+\|h\|\beta^*(\|h\|).$$

This completes the proof. □

To deal with a family of mappings, the following conditions are introduced: Let C be a subset of a real Banach space X and let $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of C such that $\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. Then $\{T_n\}$ is said to satisfy the *AKTT-condition* [6] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\|: z\in B\}<\infty.$$

Lemma 2.4.15. [6] *Let C be a nonempty and closed subset of a Banach space X and let $\{T_n\}$ be a family of mappings of C into itself which satisfies the AKTT-condition. Then, for each $x\in C$, $\{T_nx\}$ converges strongly to a point in C. Moreover, let the mapping T be defined by*

$$Tx=\lim_{n\to\infty}T_nx\quad\forall x\in C.$$

Then for each bounded subset B of C,

$$\lim_{n\to\infty}\sup\{\|Tz-T_nz\|: z\in B\}=0.$$

In the sequel, we will write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and T is defined by Lemma 2.4.15 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

We next give examples which satisfy the AKTT-condition for a family of nonexpansive mappings.

Example 2.4.16. Let $T_1, T_2, \cdots$, be an infinite family of nonexpansive mappings of C into itself and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_i < 1$ for all $i \in \mathbb{N}$. Moreover, let W_n and W be the W-mappings [94] generated by $T_1, T_2, \cdots, T_n$ and $\gamma_1, \gamma_2, \cdots, \gamma_n$, and $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$. Then $\big(\{W_n\}, W\big)$ satisfies the AKTT-condition [81, 94].

Example 2.4.17. Let $T_1, T_2, \cdots$ be an infinite family of nonexpansive mappings of C into itself. For each $n \in \mathbb{N}$, define the mapping $V_n : C \to C$ by

$$V_n x = \sum_{i=1}^{n} \lambda_n^i T_i x, \quad \forall x \in C,$$

where $\{\lambda_n^i\}$ is a family of nonnegative numbers satisfying the following conditions:

(1) $\sum_{i=1}^{n} \lambda_n^i = 1$ for each $n \in \mathbb{N}$;

(2) $\lambda^i := \lim_{n\to\infty} \lambda_n^i > 0$ for each $i \in \mathbb{N}$;

(3) $\sum_{n=1}^{\infty} \sum_{i=1}^{n} |\lambda_{n+1}^i - \lambda_n^i| < +\infty$.

Let $V : C \to C$ be the mapping defined by

$$Vx = \sum_{i=1}^{\infty} \lambda^i T_i x, \quad \forall x \in C.$$

Then $\big(\{V_n\}, V\big)$ satisfies the AKTT-condition [6].

The following results can be found in [14, 15].

Lemma 2.4.18. [14, 15] *Let C be a closed and convex subset of a smooth Banach space X. Suppose that $\{T_n\}_{n=1}^{\infty}$ is a family of λ-strictly pseudocontractive mappings from C into X with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\{\mu_n\}_{n=1}^{\infty}$ is a real sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \mu_n = 1$. Then the following conclusions hold:*

(1) *$G := \sum_{n=1}^{\infty} \mu_n T_n : C \to X$ is a λ-strictly pseudocontractive mapping;*

(2) *$F(G) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.4.19. [15] *Let C be a closed and convex subset of a smooth Banach space X. Suppose that $\{S_k\}_{k=1}^{\infty}$ is a countable family of λ-strictly pseudocontractive mappings of C into itself with $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by*

$$T_n x = \sum_{k=1}^{n} \mu_n^k S_k x, \quad x \in C,$$

where $\{\mu_n^k\}$ is a family of nonnegative numbers satisfying

(1) $\sum_{k=1}^{n} \mu_n^k = 1$ *for all* $n \in \mathbb{N}$;

(2) $\mu^k := \lim_{n\to\infty} \mu_n^k > 0$ *for all* $k \in \mathbb{N}$;

(3) $\sum_{n=1}^{\infty} \sum_{k=1}^{n} |\mu_{n+1}^k - \mu_n^k| < +\infty$.

Then

(1) *Each* T_n *is a* λ*-strictly pseudocontractive mapping.*

(2) $\{T_n\}$ *satisfies AKTT-condition.*

(3) *If* $T : C \to C$ *is defined by*

$$Tx = \sum_{k=1}^{\infty} \mu^k S_k x, \quad x \in C.$$

Then $Tx = \lim_{n\to\infty} T_n x$ *and* $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k)$.

Using Lemma 2.3.6, we can prove the following lemma.

Lemma 2.4.20. *Let* X *be a uniformly smooth Banach space and* C *a nonempty, bounded and convex subset of* X. *Let* $T : C \to X$ *be a* λ*-strict pseudocontractions for some* $0 < \lambda < 1$. *Assume that* $\Phi(t) \le 2t^2$, $t \in [0, \infty)$ *where* Φ *is a function appearing in* (2.3.1). *For* $\alpha \in (0, 1)$, *we define* $T_\alpha = (1 - \alpha)I + \alpha T$. *Then, as* $\alpha \in (0, \lambda]$, T_α *is nonexpansive such that* $F(T_\alpha) = F(T)$.

Proof. For $x, y \in C$, by Lemma 2.3.6, we have

$$\begin{aligned}
\|T_\alpha x - T_\alpha y\|^2 &= \|(x - y) + \alpha(Tx - Ty - (x - y))\|^2 \\
&\le \|x - y\|^2 + 2\alpha\langle Tx - Ty - (x - y), j(x - y)\rangle \\
&\quad + \Phi\Big(\big\|\alpha(Tx - Ty - (x - y))\big\|\Big) \\
&\le \|x - y\|^2 - 2\alpha\lambda\|Tx - Ty - (x - y)\|^2 \\
&\quad + 2\alpha^2\|Tx - Ty - (x - y)\|^2 \\
&= \|x - y\|^2 - 2\alpha(\lambda - \alpha)\|Tx - Ty - (x - y)\|^2 \\
&\le \|x - y\|^2,
\end{aligned}$$

which implies that T_α is nonexpansive. □

Motivated by [8], we next study the class of Lipschitz and quasi-nonexpansive mappings. Let C be a nonempty subset of a Banach space X and let $T : C \to X$ be a mapping. Then T is called

(1) *quasi-nonexpansive* if $F(T) \ne \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for all $x \in C$ and $p \in F(T)$;

(2) *Lipschitz* if there exists $L > 0$ such that $\|Tx - Ty\| \le L\|x - y\|$ for all $x, y \in C$.

It is remarked that the class of Lipschitz and quasi-nonexpansive mappings includes that of nonexpansive mappings as special cases.

The following example is a Lipschitz and quasi-nonexpansive mapping but not nonexpansive.

Example 2.4.21. Let $X = \mathbb{R}$ and $C = [0, 1.5]$. Define $T : C \to \mathbb{R}$ by

$$Tx = \begin{cases} 0, & x \in [0, 1), \\ 2x - 2, & x \in [1, 1.5]. \end{cases}$$

It is clear that $F(T) = \{0\}$ and T is a quasi-nonexpansive and Lipschitz mapping. Indeed, $|Tx - 0| = |Tx| \leq |x| = |x - 0|$ for all $x \in C$ and $|Tx - Ty| \leq 2|x - y|$ for all $x, y \in C$. However, T is not nonexpansive. In fact, if $x = 1$ and $y = 1.5$, then $|Tx - Ty| = 1 > 0.5 = |x - y|$.

Lemma 2.4.22. [45] *Let C be a closed and convex subset of a strictly convex Banach space X, T a quasi-nonexpansive mapping of C into C. Then $F(T)$ is a nonempty, closed and convex set on which T is continuous.*

We first prove some useful lemmas concerning the W-mapping of Lipschitz and quasi-nonexpansive mappings in a strictly convex Banach space.

Lemma 2.4.23. *Let C be a nonempty, closed and convex subset of a strictly convex Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive and L_i-Lipschitz mappings of C into itself such that $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\beta_1, \beta_2, \cdots, \beta_N$ be real numbers such that $0 < \beta_i < 1$ for all $i = 1, 2, \cdots, N-1$, $0 < \beta_N \leq 1$ and $\sum_{i=1}^N \beta_i = 1$. Let W be the W-mapping generated by $T_1, T_2, \cdots, T_N$ and $\beta_1, \beta_2, \cdots, \beta_N$. Then the followings hold:*

(1) $F(W) = \bigcap_{i=1}^N F(T_i)$;

(2) *W is quasi-nonexpansive and Lipschitz.*

Proof. (1) Since $F \subset F(W)$ is trivial, it suffices to show that $F(W) \subset F$. To this end, let $p \in F(W)$ and $x^* \in F$. Then we have

$$\begin{aligned}
\|p - x^*\| &= \|Wp - x^*\| = \|\beta_N(T_N U_{N-1}p - x^*) + (1 - \beta_N)(p - x^*)\| \\
&\leq \beta_N\|U_{N-1}p - x^*\| + (1 - \beta_N)\|p - x^*\| \\
&= \beta_N\|\beta_{N-1}(T_{N-1}U_{N-2}p - x^*) + (1 - \beta_{N-1})(p - x^*)\| \\
&\quad + (1 - \beta_N)\|p - x^*\| \\
&\leq \beta_N\beta_{N-1}\|U_{N-2}p - x^*\| + (1 - \beta_N\beta_{N-1})\|p - x^*\| \\
&= \beta_N\beta_{N-1}\|\beta_{N-2}(T_{N-2}U_{N-3}p - x^*) + (1 - \beta_{N-2})(p - x^*)\| \\
&\quad + (1 - \beta_N\beta_{N-1})\|p - x^*\| \\
&\leq \beta_N\beta_{N-1}\beta_{N-2}\|U_{N-3}p - x^*\| + (1 - \beta_N\beta_{N-1}\beta_{N-2})\|p - x^*\| \\
&\ \vdots \\
&= \beta_N\beta_{N-1}\cdots\beta_3\|\beta_2(T_2U_1p - x^*) + (1 - \beta_2)(p - x^*)\| \\
&\quad + (1 - \beta_N\beta_{N-1}\cdots\beta_3)\|p - x^*\| \\
&\leq \beta_N\beta_{N-1}\cdots\beta_2\|T_2U_1p - x^*\| + (1 - \beta_N\beta_{N-1}\cdots\beta_2)\|p - x^*\| \\
&\leq \beta_N\beta_{N-1}\cdots\beta_2\|U_1p - x^*\| + (1 - \beta_N\beta_{N-1}\cdots\beta_2)\|p - x^*\| \\
&= \beta_N\beta_{N-1}\cdots\beta_2\|\beta_1(T_1p - x^*) + (1 - \beta_1)(p - x^*)\| \\
&\quad + (1 - \beta_N\beta_{N-1}\cdots\beta_2)\|p - x^*\| \\
&\leq \beta_N\beta_{N-1}\cdots\beta_2\beta_1\|T_1p - x^*\| + (1 - \beta_N\beta_{N-1}\cdots\beta_2\beta_1)\|p - x^*\| \\
&\leq \beta_N\beta_{N-1}\cdots\beta_2\beta_1\|p - x^*\| + (1 - \beta_N\beta_{N-1}\cdots\beta_2\beta_1)\|p - x^*\| \\
&= \|p - x^*\|. \qquad (2.4.5)
\end{aligned}$$

This shows that

$$\|p-x^*\| = \beta_N\beta_{N-1}\cdots\beta_2\|\beta_1(T_1p-x^*)+(1-\beta_1)(p-x^*)\|+(1-\beta_N\beta_{N-1}\cdots\beta_2)\|p-x^*\|,$$

hence

$$\|p-x^*\| = \|\beta_1(T_1p-x^*)+(1-\beta_1)(p-x^*)\|.$$

Again by (2.4.5), we see that $\|p-x^*\| = \|T_1p-x^*\|$. Hence

$$\|p-x^*\| = \|T_1p-x^*\| = \|\beta_1(T_1p-x^*)+(1-\beta_1)(p-x^*)\|. \tag{2.4.6}$$

Applying Lemma 2.2.11 to (2.4.6), we get that $T_1p = p$ and hence $U_1p = p$.
Again by (2.4.5), we have

$$\|p-x^*\| = \beta_N\beta_{N-1}\cdots\beta_3\|\beta_2(T_2U_1p-x^*)+(1-\beta_2)(p-x^*)\|+(1-\beta_N\beta_{N-1}\cdots\beta_3)\|p-x^*\|,$$

hence

$$\|p-x^*\| = \|\beta_2(T_2U_1p-x^*)+(1-\beta_2)(p-x^*)\|.$$

From (2.4.5), we know that $\|U_1p-x^*\| = \|T_2U_1p-x^*\|$. Since $U_1p = p$, we have

$$\|p-x^*\| = \|T_2p-x^*\| = \|\beta_2(T_2p-x^*)+(1-\beta_2)(p-x^*)\|. \tag{2.4.7}$$

Applying Lemma 2.2.11 to (2.4.7), we get that $T_2p = p$ and hence $U_2p = p$.

By proving in the same manner, we can conclude that $T_ip = p$ and $U_ip = p$ for all $i = 1, 2, \cdots, N-1$. Finally, we also have

$$\begin{aligned}\|p-T_Np\| &\leq \|p-Wp\| + \|Wp-T_Np\| \\ &= \|p-Wp\| + (1-\beta_N)\|p-T_Np\|,\end{aligned}$$

which yields that $p = T_Np$ since $p \in F(W)$. Hence $p \in F := \bigcap_{i=1}^N F(T_i)$.

(2) For each $x \in C$ and $z \in F$, we observe that

$$\|T_1x-z\| \leq \|x-z\|.$$

Let $k \in \{2, 3, \cdots, N\}$. Then

$$\begin{aligned}\|U_kx-z\| &= \|\beta_kT_kU_{k-1}x+(1-\beta_k)x-z\| \\ &\leq \beta_k\|U_{k-1}x-z\| + (1-\beta_k)\|x-z\|.\end{aligned}$$

So we have

$$\begin{aligned}\|Wx-z\| &= \|U_Nx-z\| \\ &\leq \beta_N\|U_{N-1}x-z\| + (1-\beta_N)\|x-z\| \\ &\leq \beta_N\Big(\beta_{N-1}\|U_{N-2}x-z\| + (1-\beta_{N-1})\|x-z\|\Big) \\ &\quad + (1-\beta_N)\|x-z\| \\ &\leq \beta_N\Bigg(\beta_{N-1}\Big(\beta_{N-2}\|U_{N-3}x-z\| + (1-\beta_{N-2})\|x-z\|\Big) \\ &\quad + (1-\beta_{N-1})\|x-z\|\Bigg) + (1-\beta_N)\|x-z\| \\ &\vdots\end{aligned}$$

$$
\begin{aligned}
&\le \beta_N\Bigg(\beta_{N-1}\Bigg(\beta_{N-2}\cdots\Big(\beta_2\big(\beta_1\|T_1x-z\|+(1-\beta_1)\|x-z\|\big)\\
&\quad+(1-\beta_2)\|x-z\|\Big)+\cdots+(1-\beta_{N-2})\|x-z\|\Bigg)\\
&\quad+(1-\beta_{N-1})\|x-z\|\Bigg)+(1-\beta_N)\|x-z\|\\
&\le \beta_N\Bigg(\beta_{N-1}\Bigg(\beta_{N-2}\cdots\Big(\beta_2\big(\beta_1\|x-z\|+(1-\beta_1)\|x-z\|\big)\\
&\quad+(1-\beta_2)\|x-z\|\Big)+\cdots+(1-\beta_{N-2})\|x-z\|\Bigg)\\
&\quad+(1-\beta_{N-1})\|x-z\|\Bigg)+(1-\beta_N)\|x-z\|\\
&= \beta_N\Bigg(\beta_{N-1}\Bigg(\beta_{N-2}\cdots\Big(\beta_3\big(\beta_2\|x-z\|+(1-\beta_2)\|x-z\|\big)\\
&\quad+(1-\beta_3)\|x-z\|\Big)+\cdots+(1-\beta_{N-2})\|x-z\|\Bigg)\\
&\quad+(1-\beta_{N-1})\|x-z\|\Bigg)+(1-\beta_N)\|x-z\|\\
&= \|x-z\|.
\end{aligned}
$$

This shows that W is a quasi-nonexpansive mapping.

Next, we show that W is a Lipschitz mapping. Note that T_i is L_i-Lipschitz for all $i=1,2,\cdots,N$. For each $x,y\in C$, we observe

$$
\begin{aligned}
\|U_1x-U_1y\| &= \|\beta_1T_1x+(1-\beta_1)x-\beta_1T_1y-(1-\beta_1)y\|\\
&\le \beta_1\|T_1x-T_1y\|+(1-\beta_1)\|x-y\|\\
&\le \big(\beta_1L_1+(1-\beta_1)\big)\|x-y\|.
\end{aligned}
$$

Let $k\in\{2,3,\cdots,N\}$, then

$$
\begin{aligned}
\|U_kx-U_ky\| &= \|\beta_kT_kU_{k-1}x+(1-\beta_k)x-\beta_kT_kU_{k-1}y-(1-\beta_k)y\|\\
&\le \beta_kL_k\|U_{k-1}x-U_{k-1}y\|+(1-\beta_k)\|x-y\|.
\end{aligned}
$$

So we have

$$
\begin{aligned}
\|Wx-Wy\| &\le \beta_NL_N\|U_{N-1}x-U_{N-1}y\|+(1-\beta_N)\|x-y\|\\
&\le \beta_NL_N\beta_{N-1}L_{N-1}\|U_{N-2}x-U_{N-2}y\|\\
&\quad+\Big(\beta_NL_N(1-\beta_{N-1})+(1-\beta_N)\Big)\|x-y\|\\
&\;\;\vdots
\end{aligned}
$$

$$
\begin{aligned}
&\leq \beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_2 L_2 \|U_1 x - U_1 y\| \\
&\quad + \Big(\beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_3 L_3 (1-\beta_2) \\
&\quad + \beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_4 L_4 (1-\beta_3) \\
&\quad + \cdots + \beta_N L_N (1-\beta_{N-1}) + (1-\beta_N)\Big)\|x-y\| \\
&\leq \beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_2 L_2 \Big(\beta_1 L_1 + (1-\beta_1)\|x-y\|\Big) \\
&\quad + \Big(\beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_3 L_3 (1-\beta_2) \\
&\quad + \beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_4 L_4 (1-\beta_3) \\
&\quad + \cdots + \beta_N L_N (1-\beta_{N-1}) + (1-\beta_N)\Big)\|x-y\| \\
&= \Big(\beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_1 L_1 \\
&\quad + \beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_2 L_2 (1-\beta_1) \\
&\quad + \beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_3 L_3 (1-\beta_2) \\
&\quad + \beta_N L_N \beta_{N-1} L_{N-1} \cdots \beta_4 L_4 (1-\beta_3) \\
&\quad + \cdots + \beta_N L_N (1-\beta_{N-1}) + (1-\beta_N)\Big)\|x-y\|. \\
&\leq \Big(L_N L_{N-1} \cdots L_1 + L_N L_{N-1} \cdots L_2 + L_N L_{N-1} \cdots L_3 \\
&\quad + L_N L_{N-1} \cdots L_4 + \cdots + L_N L_{N-1} + L_N + 1\Big)\|x-y\|.
\end{aligned}
$$

Since $L_i > 0$ for all $i = 1, 2, \cdots, N$, W is a Lipschitz mapping. □

Lemma 2.4.24. *Let C be a nonempty, closed and convex subset of a Banach space X. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive and L_i-Lipschitz mappings of C into itself and $\{\beta_{n,i}\}_{i=1}^N$ sequences in $[0,1]$ such that $\beta_{n,i} \to \beta_i$ as $n \to \infty$. Moreover, for every $n \in \mathbb{N}$, let W and W_n be the W-mappings generated by $T_1, T_2, \cdots, T_N$ and $\beta_1, \beta_2, \cdots, \beta_N$, and $T_1, T_2, \cdots, T_N$ and $\beta_{n,1}, \beta_{n,2}, \cdots, \beta_{n,N}$, respectively. Then*

$$\lim_{n\to\infty} \|W_n x - Wx\| = 0, \quad \forall x \in C.$$

Proof. Let $x \in C$ and U_k and $U_{n,k}$ be generated by $T_1, T_2, \cdots, T_k$ and $\beta_1, \beta_2, \cdots, \beta_k$, and $T_1, T_2, \cdots, T_k$ and $\beta_{n,1}, \beta_{n,2}, \cdots, \beta_{n,k}$, respectively. Then

$$\|U_{n,1}x - U_1 x\| = \|(\beta_{n,1} - \beta_1)(T_1 x - x)\| \leq |\beta_{n,1} - \beta_1| \|T_1 x - x\|.$$

Let $k \in \{2, 3, \cdots, N\}$ and $M = \max\{\|T_k U_{k-1} x\| + \|x\| : k = 2, 3, \cdots, N\}$. Then

$$
\begin{aligned}
\|U_{n,k}x - U_k x\| &= \|\beta_{n,k} T_k U_{n,k-1} x + (1-\beta_{n,k})x - \beta_k T_k U_{k-1} - (1-\beta_k)x\| \\
&= \|\beta_{n,k} T_k U_{n,k-1} x - \beta_{n,k} x - \beta_k T_k U_{k-1} + \beta_k x\| \\
&\leq \beta_{n,k} \|T_k U_{n,k-1} x - T_k U_{k-1} x\| + |\beta_{n,k} - \beta_k| \|T_k U_{k-1} x\| \\
&\quad + |\beta_{n,k} - \beta_k| \|x\| \\
&\leq L_k \|U_{n,k-1} x - U_{k-1} x\| + |\beta_{n,k} - \beta_k| M.
\end{aligned}
$$

It follows that

$$\begin{aligned}
\|W_n x - Wx\| &= \|U_{n,N}x - U_N x\| \\
&\leq L_N\|U_{n,N-1}x - U_{N-1}x\| + |\beta_{n,N} - \beta_N|M \\
&\leq L_N\Big(L_{N-1}\|U_{n,N-2}x - U_{N-2}x\| + |\beta_{n,N-1} - \beta_{N-1}|M\Big) \\
&\quad + |\beta_{n,N} - \beta_N|M \\
&= L_N L_{N-1}\|U_{n,N-2}x - U_{N-2}x\| + L_N|\beta_{n,N-1} - \beta_{N-1}|M \\
&\quad + |\beta_{n,N} - \beta_N|M \\
&\ \vdots \\
&\leq L_N L_{N-1}\cdots L_3\Big(L_2\|U_{n,1}x - U_1 x\| + |\beta_{n,2} - \beta_2|M\Big) \\
&\quad + L_N L_{N-1}\cdots L_4|\beta_{n,3} - \beta_3|M + \cdots + L_N|\beta_{n,N-1} - \beta_{N-1}|M \\
&\quad + |\beta_{n,N} - \beta_N|M \\
&\leq L_N L_{N-1}\cdots L_2|\beta_{n,1} - \beta_1|\|T_1 x - x\| + L_N L_{N-1}\cdots L_3|\beta_{n,2} - \beta_2|M \\
&\quad + L_N L_{N-1}\cdots L_4|\beta_{n,3} - \beta_3|M + \cdots + L_N|\beta_{n,N-1} - \beta_{N-1}|M \\
&\quad + |\beta_{n,N} - \beta_N|M.
\end{aligned}$$

Since $\beta_{n,i} \to \beta_i$ as $n \to \infty$ $(i = 1, 2, \cdots, N)$, we obtain the desired result. □

We next recall some useful lemmas concerning the generalized metric projection in strictly convex, reflexive and smooth Banach spaces.

Let X be a smooth Banach space. The function $\phi : X \times X \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in X$.

Remark 2.4.25. We know the following: for each $x, y, z \in X$,

(1) $\big(\|x\| - \|y\|\big)^2 \leq \phi(x, y) \leq \big(\|x\| + \|y\|\big)^2$;

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$;

(3) $\phi(x, y) = \|x - y\|^2$ in a Hilbert space.

Lemma 2.4.26. [48] *Let X be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences of X such that $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let X be a reflexive, strictly convex and smooth Banach space and let C be a nonempty, closed and convex subset of X. The *generalized projection mapping*, introduced by Alber [3], is a mapping $\Pi_C : X \to C$, that assigns to an arbitrary point $x \in X$ the minimum point of the functional $\phi(y, x)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y, x) : y \in C\}.$$

In fact, we have the following result:

Lemma 2.4.27. [3] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex, and smooth Banach space X and let $x \in X$. Then, there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.*

Lemma 2.4.28. [3, 48] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space X, $x \in X$, and $z \in C$. Then $z = \Pi_C x$ if and only if $\big\langle Jx - Jz, y - z\big\rangle \leq 0$ for all $y \in C$.*

Lemma 2.4.29. [3, 48] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space X and let $x \in X$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x) \quad \forall y \in C.$$

Lemma 2.4.30. [83] *Let X be a uniformly convex and uniformly smooth Banach space and let C be a closed and convex subset of X. Then, for points $w, x, y, z \in X$ and a real number $a \in \mathbb{R}$, the set $K := \{v \in C : \phi(v, y) \leq \phi(v, x) + \langle v, Jz - Jw\rangle + a\}$ is closed and convex.*

Lemma 2.4.31. [64] *Let X be a smooth and strictly convex Banach space and let C be a nonempty, closed and convex subset of X. Let T be a mapping from C into itself such that $F(T)$ is nonempty and $\phi(u, Tx) \leq \phi(u, x)$ for all $(u, x) \in F(T) \times C$. Then $F(T)$ is closed and convex.*

Lemma 2.4.32. [48] *Let X be a uniformly convex and uniformly smooth Banach space and C a nonempty, closed and convex subset of X. Then Π_C is uniformly norm-to-norm continuous on every bounded set.*

Let $\{C_n\}$ be a sequence of nonempty, closed and convex subset of a reflexive Banach space X. We define two subsets $\mathrm{s}-\mathrm{Li}_n C_n$ and $\mathrm{w}-\mathrm{Ls}_n C_n$ as follows: $x \in \mathrm{s}-\mathrm{Li}_n C_n$ if and only if there exists $\{x_n\} \subset X$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \mathrm{w}-\mathrm{Ls}_n C_n$ if and only if there exists a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. We define the Mosco convergence [66] of $\{C_n\}$ as follows: If C_0 satisfies that $C_0 = \mathrm{s}-\mathrm{Li}_n C_n = \mathrm{w}-\mathrm{Ls}_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco and we write $C_0 = \mathrm{M}-\lim_{n\to\infty} C_n$ (see [11]).

Lemma 2.4.33. [44] *Let X be a smooth, reflexive and strictly convex Banach space having the Kadec-Klee property. Let $\{R_n\}$ be a sequence of nonempty, closed and convex subset of X. If $R_0 = \mathrm{M}-\lim_{n\to\infty} R_n$ exists and is nonempty, then $\{\Pi_{R_n} x\}$ converges strongly to $\Pi_{R_0} x$ for each $x \in C$.*

We also make use of the following mapping V studied in Alber [3]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2$$

for all $x \in X$ and $x^* \in X^*$, that is, $V(x, x^*) = \phi\big(x, J^{-1}(x^*)\big)$.

Lemma 2.4.34. [54] *Let X be a reflexive, strictly convex and smooth Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*)$$

for all $x \in X$ and $x^*, y^* \in X^*$.

Lemma 2.4.35. [105] *Let C be a closed and convex subset of a uniformly smooth, strictly convex, and reflexive Banach space X, and let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions* (A1)-(A4). *For all $r > 0$ and $x \in X$, define the mapping $T_r : X \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\}.$$

Then, the following statements hold:

(1) *T_r is single-valued;*

(2) *T_r is of firmly nonexpansive-type* [55], i.e., *for all $x, y \in X$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(3) *$F(T_r) = EP(f)$;*

(4) *$EP(f)$ is closed and convex.*

Lemma 2.4.36. [105] *Let C be a closed and convex subset of a smooth, strictly and reflexive Banach space X, let f be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies conditions* (A1) $-$ (A4), *let $r > 0$. Then, for all $x \in X$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x).$$

Lemma 2.4.37. [54] *Let X be a reflexive, strictly convex and smooth Banach space, let $z \in X$ and let $\{t_i\}_{i=1}^m \subset (0, 1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in X such that*

$$\phi\big(z, J^{-1}(\sum_{i=1}^m t_i Jx_i)\big) = \sum_{i=1}^m t_i \phi(z, x_i),$$

then $x_1 = x_2 = \cdots = x_m$.

Let X be a smooth, strictly convex and reflexive Banach space and C a nonempty, closed and convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of relatively quasi-nonexpansive self-mappings of C such that $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. For each $n \in \mathbb{N}$, we consider the mappings V_n, W_n and K_n defined as follows:

$$V_n = \Pi_C J^{-1}(\beta_{0,n} J + \beta_{1,n} JT_1 + \cdots + \beta_{N,n} JT_N), \tag{2.4.8}$$

where Π_C is the generalized projection of X onto C, J is the duality mapping of X and $\beta_{0,n}, \beta_{1,n}, \cdots, \beta_{N,n}$ are real sequences in $(0, 1)$ with $\beta_{0,n} + \cdots + \beta_{N,n} = 1$.

$$\begin{aligned}
U_{1,n} &= \Pi_C J^{-1}\big(\beta_{1,n} JT_1 + (1 - \beta_{1,n})J\big),\\
U_{2,n} &= \Pi_C J^{-1}\big(\beta_{2,n} JT_2 U_{1,n} + (1 - \beta_{2,n})J\big),\\
&\ \ \vdots\\
U_{N-1,n} &= \Pi_C J^{-1}\big(\beta_{N-1,n} JT_{N-1} U_{N-2,n} + (1 - \beta_{N-1,n})J\big),\\
W_n &= U_{N,n} = \Pi_C J^{-1}\big(\beta_{N,n} JT_N U_{N-1,n} + (1 - \beta_{N,n})J\big),
\end{aligned} \tag{2.4.9}$$

where $\beta_{1,n}, \beta_{2,n}, \cdots, \beta_{N,n}$ are real sequences in $(0,1)$. See [3, 5] for a class of relatively nonexpansive mappings.

$$\begin{aligned} U_{1,n} &= \Pi_C J^{-1}\big(\beta_{1,n} JT_1 + (1-\beta_{1,n})J\big), \\ U_{2,n} &= \Pi_C J^{-1}\big(\beta_{2,n} JT_2U_{1,n} + (1-\beta_{2,n})JU_{1,n}\big), \\ &\vdots \\ U_{N-1,n} &= \Pi_C J^{-1}\big(\beta_{N-1,n} JT_{N-1}U_{N-2,n} + (1-\beta_{N-1,n})JU_{N-2,n}\big), \\ K_n &= U_{N,n} = \Pi_C J^{-1}\big(\beta_{N,n} JT_NU_{N-1,n} + (1-\beta_{N,n})JU_{N-1,n}\big), \quad (2.4.10) \end{aligned}$$

where $\beta_{1,n}, \beta_{2,n}, \cdots, \beta_{N,n}$ are real sequences in $(0,1)$.

To study a countable family of relatively quasi-nonexpansive mappings, we make use of the following condition: let C be a closed subset of a Banach space X. A family of mappings $\{T_n\}_{n=1}^{\infty}$ of C into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ is said to satisfy the *$(*)$-condition* [16] if for each sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_n - T_nz_n\| = 0 \ \text{ and } \ z_n \to z \text{ imply } z \in \bigcap_{n=1}^{\infty} F(T_n).$$

We next prove the crucial lemmas concerning the mappings defined as above.

Lemma 2.4.38. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space X. For each $n \in \mathbb{N}$, let V_n be defined as in (2.4.8). Then the followings hold:*

(1) $F(V_n) = \bigcap_{i=1}^{N} F(T_i)$.

(2) *V_n is a relatively quasi-nonexpansive mapping.*

Proof. (1) Since $\bigcap_{i=1}^{N} F(T_i) \subset F(V_n)$ is obvious, it suffices to show that $F(V_n) \subset \bigcap_{i=1}^{N} F(T_i)$. To this end, let $q \in F(V_n)$ and $p \in \bigcap_{i=1}^{N} F(T_i)$. So we have by the definition of ϕ that

$$\begin{aligned} \phi(p,q) = \phi(p, V_nq) &\leq \phi(p, J^{-1}(\beta_{0,n}Jq + \beta_{1,n}JT_1q + \cdots + \beta_{N,n}JT_Nq)) \\ &\leq \beta_{0,n}\phi(p,q) + \beta_{1,n}\phi(p,T_1q) + \cdots + \beta_{N,n}\phi(p,T_Nq) \\ &\leq \phi(p,q). \end{aligned}$$

By Lemma 2.4.37, we get that $q = T_1q = \cdots = T_Nq$. Thus $q \in \bigcap_{i=1}^{N} F(T_i)$.

(2) The proof is directly obtained from (1). □

Lemma 2.4.39. *Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space X. For each $n \in \mathbb{N}$, let $V_n : C \to C$ be defined as in (2.4.8) and let $0 < a \leq \beta_{0,n} \leq b < 1$ and $0 < a \leq \beta_{i,n} \leq b < 1$ for all $i = 1, 2, \cdots, N$. If either T_i is closed for all $i = 1, 2, \cdots, N$ or $\widehat{F}(T_i) = F(T_i)$ for all $i = 1, 2, \cdots, N$, then $\{V_n\}$ satisfies the $(*)$-condition.*

Proof. Let $p \in \bigcap_{i=1}^{N} F(T_i)$ and let $\{z_n\}$ be a sequence in C such that $z_n \to z$ and $\lim_{n\to\infty} \|z_n - V_nz_n\| = 0$. From Lemma 2.3.5 and X^* is uniformly convex, then there exists a strictly increasing, continuous and convex function g^* with $g^*(0) = 0$ such that

$$\begin{aligned}
\phi(p, V_n z_n) &\leq \phi(p, J^{-1}(\beta_{0,n} Jz_n + \beta_{1,n} JT_1 z_n + \cdots + \beta_{N,n} JT_N z_n)) \\
&= \|p\|^2 - 2\langle p, \beta_{0,n} Jz_n + \beta_{1,n} JT_1 z_n + \cdots + \beta_{N,n} JT_N z_n \rangle \\
&\quad + \|\beta_{0,n} Jz_n + \beta_{1,n} JT_1 z_n + \cdots + \beta_{N,n} JT_N z_n\|^2 \\
&\leq \beta_{0,n}\phi(p, z_n) + \beta_{1,n}\phi(p, T_1 z_n) + \cdots + \beta_{N,n}\phi(p, T_N z_n) \\
&\quad - \beta_{0,n}\beta_{1,n} g^*(\|Jz_n - JT_1 z_n\|) \\
&\leq \phi(p, z_n) - \beta_{0,n}\beta_{1,n} g^*(\|Jz_n - JT_1 z_n\|),
\end{aligned}$$

which implies

$$\begin{aligned}
\beta_{0,n}\beta_{1,n} g^*(\|Jz_n - JT_1 z_n\|) &\leq \phi(p, z_n) - \phi(p, V_n z_n) \\
&= \|z_n\|^2 - \|V_n z_n\|^2 - 2\langle p, JV_n z_n - Jz_n \rangle \\
&\leq \|z_n - V_n z_n\|(\|z_n\| + \|V_n z_n\|) \\
&\quad + 2\|p\|\|Jz_n - JV_n z_n\|.
\end{aligned}$$

Since $\beta_{0,n}\beta_{1,n} \geq a^2 > 0$, $\{z_n\}$ is bounded and $\lim_{n\to\infty} \|z_n - V_n z_n\| = 0$, it follows from the properties of g^* that

$$\lim_{n\to\infty} \|Jz_n - JT_1 z_n\| = 0.$$

Hence, by the uniform continuity of J, we also have

$$\lim_{n\to\infty} \|z_n - T_1 z_n\| = 0.$$

By changing the role of vectors and proving in the same way, we can conclude that

$$\lim_{n\to\infty} \|z_n - T_2 z_n\| = \cdots = \lim_{n\to\infty} \|z_n - T_N z_n\| = 0.$$

Hence $\lim_{n\to\infty} \|z_n - T_i z_n\| = 0$ for all $i = 1, 2, \cdots, N$. If T_i is closed for all $i = 1, 2, \cdots, N$, then $z \in \bigcap_{i=1}^N F(T_i)$. On the other hand, we see that $z \in \widehat{F}(T_i)$ for all $i = 1, 2, \cdots, N$. So if $\widehat{F}(T_i) = F(T_i)$ for all $i = 1, 2, \cdots, N$, then $z \in \bigcap_{i=1}^N F(T_i)$. By Lemma 2.4.38 (1), we get that $z \in F(V_n)$. Thus, the proof is complete. □

Lemma 2.4.40. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space X. For each $n \in \mathbb{N}$, let W_n be defined as in (2.4.9). Then $F(W_n) = \bigcap_{i=1}^N F(T_i)$.*

Proof. $\bigcap_{i=1}^N F(T_i) \subset F(W_n)$ is obvious. Let $q \in F(W_n)$ and $p \in \bigcap_{i=1}^N F(T_i)$. Then

$$\begin{aligned}
\phi(p, q) &= \phi(p, W_n q) \\
&\leq \phi\Big(p, J^{-1}\big(\beta_{N,n} JT_N U_{N-1,n} q + (1 - \beta_{N,n}) Jq\big)\Big) \\
&\leq \beta_{N,n}\phi(p, T_N U_{N-1,n} q) + (1 - \beta_{N,n})\phi(p, q) \\
&\leq \beta_{N,n}\phi(p, U_{N-1,n} q) + (1 - \beta_{N,n})\phi(p, q) \\
&\;\;\vdots \\
&\leq \beta_{N,n}\Bigg(\beta_{N-1,n}\bigg(\cdots\Big(\beta_{3,n}\phi\big(p, J^{-1}(\beta_{2,n} JT_2 U_{1,n} q + (1 - \beta_{2,n}) Jq)\big) \\
&\quad + (1 - \beta_{3,n})\phi(p, q)\Big) + \cdots\bigg) + (1 - \beta_{N-1,n})\phi(p, q)\Bigg) + (1 - \beta_{N,n})\phi(p, q)
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_{N,n}\Bigg(\beta_{N-1,n}\Bigg(\cdots\beta_{3,n}\Big(\beta_{2,n}\phi(p,T_2U_{1,n}q)+(1-\beta_{2,n})\phi(p,q)\Big)\\
&\quad +(1-\beta_{3,n})\phi(p,q)\Big)+\cdots\Bigg)+(1-\beta_{N-1,n})\phi(p,q)\Bigg)+(1-\beta_{N,n})\phi(p,q)\\
&\leq \beta_{N,n}\Bigg(\beta_{N-1,n}\Bigg(\cdots\beta_{3,n}\Big(\beta_{2,n}\phi(p,U_{1,n}q)+(1-\beta_{2,n})\phi(p,q)\Big)\\
&\quad +(1-\beta_{3,n})\phi(p,q)\Big)+\cdots\Bigg)+(1-\beta_{N-1,n})\phi(p,q)\Bigg)+(1-\beta_{N,n})\phi(p,q)\\
&\leq \beta_{N,n}\Bigg(\beta_{N-1,n}\Bigg(\cdots\Big(\beta_{2,n}\phi\big(p,J^{-1}(\beta_{1,n}JT_1q+(1-\beta_{1,n})Jq)\big)\\
&\quad +(1-\beta_{2,n})\phi(p,q)\Big)+\cdots\Bigg)+(1-\beta_{N-1,n})\phi(p,q)\Bigg)+(1-\beta_{N,n})\phi(p,q)\\
&\leq \beta_{N,n}\Bigg(\beta_{N-1,n}\Bigg(\cdots\Big(\beta_{2,n}\big(\beta_{1,n}\phi(p,T_1q)+(1-\beta_{1,n})\phi(p,q)\big)\\
&\quad +(1-\beta_{2,n})\phi(p,q)\Big)+\cdots\Bigg)+(1-\beta_{N-1,n})\phi(p,q)\Bigg)+(1-\beta_{N,n})\phi(p,q)\\
&\leq \beta_{N,n}\Bigg(\beta_{N-1,n}\Bigg(\cdots\Big(\beta_{2,n}\big(\beta_{1,n}\phi(p,q)+(1-\beta_{1,n})\phi(p,q)\big)\\
&\quad +(1-\beta_{2,n})\phi(p,q)\Big)+\cdots\Bigg)+(1-\beta_{N-1,n})\phi(p,q)\Bigg)+(1-\beta_{N,n})\phi(p,q)\\
&= \phi(p,q). \qquad (2.4.11)
\end{aligned}
$$

This shows that

$$\beta_{1,n}\phi(p,T_1q)+(1-\beta_{1,n})\phi(p,q)=\phi\Big(p,J^{-1}\big(\beta_{1,n}JT_1q+(1-\beta_{1,n})Jq\big)\Big).$$

From Lemma 2.4.37, we have $q=T_1q$ and hence $q\in F(T_1)$. Again, from (2.4.11) we see that

$$\beta_{2,n}\phi(p,T_2U_{1,n}q)+(1-\beta_{2,n})\phi(p,U_{1,n}q)=\phi(p,J^{-1}(\beta_{2,n}JT_2U_{1,n}q+(1-\beta_{2,n})Jq).$$

We note, by (2.4.9), that $U_{1,n}q=q$ for all $n\in\mathbb{N}$. So Lemma 2.4.37 implies that $q=T_2q$; consequently $q\in F(T_2)$. Similarly, we can show that $q\in F(T_i)$ for all $i=3,4,\cdots,N$. Hence $q\in\bigcap_{i=1}^{N}F(T_i)$. This completes the proof. □

Lemma 2.4.41. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space X. For each $n\in\mathbb{N}$, let W_n be defined as in (2.4.9). Then W_n is a relatively quasi-nonexpansive mapping.*

Proof. Lemma 2.4.40 asserts that $F(W_n)=\bigcap_{i=1}^{N}F(T_i)\neq\emptyset$. Let $x\in C$ and $p\in\bigcap_{i=1}^{N}F(T_i)$. Since T_i is relatively quasi-nonexpansive for all $i=1,2,\cdots,N$, it is easy to see that $\phi(p,W_nx)\leq\phi(p,x)$. Thus W_n is a relatively quasi-nonexpansive mapping. □

Lemma 2.4.42. *Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space X. For each $n \in \mathbb{N}$, let W_n be defined as in (2.4.9) and let $0 < a \leq \beta_{i,n} \leq b < 1$ for all $i = 1, 2, \cdots, N$. If either T_i is closed for all $i = 1, 2, \cdots, N$ or $\widehat{F}(T_i) = F(T_i)$ for all $i = 1, 2, \cdots, N$, then $\{W_n\}$ satisfies the $(*)$-condition.*

Proof. Let $p \in \bigcap_{i=1}^{N} F(T_i)$ and let $\{z_n\} \subset C$ be such that $z_n \to z$ and $\lim_{n\to\infty} \|z_n - W_n z_n\| = 0$. From Lemma 2.3.5 and X^* is uniformly convex, then there exists a strictly increasing, continuous and convex function g^* with $g^*(0) = 0$ such that

$$\begin{aligned}
\phi(p, W_n z_n) &\leq \phi\Big(p, J^{-1}\big(\beta_{N,n} J T_N U_{N-1,n} z_n + (1 - \beta_{N,n}) J z_n\big)\Big) \\
&= \|p\|^2 - 2\big\langle p, \beta_{N,n} J T_N U_{N-1,n} z_n + (1 - \beta_{N,n}) J z_n \big\rangle \\
&\quad + \|\beta_{N,n} J T_N U_{N-1,n} z_n + (1 - \beta_{N,n}) J z_n\|^2 \\
&\leq \|p\|^2 - 2\big\langle p, \beta_{N,n} J T_N U_{N-1,n} z_n + (1 - \beta_{N,n}) J z_n \big\rangle \\
&\quad + \beta_{N,n}\|T_N U_{N-1,n} z_n\|^2 + (1 - \beta_{N,n})\|z_n\|^2 \\
&\quad - \beta_{N,n}(1 - \beta_{N,n}) g^*(\|J z_n - J T_N U_{N-1,n} z_n\|) \\
&= \beta_{N,n}\phi(p, T_N U_{N-1,n} z_n) + (1 - \beta_{N,n})\phi(p, z_n) \\
&\quad - \beta_{N,n}(1 - \beta_{N,n}) g^*(J z_n - J T_N U_{N-1,n} z_n\|) \\
&\leq \beta_{N,n}\phi(p, U_{N-1,n} z_n) + (1 - \beta_{N,n})\phi(p, z_n) \\
&\quad - \beta_{N,n}(1 - \beta_{N,n}) g^*(\|J z_n - J T_N U_{N-1,n} z_n\|) \\
&\leq \beta_{N,n}\Big(\beta_{N-1,n}\phi(p, U_{N-2,n} z_n) + (1 - \beta_{N-1,n})\phi(p, z_n)\Big) \\
&\quad + (1 - \beta_{N,n})\phi(p, z_n) \\
&\quad - \beta_{N,n}\beta_{N-1,n}(1 - \beta_{N-1,n}) g^*(\|J z_n - J T_{N-1} U_{N-2,n} z_n\|) \\
&\quad - \beta_{N,n}(1 - \beta_{N,n}) g^*(\|J z_n - J T_N U_{N-1,n} z_n\|) \\
&\;\vdots \\
&\leq \phi(p, z_n) - \prod_{i=1}^{N} \beta_{i,n}(1 - \beta_{1,n}) g^*(\|J z_n - J T_1 z_n\|) \\
&\quad - \prod_{i=2}^{N} \beta_{i,n}(1 - \beta_{2,n}) g^*(\|J z_n - J T_2 U_{1,n} z_n\|) \\
&\quad - \cdots - \beta_{N,n}\beta_{N-1,n}(1 - \beta_{N-1,n}) g^*(\|J z_n - J T_{N-1} U_{N-2,n} z_n\|) \\
&\quad - \beta_{N,n}(1 - \beta_{N,n}) g^*(\|J z_n - J T_N U_{N-1,n} z_n\|), \qquad (2.4.12)
\end{aligned}$$

which implies

$$\prod_{i=1}^{N} \beta_{i,n}(1 - \beta_{1,n}) g^*(\|J z_n - J T_1 z_n\|) \;\leq\; \phi(p, z_n) - \phi(p, W_n z_n).$$

Since $\prod_{i=1}^{N} \beta_{i,n}(1 - \beta_{1,n}) \geq a^N(1 - b) > 0$ and $\lim_{n\to\infty} \|z_n - W_n z_n\| = 0$, it follows from the properties of g^* and the uniform continuity of J that

$$\lim_{n\to\infty} \|z_n - T_1 z_n\| = 0.$$

If T_i is closed for all $i = 1, 2, \cdots, N$, we get that $z \in F(T_1)$. Also, by (2.4.12), we obtain

$$\lim_{n\to\infty} \|z_n - T_2 U_{1,n} z_n\| = \cdots = \lim_{n\to\infty} \|z_n - T_N U_{N-1,n} z_n\| = 0. \qquad (2.4.13)$$

We observe

$$\begin{aligned}\phi(z_n, U_{1,n}z_n) &\le \phi\Big(z_n, J^{-1}\big(\beta_{1,n}JT_1z_n + (1-\beta_{1,n})Jz_n\big)\Big)\\ &\le \beta_{1,n}\phi(z_n, T_1z_n) + (1-\beta_{1,n})\phi(z_n, z_n).\end{aligned}$$

So, by Lemma 2.4.26, we get

$$\lim_{n\to\infty}\|z_n - U_{1,n}z_n\| = 0. \tag{2.4.14}$$

Therefore $U_{1,n}z_n \to z$ as $n \to \infty$. From (4.2.11) and (2.4.14) we see that

$$\|T_2U_{1,n}z_n - U_{1,n}z_n\| \le \|T_2U_{1,n}z_n - z_n\| + \|z_n - U_{1,n}z_n\| \to 0, \tag{2.4.15}$$

as $n \to \infty$. Since T_2 is closed, $z \in F(T_2)$. Similarly, we can show that $z \in F(T_i)$ for all $i = 3, 4, \cdots, N$. Thus $z \in \bigcap_{i=1}^N F(T_i)$. From Lemma 2.4.40, we can conclude that $z \in F(W_n)$. If $\widehat{F}(T_i) = F(T_i)$ for all $i = 1, 2, \cdots, N$, by using the same proof as in the first case, we can show that $z \in F(W_n)$. This completes the proof. □

Lemma 2.4.43. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space X. For each $n \in \mathbb{N}$, let K_n be defined as in* (2.4.10). *Then $F(K_n) = \bigcap_{i=1}^N F(T_i)$.*

Proof. Let $q \in F(K_n)$ and let $p \in \bigcap_{i=1}^N F(T_i)$. So we have

$$\begin{aligned}\phi(p,q) &= \phi(p, K_nq)\\ &\le \phi\Big(p, J^{-1}\big(\beta_{N,n}JT_NU_{N-1,n}q + (1-\beta_{N,n})JU_{N-1,n}q\big)\Big)\\ &\le \beta_{N,n}\phi(p, T_NU_{N-1,n}q) + (1-\beta_{N,n})\phi(p, U_{N-1,n}q)\\ &\le \phi(p, U_{N-1,n}q)\\ &\ \vdots\\ &\le \phi\Big(p, J^{-1}\big(\beta_{2,n}JT_2U_{1,n}q + (1-\beta_{2,n})JU_{1,n}q\big)\Big)\\ &\le \beta_{2,n}\phi(p, T_2U_{1,n}q) + (1-\beta_{2,n})\phi(p, U_{1,n}q)\\ &\le \phi(p, U_{1,n}q)\\ &\le \phi\Big(p, J^{-1}\big(\beta_{1,n}JT_1q + (1-\beta_{1,n})Jq\big)\Big)\\ &\le \beta_{1,n}\phi(p, T_1q) + (1-\beta_{1,n})\phi(p,q)\\ &\le \phi(p,q),\end{aligned} \tag{2.4.16}$$

which implies $\beta_{1,n}\phi(p, T_1q) + (1-\beta_{1,n})\phi(p,q) = \phi\Big(p, J^{-1}\big(\beta_{1,n}JT_1q + (1-\beta_{1,n})Jq\big)\Big)$. By Lemma 2.4.37, we obtain that $q = T_1q$ and hence $q \in F(T_1)$. From (5.1.18) we see that $\beta_{2,n}\phi(p, T_2U_{1,n}q) + (1-\beta_{2,n})\phi(p, U_{1,n}q) = \phi(p, J^{-1}(\beta_{2,n}JT_2U_{1,n}q + (1-\beta_{2,n})JU_{1,n}q)$. Also, Lemma 2.4.37 implies that $U_{1,n}q = T_2U_{1,n}q$. Since $q = U_{1,n}q$ by (2.4.10), we have $q \in F(T_2)$. Similarly, we can verify that $q \in F(T_i)$ for all $i = 3, 4, \cdots, N$ and hence $q \in \bigcap_{i=1}^N F(T_i)$. This completes the proof. □

Lemma 2.4.44. *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space X. For each $n \in \mathbb{N}$, let K_n be defined as in* (2.4.10). *Then K_n is a relatively quasi-nonexpansive mapping.*

Proof. From Lemma 2.4.43 and by the relative quasi-nonexpansiveness of T_i for all $i = 1, 2, \cdots, N$, we immediately obtain the desired result. □

Lemma 2.4.45. *Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space X. For each $n \in \mathbb{N}$, let K_n be defined as in (2.4.10) and let $0 < a \leq \beta_{i,n} \leq b < 1$ for all $i = 1, 2, \cdots, N$. If either T_i is closed for all $i = 1, 2, \cdots, N$ or $\widehat{F}(T_i) = F(T_i)$ for all $i = 1, 2, \cdots, N$, then $\{K_n\}$ satisfies the $(*)$-condition.*

Proof. Let $p \in \bigcap_{i=1}^{N} F(T_i)$ and let $\{z_n\}$ be a sequence in C such that $z_n \to z$ and $\lim_{n\to\infty} \|z_n - K_n z_n\| = 0$. For each $n \in \mathbb{N}$, let $U_{0,n}$ be the identity mapping. From Lemma 2.3.5 and X^* is uniformly convex, then there exists a strictly increasing, continuous and convex function g^* with $g^*(0) = 0$ such that

$$\begin{aligned}
\phi(p, K_n z_n) &\leq \phi\Big(p, J^{-1}\big(\beta_{N,n} JT_N U_{N-1,n} z_n + (1-\beta_{N,n}) JU_{N-1,n} z_n\big)\Big) \\
&= \|p\|^2 - 2\langle p, \beta_{N,n} JT_N U_{N-1,n} z_n + (1-\beta_{N,n}) JU_{N-1,n} z_n\rangle \\
&\quad + \|\beta_{N,n} JT_N U_{N-1,n} z_n + (1-\beta_{N,n}) JU_{N-1,n} z_n\|^2 \\
&\leq \|p\|^2 - 2\langle p, \beta_{N,n} JT_N U_{N-1,n} z_n + (1-\beta_{N,n}) JU_{N-1,n} z_n\rangle \\
&\quad + \beta_{N,n}\|T_N U_{N-1,n} z_n\|^2 + (1-\beta_{N,n})\|U_{N-1,n} z_n\|^2 \\
&\quad - \beta_{N,n}(1-\beta_{N,n}) g^*(\|U_{N-1,n} z_n - T_N U_{N-1,n} z_n\|) \\
&= \beta_{N,n}\phi(p, T_N U_{N-1,n} z_n) + (1-\beta_{N,n})\phi(p, U_{N-1,n} z_n) \\
&\quad - \beta_{N,n}(1-\beta_{N,n}) g^*(\|JU_{N-1,n} z_n - JT_N U_{N-1,n} z_n\|) \\
&\leq \phi(p, U_{N-1,n} z_n) - \beta_{N,n}(1-\beta_{N,n}) g^*(\|JU_{N-1,n} z_n - JT_N U_{N-1,n} z_n\|) \\
&\ \ \vdots \\
&\leq \phi(p, z_n) - \sum_{i=1}^{N} \beta_{i,n}(1-\beta_{i,n}) g^*(\|JT_i U_{i-1,n} z_n - JU_{i-1,n} z_n\|),
\end{aligned}$$

which yields

$$\sum_{i=1}^{N} \beta_{i,n}(1-\beta_{i,n}) g^*(\|JT_i U_{i-1,n} z_n - JU_{i-1,n} z_n\|) \ \leq\ \phi(p, z_n) - \phi(p, K_n z_n).$$

Since $\beta_{i,n}(1-\beta_{i,n}) \geq a(1-b) > 0$ for all $i = 1, 2, \cdots, N$ and $\lim_{n\to\infty} \|z_n - K_n z_n\| = 0$, it follows from the properties of g^* and the uniform continuity of J that

$$\lim_{n\to\infty} \|T_i U_{i-1,n} z_n - U_{i-1,n} z_n\| = 0, \tag{2.4.17}$$

for all $i = 1, 2, \cdots, N$. In particular, we have

$$\lim_{n\to\infty} \|T_1 z_n - z_n\| = 0. \tag{2.4.18}$$

We observe that

$$\begin{aligned}
\phi(z_n, U_{1,n} z_n) &\leq \phi\Big(z_n, J^{-1}\big(\beta_{1,n} JT_1 z_n + (1-\beta_{1,n}) Jz_n\big)\Big) \\
&\leq \beta_{1,n}\phi(z_n, T_1 z_n) + (1-\beta_{1,n})\phi(z_n, z_n),
\end{aligned}$$

which implies from Lemma 2.4.26 and (2.4.18) that

$$\lim_{n\to\infty} \|z_n - U_{1,n} z_n\| = 0. \tag{2.4.19}$$

From (4.2.16) and (2.4.17), we have

$$\|T_2 U_{1,n} z_n - z_n\| \leq \|T_2 U_{1,n} z_n - U_{1,n} z_n\| + \|U_{1,n} z_n - z_n\| \to 0, \tag{2.4.20}$$

as $n \to \infty$. Hence

$$\begin{aligned}\phi(z_n, U_{2,n}z_n) &\leq \phi\Big(z_n, J^{-1}\big(\beta_{2,n}JT_2U_{1,n}z_n + (1-\beta_{2,n})JU_{1,n}z_n\big)\Big)\\ &\leq \beta_{2,n}\phi(z_n, T_2U_{1,n}z_n) + (1-\beta_{2,n})\phi(z_n, U_{1,n}z_n),\end{aligned}$$

which implies from (4.2.16) and (2.4.20) that

$$\lim_{n\to\infty} \|z_n - U_{2,n}z_n\| = 0.$$

By proving in the same manner, we can conclude that

$$\lim_{n\to\infty} \|z_n - U_{i,n}z_n\| = 0.$$

for all $i = 1, 2, \cdots, N$. Since $z_n \to z$, $U_{i,n}z_n \to z$. If T_i is closed for all $i = 1, 2, \cdots, N$, it follows from (2.4.17) that $z \in \bigcap_{i=1}^{N} F(T_i)$. If $\widehat{F}(T_i) = F(T_i)$ for all $i = 1, 2, \cdots, N$, by (2.4.17), $z \in \bigcap_{i=1}^{N} \widehat{F}(T_i) = \bigcap_{i=1}^{N} F(T_i)$. From Lemma 2.4.43, we can conclude that $z \in F(K_n)$. This completes the proof. □