Chapter 3
Approximation Methods for Common Fixed Points of Strict Pseudocontractions in Banach Spaces

In this chapter, we study Mann-type and Halpern-type iterations for a countable family of strict pseudocontractions in the framework of Banach spaces. We prove that the sequences generated by the proposed algorithms converge weakly and strongly to common fixed points of mappings.

3.1 Weak Convergence Theorems for a Countable Family of Strict Pseudocontractions in Banach Spaces

Let X be a Banach space and let C be a nonempty, closed and convex subset of X. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a countable family of mappings. We consider the following Mann-type iteration: $x_1 \in C$ and

$$x_{n+1} \;=\; (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1, \tag{3.1.1}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

Further, we also consider the following Halpern-type iteration: $x_1 \in C$ and

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n T_n x_n, \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \quad n \geq 1, \end{cases} \tag{3.1.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in $(0, 1)$.

Proposition 3.1.1. *Let X be a Banach space and let C be a nonempty, closed and convex subset of X. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that the control sequence $\{\alpha_n\} \subset (0, 1)$ satisfies the following conditions:*

(a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(b) $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$.

Let $\{x_n\}$ be generated by (3.1.1). If $\{T_n\}$ satisfies the AKTT-condition, then

(1) $\lim_{n\to\infty} \|x_n - p\|$ *exists for all* $p \in F$;

(2) $\liminf_{n\to\infty} \|x_n - T_n x_n\| = 0$.

Proof. Let $p \in F$ and put $L = \frac{\lambda+1}{\lambda}$. First, we observe that

$$\|x_{n+1} - p\| \leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|T_n x_n - p\| \leq (1+L)\|x_n - p\|$$

and

$$\|x_{n+1} - x_n\| = \alpha_n \|T_n x_n - x_n\| \leq \alpha_n (1 + L)\|x_n - p\|.$$

Since T_n is a λ-strict pseudocontraction, there exists $j(x_{n+1} - p) \in J(x_{n+1} - p)$ such that

$$\langle (I - T_n)x_{n+1} - (I - T_n)p, j(x_{n+1} - p)\rangle \geq \lambda \|(I - T_n)x_{n+1} - (I - T_n)p\|^2,$$

which implies

$$\langle x_{n+1} - T_n x_{n+1}, j(x_{n+1} - p)\rangle \geq \lambda \|x_{n+1} - T_n x_{n+1}\|^2.$$

From Lemma 2.3.1 (2), it follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(x_n - p) + \alpha_n (T_n x_n - x_n)\|^2 \\
&\leq \|x_n - p\|^2 + 2\alpha_n \langle T_n x_n - x_n, j(x_{n+1} - p)\rangle \\
&= \|x_n - p\|^2 + 2\alpha_n \langle T_n x_n - T_n x_{n+1}, j(x_{n+1} - p)\rangle \\
&\quad + 2\alpha_n \langle T_n x_{n+1} - x_{n+1}, j(x_{n+1} - p)\rangle + 2\alpha_n \langle x_{n+1} - x_n, j(x_{n+1} - p)\rangle \\
&\leq \|x_n - p\|^2 + 2\alpha_n L \|x_n - x_{n+1}\| \|x_{n+1} - p\| \\
&\quad - 2\alpha_n \lambda \|T_n x_{n+1} - x_{n+1}\|^2 + 2\alpha_n \|x_n - x_{n+1}\| \|x_{n+1} - p\| \\
&\leq \|x_n - p\|^2 + 2\alpha_n^2 L (1 + L)^2 \|x_n - p\|^2 \\
&\quad - 2\alpha_n \lambda \|T_n x_{n+1} - x_{n+1}\|^2 + 2\alpha_n^2 (1 + L)^2 \|x_n - p\|^2 \qquad (3.1.3) \\
&= \|x_n - p\|^2 + 2\alpha_n^2 (1 + L)^3 \|x_n - p\|^2 - 2\alpha_n \lambda \|T_n x_{n+1} - x_{n+1}\|^2.
\end{aligned}$$

This implies that

$$\|x_{n+1} - p\|^2 \leq (1 + 2\alpha_n^2 (1 + L)^3)\|x_n - p\|^2.$$

Hence, by condition (b), we have from Lemma 2.4.9 that $\lim_{n\to\infty} \|x_n - p\|$ exists; consequently, $\{x_n\}$ is bounded. Moreover, by (3.1.3), we also have

$$\sum_{n=1}^{\infty} \alpha_n \lambda \|T_n x_{n+1} - x_{n+1}\|^2 \leq \sum_{n=1}^{\infty} \Big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2 \Big) + 2(1+L)^3 M_1^2 \sum_{n=1}^{\infty} \alpha_n^2 < +\infty,$$

where $M_1 = \sup_{n\geq 1}\{\|x_n - p\|\}$. It follows, by condition (a), that $\liminf_{n\to\infty} \|T_n x_{n+1} - x_{n+1}\| = 0$. Further, since $\{x_n\}$ is bounded,

$$\begin{aligned}
\|x_{n+1} - T_{n+1} x_{n+1}\| &\leq \|x_{n+1} - T_n x_{n+1}\| + \|T_n x_{n+1} - T_{n+1} x_{n+1}\| \\
&\leq \|x_{n+1} - T_n x_{n+1}\| + \sup_{z \in \{x_n\}} \|T_n z - T_{n+1} z\|.
\end{aligned}$$

Since $\{T_n\}$ satisfies the AKTT-condition, it follows that $\liminf_{n\to\infty} \|x_n - T_n x_n\| = 0$. This completes the proof of (1) and (2). □

In our more general setting, in this section we will assume that:

$$\beta^*(t) \leq 2t, \qquad (3.1.4)$$

where β^* is a function appearing in Lemma 2.4.14.

Proposition 3.1.2. *Let X be a Banach space with the Fréchet differentiable norm and let C be a nonempty, closed and convex subset of X. Let $\{T_n\}_{n=1}^{\infty}: C \to C$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that the control sequence $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:*

(a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(b) $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$.

Let $\{x_n\}$ be generated by (3.1.1). *If $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition, then*

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = \lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Proof. Let $p \in F$ and put $M_2 = \sup_{n\geq 1}\big\{\|x_n - T_n x_n\|\big\} > 0$. Then by (2.4.3) and (3.1.4) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(x_n - p) + \alpha_n(T_n x_n - x_n)\|^2 \\
&\leq \|x_n - p\|^2 + 2\alpha_n\langle T_n x_n - x_n, j(x_n - p)\rangle \\
&\quad + \alpha_n\|T_n x_n - x_n\|\beta^*(\alpha_n\|T_n x_n - x_n\|) \\
&\leq \|x_n - p\|^2 - 2\alpha_n\lambda\|x_n - T_n x_n\|^2 + 2\alpha_n^2\|x_n - T_n x_n\|^2 \\
&\leq \|x_n - p\|^2 - 2\alpha_n\lambda\|x_n - T_n x_n\|^2 + 2\alpha_n^2 M_2^2.
\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \alpha_n\|x_n - T_n x_n\|^2 < +\infty.$$

Observe that

$$\begin{aligned}
\|x_n - T_{n+1}x_{n+1}\|^2 &= \|(x_n - T_n x_n) + (T_n x_n - T_{n+1}x_{n+1})\|^2 \\
&\leq \|x_n - T_n x_n\|^2 + 2\langle T_n x_n - T_{n+1}x_{n+1}, j(x_n - T_{n+1}x_{n+1})\rangle \\
&= \|x_n - T_n x_n\|^2 + 2\langle T_n x_n - T_n x_{n+1}, j(x_n - T_{n+1}x_{n+1})\rangle \\
&\quad + 2\langle T_n x_{n+1} - T_{n+1}x_{n+1}, j(x_n - T_{n+1}x_{n+1})\rangle \\
&\leq \|x_n - T_n x_n\|^2 + 2L\|x_n - x_{n+1}\|\|x_n - T_{n+1}x_{n+1}\| \\
&\quad + 2\|T_n x_{n+1} - T_{n+1}x_{n+1}\|\|x_n - T_{n+1}x_{n+1}\| \\
&\leq \|x_n - T_n x_n\|^2 + 2L\|x_n - x_{n+1}\|\|x_n - T_n x_n\| \\
&\quad + 2L\|x_n - x_{n+1}\|\|T_n x_n - T_n x_{n+1}\| \\
&\quad + 2L\|x_n - x_{n+1}\|\|T_n x_{n+1} - T_{n+1}x_{n+1}\| \\
&\quad + 2\|T_n x_{n+1} - T_{n+1}x_{n+1}\|\|x_n - x_{n+1}\| \\
&\quad + 2\|T_n x_{n+1} - T_{n+1}x_{n+1}\|\|x_{n+1} - T_{n+1}x_{n+1}\| \\
&\leq \|x_n - T_n x_n\|^2 + (2L\alpha_n + 2L^2\alpha_n^2)\|x_n - T_n x_n\|^2 \\
&\quad + (2LM_2\alpha_n + 2M_2\alpha_n + 2M_2)\|T_n x_{n+1} - T_{n+1}x_{n+1}\| \\
&\leq \|x_n - T_n x_n\|^2 + 2L(1+L)\alpha_n\|x_n - T_n x_n\|^2 \\
&\quad + 2M_2(L+2)\|T_n x_{n+1} - T_{n+1}x_{n+1}\|. \qquad (3.1.5)
\end{aligned}$$

From (3.1.5) we have

$$
\begin{aligned}
\|x_{n+1}-T_{n+1}x_{n+1}\|^2 &\le (1-\alpha_n)\|x_n-T_{n+1}x_{n+1}\|^2+\alpha_n\|T_nx_n-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_{n+1}x_{n+1}\|^2\\
&\quad+\alpha_n\big(\|T_nx_n-T_nx_{n+1}\|+\|T_nx_{n+1}-T_{n+1}x_{n+1}\|\big)^2\\
&= \|x_n-T_{n+1}x_{n+1}\|^2+\alpha_n\|T_nx_n-T_nx_{n+1}\|^2\\
&\quad+2\alpha_n\|T_nx_n-T_nx_{n+1}\|\|T_nx_{n+1}-T_{n+1}x_{n+1}\|\\
&\quad+\alpha_n\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_{n+1}x_{n+1}\|^2+\alpha_n^2L^2\|x_n-T_nx_n\|^2\\
&\quad+2\alpha_n^2L\|x_n-T_nx_n\|\|T_nx_{n+1}-T_{n+1}x_{n+1}\|\\
&\quad+\alpha_n\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_{n+1}x_{n+1}\|^2+\alpha_n^2L^2M_2^2\\
&\quad+2LM_2\|T_nx_{n+1}-T_{n+1}x_{n+1}\|+\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_nx_n\|^2+2L(1+L)\alpha_n\|x_n-T_nx_n\|^2\\
&\quad+2M_2(L+2)\|T_nx_{n+1}-T_{n+1}x_{n+1}\|+\alpha_n^2L^2M_2^2\\
&\quad+2LM_2\|T_nx_{n+1}-T_{n+1}x_{n+1}\|+\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_nx_n\|^2+2L(1+L)\alpha_n\|x_n-T_nx_n\|^2\\
&\quad+\alpha_n^2L^2M_2^2+2M_2(2L+2)\|T_nx_{n+1}-T_{n+1}x_{n+1}\|\\
&\quad+\|T_nx_{n+1}-T_{n+1}x_{n+1}\|^2\\
&\le \|x_n-T_nx_n\|^2+2L(1+L)\alpha_n\|x_n-T_nx_n\|^2\\
&\quad+\alpha_n^2L^2M_2^2+2M_2(2L+2)\sup_{z\in\{x_n\}}\|T_nz-T_{n+1}z\|\\
&\quad+\sup_{z\in\{x_n\}}\|T_nz-T_{n+1}z\|^2.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}\alpha_n\|x_n-T_nx_n\|^2<+\infty$, $\sum_{n=1}^{\infty}\alpha_n^2<+\infty$ and $\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z-T_nz\|:z\in\{x_n\}\}<+\infty$, it follows that $\lim_{n\to\infty}\|x_n-T_nx_n\|$ exists. From Proposition 3.1.1 (2), we can conclude that $\lim_{n\to\infty}\|x_n-T_nx_n\|=0$. Since

$$
\begin{aligned}
\|x_n-Tx_n\| &\le \|x_n-T_nx_n\|+\|T_nx_n-Tx_n\|\\
&\le \|x_n-T_nx_n\|+\sup_{z\in\{x_n\}}\|T_nz-Tz\|,
\end{aligned}
$$

it follows from Lemma 2.4.15 that $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. This completes the proof. □

Theorem 3.1.3. *Let X be a uniformly convex Banach space with the Fréchet differentiable norm and let C be a nonempty, closed and convex subset of X. Let $\{T_n\}_{n=1}^{\infty}:C\to C$ be a family of λ-strict pseudocontractions for some $0<\lambda<1$ such that $F:=\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. Assume that the control sequence $\{\alpha_n\}\subset(0,\lambda]$ satisfies the following conditions:*

(a) $\sum_{n=1}^{\infty}\alpha_n=+\infty$;

(b) $\sum_{n=1}^{\infty}\alpha_n^2<+\infty$.

If $\big(\{T_n\},T\big)$ satisfies the AKTT-condition, then $\{x_n\}$ generated by (3.1.1) converges weakly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Proof. Let $p \in F$ and define $S_n : C \to C$ by

$$S_n x = (1-\alpha_n)x + \alpha_n T_n x, \quad x \in C.$$

Then $\bigcap_{n=1}^{\infty} F(S_n) = F = F(T)$. From (2.4.3), we have for bounded $x, y \in C$ that

$$\begin{aligned}
\|S_n x - S_n y\|^2 &= \|x - y - \alpha_n[x - y - (T_n x - T_n y)]\|^2 \\
&\le \|x-y\|^2 - 2\alpha_n\langle (I-T_n)x - (I-T_n)y, j(x-y)\rangle \\
&\quad +\alpha_n\|x - y - (T_n x - T_n y)\|\beta^*\big(\alpha_n\|x - y - (T_n x - T_n y)\|\big) \\
&\le \|x-y\|^2 - 2\alpha_n\lambda\|x - y - (T_n x - T_n y)\|^2 \\
&\quad + 2\alpha_n^2\|x - y - (T_n x - T_n y)\|^2 \\
&= \|x-y\|^2 - 2\alpha_n(\lambda - \alpha_n)\|x - y - (T_n x - T_n y)\|^2 \\
&\le \|x-y\|^2.
\end{aligned}$$

This implies that S_n is nonexpansive. From Proposition 3.1.1 (1), $\{x_n\}$ is bounded. From Proposition 3.1.2, we know that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Applying Lemma 2.4.4, we also have $\omega_\omega(x_n) \subset F(T)$.

Finally, we will show that $\omega_\omega(x_n)$ is a singleton. Suppose that $x^*, y^* \in \omega_\omega(x_n) \subset F(T)$. Hence $x^*, y^* \in \bigcap_{n=1}^{\infty} F(S_n)$. By Lemma 2.4.13, $\lim_{n\to\infty}\langle x_n, j(x^* - y^*)\rangle$ exists. Suppose that $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. Then

$$\|x^* - y^*\|^2 = \langle x^* - y^*, j(x^* - y^*)\rangle = \lim_{k\to\infty}\langle x_{n_k} - x_{m_k}, j(x^* - y^*)\rangle = 0.$$

Hence $x^* = y^*$; consequently, $x_n \rightharpoonup x^* \in \bigcap_{n=1}^{\infty} F(S_n) = F$. This completes the proof. □

Using Lemma 2.4.19 we obtain the following result:

Corollary 3.1.4. *Let X be a uniformly convex Banach space with the Fréchet differentiable norm and let C be a nonempty, closed and convex subset of X. Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of λ_k-strict pseudocontractions of C into itself such that $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$ and* $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. *Define the sequence $\{x_n\}$ by $x_1 \in C$,*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n\sum_{k=1}^{n}\mu_n^k S_k x_n, \quad n \ge 1,$$

where $\{\alpha_n\} \subset (0, \lambda]$ satisfying $\sum_{n=1}^{\infty}\alpha_n = +\infty$ and $\sum_{n=1}^{\infty}\alpha_n^2 < +\infty$, and $\{\mu_n^k\}$ satisfies conditions (1)-(3) *of Lemma* 2.4.19. *Then $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}_{k=1}^{\infty}$.*

Remark 3.1.5. Theorem 3.1.3 and Corollary 3.1.4 extend and improve Theorem 3.3 and Theorem 3.4 of Chidume-Shahzad [34] in the following senses:

(1) from real uniformly smooth and uniformly convex Banach spaces to real uniformly convex Banach spaces with Fréchet differentiable norms;

(2) from finite strict pseudocontractions to infinite strict pseudocontractions.

Using Opial's condition, we also obtain the results in a reflexive Banach space.

Theorem 3.1.6. *Let X be a Fréchet smooth and reflexive Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of X. Let $\{T_n\}_{n=1}^{\infty}$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that the control sequence $\{\alpha_n\} \subset (0, \lambda]$ satisfies the following conditions:*

(a) $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(b) $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$.

If $(\{T_n\}, T)$ satisfies the AKTT-condition, then $\{x_n\}$ generated by (3.1.1) converges weakly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

Proof. Let $p \in F$. From Proposition 3.1.1 (1), we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. Since X has the Fréchet differentiable norm, by Proposition 3.1.2, we know that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. It follows from Proposition 2.4.5 that $\omega_\omega(x_n) \subset F(T) = F$. Finally, we show that $\omega_\omega(x_n)$ is a singleton. Let $x^*, y^* \in \omega_\omega(x_n)$ and $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be subsequences of $\{x_n\}$ chosen such that $x_{n_k} \rightharpoonup x^*$ and $x_{m_k} \rightharpoonup y^*$. If $x^* \neq y^*$, then Opial's condition of X implies that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{k\to\infty} \|x_{n_k} - x^*\| < \lim_{k\to\infty} \|x_{n_k} - y^*\| = \lim_{k\to\infty} \|x_{m_k} - y^*\| \\ &< \lim_{k\to\infty} \|x_{m_k} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|.\end{aligned}$$

This is a contradiction and thus the proof is complete. □

Corollary 3.1.7. *Let X be a Fréchet smooth and reflexive Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of X. Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of λ_k-strict pseudocontractions of C into itself such that $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Define the sequence $\{x_n\}$ by $x_1 \in C$,*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n \sum_{k=1}^{n} \mu_n^k S_k x_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subset [0, \lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$, and $\{\mu_n^k\}$ satisfies conditions (1)-(3) of Lemma 2.4.19. Then $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}_{k=1}^{\infty}$.

3.2 Weak and Strong Convergence Theorems for a Countable Family of Strict Pseudocontractions in Banach Spaces

In this section, we study weak and strong convergence of Mann-type and Halpern-type iterations for a countable family of strict pseudocontractions in uniformly smooth Banach spaces. We give an affirmative answer raised by Zhou [122] in 2010.

Proposition 3.2.1. *Let X be a uniformly smooth Banach space and C a nonempty, closed and convex subset of X. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that $\Phi(t) \leq 2t^2$, $t \in [0, \infty)$ where Φ is a function appearing in (2.3.1). Let $\{\alpha_n\}$ be a real sequence in $(0, \lambda]$ which* *satisfies the conditions* (i) $\sum_{n=1}^{\infty} \alpha_n = +\infty$ *and* (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$. *Let* $\{x_n\}$ *be generated by* (3.1.1). *If* $\big(\{T_n\}, T\big)$ *satisfies the AKTT-condition, then*

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = \lim_{n\to\infty} \|x_n - T x_n\| = 0.$$

Proof. Since, by Proposition 3.1.1 (1), $\{x_n\}$ is bounded, there exists $r > 0$ such that $\{x_n\} \subset G := \overline{B_r(0)} \cap C$. For given $\{\alpha_n\} \subset (0, \lambda]$, we define

$$\beta_n = \frac{\lambda - \alpha_n}{\lambda},$$

then $\beta_n \in [0, 1)$ for all $n \geq 1$, $\alpha_n = \lambda(1 - \beta_n)$ and $1 - \alpha_n = 1 - \lambda(1 - \beta_n)$. Hence,

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_n x_n \\ &= [1 - \lambda(1-\beta_n)]x_n + \lambda(1-\beta_n)T_n x_n \\ &= \beta_n x_n + (1-\beta_n)[(1-\lambda)x_n + \lambda T_n x_n] \\ &= \beta_n x_n + (1-\beta_n)C_n x_n, \end{aligned}$$

where $C_n = (1-\lambda)I + \lambda T_n : G \to C$. By Lemma 2.4.20, we know that C_n is nonexpansive for all $n \geq 1$. Moreover, we see that $\{C_n\}$ satisfies the AKTT-condition. Indeed,

$$\sup_{z\in B} \|C_{n+1}z - C_n z\| = \lambda \sup_{z\in B} \|T_{n+1}z - T_n z\|,$$

which implies $\sum_{n=1}^{\infty} \sup_{z\in B} \|C_{n+1}z - C_n z\| < +\infty$. On the other hand, we see that

$$\begin{aligned} \|x_{n+1} - C_{n+1}x_{n+1}\| &\leq \beta_n\|x_n - C_{n+1}x_{n+1}\| + (1-\beta_n)\|C_n x_n - C_{n+1}x_{n+1}\| \\ &\leq \beta_n\|x_n - C_n x_n\| + \beta_n\|C_n x_n - C_{n+1}x_n\| \\ &\quad + \beta_n\|C_{n+1}x_n - C_{n+1}x_{n+1}\| + (1-\beta_n)\|C_n x_n - C_{n+1}x_n\| \\ &\quad + (1-\beta_n)\|C_{n+1}x_n - C_{n+1}x_{n+1}\| \\ &\leq \beta_n\|x_n - C_n x_n\| + \|C_n x_n - C_{n+1}x_n\| + \|x_{n+1} - x_n\| \\ &\leq \|x_n - C_n x_n\| + \sup_{z\in\{x_n\}} \|C_n z - C_{n+1}z\|. \end{aligned}$$

This implies that $\lim_{n\to\infty} \|x_n - C_n x_n\|$ exists. Hence $\lim_{n\to\infty} \|x_n - T_n x_n\|$ also exists. From Proposition 3.1.1, we conclude that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Since

$$\begin{aligned} \|x_n - T x_n\| &\leq \|x_n - T_n x_n\| + \|T_n x_n - T x_n\| \\ &\leq \|x_n - T_n x_n\| + \sup_{z\in\{x_n\}} \|T_n z - T z\|, \end{aligned}$$

it follows from Lemma 2.4.15 that $\lim_{n\to\infty} \|x_n - T x_n\| = 0$. This completes the proof. □

Theorem 3.2.2. *Let* C *be a nonempty, closed and convex subset of a uniformly smooth Banach space* X *that either is uniformly convex or satisfies Opial's condition. Let* $\{T_n\}_{n=1}^{\infty} : C \to C$ *be a family of* λ*-strict pseudocontractions for some* $0 < \lambda < 1$ *such that* $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. *Assume that* $\Phi(t) \leq 2t^2$, $t \in [0, \infty)$ *where* Φ *is a function appearing in* (2.3.1). *Let* $\{\alpha_n\}$ *be a real sequence in* $(0, \lambda]$ *which satisfies the conditions* (i) $\sum_{n=1}^{\infty} \alpha_n = +\infty$ *and* (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$. *Let* $\{x_n\}$ *be generated by* (3.1.1). *If* $\big(\{T_n\}, T\big)$ *satisfies the AKTT-condition, then* $\{x_n\}$ *converges weakly to a common fixed point of* $\{T_n\}_{n=1}^{\infty}$.

Proof. First, we assume that X is uniformly convex. Since X is uniformly smooth, X has a Fréchet differentiable norm. Define the mapping $S_n : C \to C$ by

$$S_n x = (1-\alpha_n)x + \alpha_n T_n x, \quad x \in C.$$

Then $\bigcap_{n=1}^{\infty} F(S_n) = F = F(T)$ and S_n is nonexpansive on G for all $n \geq 1$. From Proposition 3.1.1 (1), we know that $\{x_n\}$ is bounded. From Proposition 3.2.1, we also know that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The uniform convexity and the Opial's condition of E ensure that the sequence $\{x_n\}$ weakly converges to a common fixed point of $\{T_n\}_{n=1}^{\infty}$. This completes the proof. □

As a direct consequence of Theorem 3.2.2, Lemma 2.4.18 and Lemma 2.4.19 we also obtain the following result.

Corollary 3.2.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space X that either is uniformly convex or satisfies Opial's condition. Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of λ_k-strict pseudocontractions of C into itself such that $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Assume that $\Phi(t) \leq 2t^2,\ t \in [0,\infty)$ where Φ is a function appearing in (2.3.1). Define the sequence $\{x_n\}$ by $x_1 \in C$,*

$$x_{n+1} \;=\; (1-\alpha_n)x_n + \alpha_n \sum_{k=1}^{n} \mu_n^k S_k x_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0,\lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$, and $\{\mu_n^k\}$ satisfies conditions (1)-(3) in Lemma 2.4.19. Then $\{x_n\}$ converges weakly to a common fixed point of $\{S_k\}_{k=1}^{\infty}$.

Theorem 3.2.4. *Let C be a nonempty, bounded, closed and convex subset of a uniformly smooth Banach space X that either is uniformly convex or satisfies Opial's condition. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that $\Phi(t) \leq 2t^2,\ t \in [0,\infty)$ where Φ is a function appearing in (2.3.1). Let $\{\alpha_n\}$ be a real sequence in $(0,\lambda]$ which satisfies $\sum_{n=1}^{\infty} \alpha_n(\lambda - \alpha_n) = +\infty$. If $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition, then $\{x_n\}$ generated by (3.1.1) converges weakly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.*

Proof. For each $n \geq 1$, define $C_n x = (1-\lambda)x + \lambda T_n x$. Since C is bounded, it follows from Lemma 2.4.20 that C_n is nonexpansive and $F(C_n) = F(T_n)$ for all $n \geq 1$. For given $\{\alpha_n\} \subset (0,\lambda]$, we define

$$\beta_n = \frac{\lambda - \alpha_n}{\lambda}.$$

Then (3.1.2) reduces to

$$x_{n+1} = \beta_n x_n + (1-\beta_n) C_n x_n, \quad n \geq 1. \tag{3.2.1}$$

Let $p \in F = \bigcap_{n=1}^{\infty} F(T_n)$. Then $p \in \bigcap_{n=1}^{\infty} C_n$. Hence

$$\begin{aligned} \|x_{n+1} - p\| &\leq \beta_n \|x_n - p\| + (1-\beta_n)\|C_n x_n - p\| \\ &\leq \|x_n - p\|, \end{aligned}$$

which implies that $\lim_{n\to\infty}\|x_n - p\|$ exists. On the other hand, we see that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(x_n-p)+\alpha_n(T_nx_n-x_n)\|^2\\ &\leq \|x_n-p\|^2+2\alpha_n\langle T_nx_n-x_n, j(x_n-p)\rangle+\Phi(\alpha_n\|T_nx_n-x_n\|)\\ &\leq \|x_n-p\|^2-2\lambda\alpha_n\|T_nx_n-x_n\|^2+2\alpha_n^2\|T_nx_n-x_n\|^2\\ &= \|x_n-p\|^2-2\alpha_n(\lambda-\alpha_n)\|T_nx_n-x_n\|^2,\end{aligned}$$

which yields

$$\alpha_n(\lambda-\alpha_n)\|T_nx_n-x_n\|^2 \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2.$$

Since $\lim_{n\to\infty}\|x_n-p\|$ exists and $\sum_{n=1}^{\infty}\alpha_n(\lambda-\alpha_n)=+\infty$, it follows that

$$\liminf_{n\to\infty}\|T_nx_n-x_n\|=0.$$

By using the same argument as in the proof of Proposition 3.1.2, we obtain that $\lim_{n\to\infty}\|T_nx_n-x_n\|$ exists and hence

$$\lim_{n\to\infty}\|T_nx_n-x_n\|=0.$$

Since $\{T_n\}$ satisfies the AKTT-condition, we can verify that

$$\lim_{n\to\infty}\|Tx_n-x_n\|=0.$$

Applying Lemma 2.4.5 and Lemma 2.4.13, we can prove that $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$. □

Remark 3.2.5. We give an affirmative answer raised by Zhou [122]. In fact, Theorems 3.2.2 and 3.2.4 mainly extend Theorem 2.1 of Zhou [122] (i) from q-uniformly smooth Banach spaces to uniformly smooth Banach spaces, and (ii) from a strict pseudocontraction to a countable family of strict pseudocontractions.

We next prove a strong convergence theorem of Halpern-type iteration for a countable family of strict pseudocontractions in the framework of uniformly smooth Banach spaces.

Theorem 3.2.6. *Let C be a nonempty, bounded, closed and convex subset of a uniformly smooth Banach space X. Let $\{T_n\}_{n=1}^{\infty}: C\to C$ be a family of λ-strict pseudocontractions for some $0<\lambda<1$ such that $F:=\bigcap_{n=1}^{\infty}F(T_n)\neq\emptyset$. Assume that $\Phi(t)\leq 2t^2$, $t\in[0,\infty)$ where Φ is a function appearing in (2.3.1). Given $u, x_1\in C$ and sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$, the following control conditions are satisfied:*

(a) *$a\leq\alpha_n\leq\lambda$ for some $a>0$ and for all $n\geq 1$;*
(b) *$\beta_n+\gamma_n+\delta_n=1$ for all $n\geq 1$;*
(c) *$\beta_n\to 0$ as $n\to\infty$, $\sum_{n=1}^{\infty}\beta_n=+\infty$;*
(d) *$\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<+\infty$;*
(e) *$0<\liminf_{n\to\infty}\gamma_n\leq\limsup_{n\to\infty}\gamma_n<1$.*

Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition. Then $\{x_n\}$ generated by (3.1.2) converges strongly to a common fixed point z of $\{T_n\}_{n=1}^{\infty}$, where $z=Q_Fu$ and $Q_F: C\to F$ is the unique sunny nonexpansive retraction from C onto F.

Proof. For each $n \geq 1$, define $S_n x = (1-\alpha_n)x + \alpha_n T_n x$, $x \in C$. From Lemma 2.4.20 and condition (a), we know that S_n is nonexpansive and $F(T_n) = F(S_n)$ for all $n \geq 1$. Moreover, $\{S_n\}$ satisfies the AKTT-condition. In fact, for any bounded subset B of C,

$$\begin{aligned}\sup_{z\in B}\|S_{n+1}z - S_n z\| &= \sup_{z\in B}\|((1-\alpha_{n+1})z + \alpha_{n+1}T_{n+1}z) - ((1-\alpha_n)z + \alpha_n T_n z)\| \\ &\leq |\alpha_{n+1}-\alpha_n|\sup_{z\in B}\|z\| + \alpha_{n+1}\sup_{z\in B}\|T_{n+1}z - T_n z\| \\ &\quad + |\alpha_{n+1}-\alpha_n|\sup_{z\in B}\|T_n z\| \\ &= |\alpha_{n+1}-\alpha_n|\big(\sup_{z\in B}\|z\| + \sup_{z\in B}\|T_n z\|\big) + \sup_{z\in B}\|T_{n+1}z - T_n z\|.\end{aligned}$$

From condition (d) and $\{T_n\}$ satisfies the AKTT-condition, we get

$$\sum_{n=1}^{\infty}\sup_{z\in B}\|S_{n+1}z - S_n z\| < +\infty. \tag{3.2.2}$$

Next, we show that $\|x_{n+1}-x_n\| \to 0$ and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$. First, observe that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|S_{n+1}x_{n+1} - S_n x_n\| \\ &\leq \|S_{n+1}x_{n+1} - S_{n+1}x_n\| + \|S_{n+1}x_n - S_n x_n\| \\ &\leq \|x_{n+1}-x_n\| + \sup_{z\in\{x_n\}}\|S_{n+1}z - S_n z\|.\end{aligned} \tag{3.2.3}$$

We now define $\omega_n = \frac{x_{n+1}-\gamma_n x_n}{1-\gamma_n}$. From (3.1.2) we have

$$\begin{aligned}\|\omega_{n+1}-\omega_n\| &= \left\|\frac{\beta_{n+1}u + \delta_{n+1}y_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n u + \delta_n y_n}{1-\gamma_n}\right\| \\ &\leq \left|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right|(\|u\| + \|y_n\|) + \frac{\delta_{n+1}}{1-\gamma_{n+1}}\|y_{n+1}-y_n\| \\ &\leq M\left|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right| + \|y_{n+1}-y_n\|,\end{aligned} \tag{3.2.4}$$

for some $M > 0$. Combining (3.2.3) and (3.2.4), we obtain

$$\|\omega_{n+1}-\omega_n\| \leq M\left|\frac{\beta_{n+1}}{1-\gamma_{n+1}} - \frac{\beta_n}{1-\gamma_n}\right| + \|x_{n+1}-x_n\| + \sup_{z\in\{x_n\}}\|S_{n+1}z - S_n z\|.$$

It follows from conditions (c), (e) and (3.2.2) that

$$\limsup_{n\to\infty}\big(\|\omega_{n+1}-\omega_n\| - \|x_{n+1}-x_n\|\big) \leq 0.$$

From Lemma 2.4.7, we have $\|\omega_n - x_n\| \to 0$ and hence $\|x_{n+1}-x_n\| = (1-\gamma_n)\|\omega_n - x_n\| \to 0$ as $n \to \infty$. It follows from (3.1.2), (c) and (e) that $\|y_n - x_n\| \to 0$. Note that $\|x_n - T_n x_n\| = \frac{1}{\alpha_n}\|y_n - x_n\|$. So from condition (a), we have

$$\lim_{n\to\infty}\|x_n - T_n x_n\| = 0.$$

Since $\{S_n\}$ satisfies the AKTT-condition, we can define $S : C \to C$ by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$. It is easy to see that S is nonexpansive and $F(S) = F(T) = F$. We observe that

$$\|x_n - S_n x_n\| = \alpha_n\|x_n - T_n x_n\| \to 0,$$

as $n \to \infty$. It follows that

$$\begin{aligned} \|x_n - Sx_n\| &\leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \\ &\leq \|x_n - S_n x_n\| + \sup_{z \in K} \|S_n z - Sz\| \to 0, \end{aligned}$$

as $n \to \infty$. For $t \in (0,1)$, we define a contraction as follows:

$$S_t x = tu + (1-t)Sx.$$

Then there exists a unique path $x_t \in C$ such that

$$x_t = tu + (1-t)Sx_t.$$

From Lemma 2.4.11 we know that $x_t \to z \in F(S)$ as $t \to 0$. Further, if we define $Q_{F(S)}u = z$, then $Q_{F(S)} : C \to F(S)$ is the unique sunny nonexpansive retraction from C onto $F(S)$. Since $F(S) = F$, we obtain that $Q_F : C \to F$ is the unique sunny nonexpansive retraction from C onto F. Note that X is uniformly smooth, S is nonexpansive and $\|x_n - Sx_n\| \to 0$ as $n \to \infty$. So, by Lemma 2.4.12, we obtain

$$\limsup_{n \to \infty} \langle u - Q_F u, J(x_n - Q_F u) \rangle \leq 0.$$

Finally, we show that $x_n \to z = Q_F u$ as $n \to \infty$. From (3.1.2) we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \beta_n \langle u - z, j(x_{n+1} - z) \rangle + \gamma_n \langle x_n - z, j(x_{n+1} - z) \rangle \\ &\quad + \delta_n \langle y_n - z, j(x_{n+1} - z) \rangle \\ &\leq \beta_n \langle u - z, j(x_{n+1} - z) \rangle + \gamma_n \|x_n - z\| \|x_{n+1} - z\| \\ &\quad + \delta_n \|y_n - z\| \|x_{n+1} - z\| \\ &\leq \beta_n \langle u - z, j(x_{n+1} - z) \rangle + (1 - \beta_n) \|x_n - z\| \|x_{n+1} - z\| \\ &\leq \beta_n \langle u - z, j(x_{n+1} - z) \rangle + \frac{1}{2}(1 - \beta_n) \|x_n - z\|^2 + \frac{1}{2} \|x_{n+1} - z\|^2. \end{aligned}$$

This implies that

$$\|x_{n+1} - z\|^2 \leq (1 - \beta_n) \|x_n - z\|^2 + 2\beta_n \langle u - z, j(x_{n+1} - z) \rangle.$$

From Lemma 2.4.8, we conclude that $x_n \to z$ as $n \to \infty$. This completes the proof. □

As a direct consequence of Theorem 3.2.6, Lemma 2.4.18 and Lemma 2.4.19 we obtain the following result.

Corollary 3.2.7. *Let C be a nonempty, bounded, closed and convex subset of a uniformly smooth Banach space X. Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of λ_k-strict pseudocontractions of C into itself such that $\bigcap_{k=1}^{\infty} F(S_k) \neq \emptyset$ and $\inf\{\lambda_k : k \in \mathbb{N}\} = \lambda > 0$. Assume that $\Phi(t) \leq 2t^2,\ t \in [0, \infty)$ where Φ is a function appearing in* (2.3.1). *For a given $u \in C$. Define the sequence $\{x_n\}$ by $x_1 \in C$,*

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n \sum_{k=1}^{n} \mu_n^k S_k x_n, \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ are real sequences in $(0,1)$ which satisfy the conditions (a)-(e) *in Theorem* 5.2.4 *and $\{\mu_n^k\}$ are real sequences which satisfy the conditions* (1)-(3) *in Lemma* 2.4.19. *Then $\{x_n\}$ converges strongly to a common fixed point of $\{S_k\}_{k=1}^{\infty}$.*

Remark 3.2.8. We give an affirmative answer raised by Zhou [122]. In fact, Theorem 3.2.6 and Corollary 3.2.7 mainly extend Theorem 2.3 of Zhou [122] (i) from q-uniformly smooth Banach spaces to uniformly smooth Banach spaces, and (ii) from a strict pseudocontraction to a countable family of strict pseudocontractions.

Remark 3.2.9. If $\lambda = \frac{1}{2}$, then for each $n \geq 1$ we can choose $\alpha_n = \delta_n = \frac{1}{2} - \frac{1}{n+2}$, $\beta_n = \frac{1}{n+2}$ and $\gamma_n = \frac{1}{2}$.