Chapter 4
Equilibrium Problems and Fixed Points of Some Generalized Nonexpansive Mappings

In this chapter, we study strong convergence of the sequences generated by the proposed algorithms for solving fixed point problems of nonexpansive mappings, quasi-nonexpansive mappings and strict pseudocontractions, and equilibrium problems. The obtained results improve and extend those announced by many authors.

4.1 A New Hybrid Algorithm for Variational Inclusions, Mixed Equilibrium Problems and a Finite Family of Quasi-nonexpansive Mappings

Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $f : C \times C \to \mathbb{R}$ be a bifunction, and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be real-valued function. For solving the equilibrium problem, let us give the following assumptions for f, φ and the set C:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, *i.e.*, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $y \in C,\ x \mapsto f(x, y)$ is weakly upper semi-continuous;
(A4) for each $x \in C,\ y \mapsto f(x, y)$ is convex;
(A5) for each $x \in C,\ y \mapsto f(x, y)$ is lower semi-continuous;
(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \cap dom\varphi$ such that for any $z \in C \setminus D_x$,

$$f(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) C is a bounded set.

Proposition 4.1.1. [25] *Let C be a nonempty, closed and convex subset of a Hilbert H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A5) *and let $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous and convex function such that $C \cap dom\varphi \neq \emptyset$. For $r > 0$ and $x \in H$, define a mapping $S_r : H \to C$ as follows:*

$$S_r(x) = \Big\{z \in C : f(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C\Big\}.$$

Assume that either (B1) *or* (B2) *holds. Then, the following conclusions hold:*

(1) *for each $x \in H,\ S_r(x) \neq \emptyset$;*
(2) *S_r is single-valued;*
(3) *S_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle;$$

(4) $F(S_r) = MEP(f, \varphi)$;
(5) $MEP(f, \varphi)$ *is closed and convex.*

Theorem 4.1.2. *Let C be a nonempty, closed and convex subset of a Hilbert space H, $f : C \times C \to \mathbb{R}$ a bifunction satisfying* (A1) – (A5), *$\varphi : C \to \mathbb{R} \cup \{+\infty\}$ a proper, lower semi-continuous and convex function, $A : H \to H$ an α-inverse strongly monotone mapping, $M : H \to 2^H$ a maximal monotone mapping, and $\{T_i\}_{i=1}^N$ a finite family of quasi-nonexpansive and L_i-Lipschitz mappings on C. Assume that $\Omega := \bigcap_{i=1}^N F(T_i) \cap MEP(f, \varphi) \cap I(A, M) \neq \emptyset$ and either* (B1) *or* (B2) *holds. Let W_n be the W-mapping generated by $T_1, T_2, \cdots, T_N$ and $\beta_{n,1}, \beta_{n,2}, \cdots, \beta_{n,N}$. For $x_0 \in H$ with $C_1 = C$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be defined by*

$$\begin{cases} f(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) W_n u_n, \\ z_n = J_{M,\lambda_n}(y_n - \lambda_n A y_n), \\ C_{n+1} = \big\{z \in C_n : \|z_n - z\| \leq \|y_n - z\| \leq \|x_n - z\|\big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{r_n\} \subset [b, \infty)$ for some $b \in (0, \infty)$ and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$.

Then $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_\Omega x_0$.

Proof. Since $0 < c \leq \lambda_n \leq d < 2\alpha$ for all $n \in \mathbb{N}$, $J_{M,\lambda_n}(I - \lambda_n A)$ is nonexpansive for all $n \in \mathbb{N}$. Hence $\bigcap_{n=1}^\infty F\big(J_{M,\lambda_n}(I - \lambda_n A)\big) = I(A, M)$ is closed and convex. From Proposition 4.1.1 (5), we know that $MEP(f, \varphi)$ is closed and convex. From Lemma 2.4.22, we also know that $F := \bigcap_{i=1}^N F(T_i)$ is closed and convex. Hence $\Omega := \bigcap_{i=1}^N F(T_i) \cap MEP(f, \varphi) \cap I(A, M)$ is a nonempty, closed and convex set.

Next, we divide the proof into seven steps.

Step 1. Show that $\Omega \subset C_n$ for all $n \in \mathbb{N}$.

First observe that C_n is closed and convex for all $n \in \mathbb{N}$. Let $p \in \Omega$. Since $u_n = S_{r_n} x_n$ and $p = J_{M,\lambda_n}(p - \lambda_n A p)$ for all $n \in \mathbb{N}$,

$$\begin{aligned} \|z_n - p\| &= \|J_{M,\lambda_n}(y_n - \lambda_n A y_n) - J_{M,\lambda_n}(p - \lambda_n A p)\| \\ &\leq \|y_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n)\|W_n u_n - p\| \\ &\leq \alpha_n \|x_n - p\| + (1 - \alpha_n)\|u_n - p\| \\ &= \alpha_n \|x_n - p\| + (1 - \alpha_n)\|S_{r_n} x_n - S_{r_n} p\| \\ &\leq \|x_n - p\|. \end{aligned}$$

It follows that $p \in C_{n+1}$ and hence $\Omega \subset C_n$ for all $n \in \mathbb{N}$.

Step 2. Show that $\lim_{n\to\infty} \|x_n - x_0\|$ exists.

Since Ω is a nonempty, closed and convex subset of C, there exists a unique element $z_0 = P_\Omega x_0 \in \Omega \subset C_n$. Since $x_n = P_{C_n} x_0$,

$$\|x_n - x_0\| \leq \|z_0 - x_0\|. \tag{4.1.1}$$

Hence $\{\|x_n - x_0\|\}$ is bounded. So are $\{y_n\}, \{z_n\}$ and $\{u_n\}$.
Noting $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|. \tag{4.1.2}$$

Combining (4.1.1) and (4.1.2), $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Step 3. Show that $\{x_n\}$ is a Cauchy sequence.

By construction of the set C_n, we know that $x_m = P_{C_m}x_0 \in C_m \subset C_n$ for $m > n$. From Lemma 2.2.51, it follows that

$$\|x_m - x_n\|^2 \leq \|x_m - x_0\|^2 - \|x_n - x_0\|^2 \to 0, \tag{4.1.3}$$

as $m, n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of H and the closedness of C, we can assume that $x_n \to q \in C$.

Step 4. Show that $q \in F$.

From (4.1.3), we get

$$\|x_{n+1} - x_n\| \to 0,$$

as $n \to \infty$. Since $x_{n+1} \in C_{n+1} \subset C_n$,

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \to 0, \tag{4.1.4}$$

as $n \to \infty$. Hence $z_n \to q$ as $n \to \infty$. By the nonexpansiveness of J_{M,λ_n} and the inverse strongly monotonicity of A, we obtain

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|y_n - \lambda_n Ay_n - (p - \lambda_n Ap)\|^2 \\
&\leq \|y_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 + c(d - 2\alpha)\|Ay_n - Ap\|^2.
\end{aligned}$$

This implies

$$\begin{aligned}
c(2\alpha - d)\|Ay_n - Ap\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\
&\leq \|x_n - z_n\|\big(\|x_n - p\| + \|z_n - p\|\big).
\end{aligned}$$

It follows from (4.1.4) that

$$\lim_{n\to\infty}\|Ay_n - Ap\| = 0. \tag{4.1.5}$$

Noting J_{M,λ_n} is 1-inverse strongly monotone, we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|J_{M,\lambda_n}(y_n - \lambda_n Ay_n) - J_{M,\lambda_n}(p - \lambda_n Ap)\|^2 \\
&\leq \big\langle (y_n - \lambda_n Ay_n) - (p - \lambda_n Ap), z_n - p\big\rangle \\
&= \frac{1}{2}\Big(\|(y_n - \lambda_n Ay_n) - (p - \lambda_n Ap)\|^2 + \|z_n - p\|^2 \\
&\quad - \|(y_n - \lambda_n Ay_n) - (p - \lambda_n Ap) - (z_n - p)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|y_n - p\|^2 + \|z_n - p\|^2 - \|(y_n - z_n) - \lambda_n(Ay_n - Ap)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|x_n - p\|^2 + \|z_n - p\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\lambda_n\langle y_n - z_n, Ay_n - Ap\rangle\Big) \\
&\leq \frac{1}{2}\Big(\|x_n - p\|^2 + \|z_n - p\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\lambda_n\|y_n - z_n\|\|Ay_n - Ap\|\Big).
\end{aligned}$$

This implies

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|y_n - z_n\|^2 + 2\lambda_n\|y_n - z_n\|\|Ay_n - Ap\|.$$

It follows that

$$\begin{aligned}\|y_n - z_n\|^2 &\leq \|x_n - z_n\|\big(\|x_n - p\| + \|z_n - p\|\big) \\ &\quad + 2d\|y_n - z_n\|\|Ay_n - Ap\|.\end{aligned}$$

From (4.1.4) and (4.1.5) we get

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{4.1.6}$$

It follows from (4.1.4) and (4.1.6) that

$$\|W_n u_n - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\| \to 0, \tag{4.1.7}$$

as $n \to \infty$. Since S_{r_n} is firmly nonexpansive and $u_n = S_{r_n}x_n$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|S_{r_n}x_n - S_{r_n}p\|^2 \\ &\leq \langle S_{r_n}x_n - S_{r_n}p, x_n - p\rangle \\ &= \langle u_n - p, x_n - p\rangle \\ &= \frac{1}{2}\Big(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\Big),\end{aligned}$$

which implies

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{4.1.8}$$

It follows from (4.1.8) that

$$\begin{aligned}\|y_n - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|W_n u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\big(\|x_n - p\|^2 - \|x_n - u_n\|^2\big) \\ &= \|x_n - p\|^2 - (1-\alpha_n)\|x_n - u_n\|^2,\end{aligned}$$

which yields

$$(1-a)\|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \|y_n - p\|^2.$$

Hence from (4.1.4) and (4.1.6), we also have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{4.1.9}$$

From (4.1.7) and (4.1.9) we get

$$\lim_{n\to\infty} \|u_n - W_n u_n\| = 0.$$

From Lemma 2.4.24 we obtain $\lim_{n\to\infty} \|u_n - Wu_n\| = 0$. From Lemma 2.4.23 (2), we know that W is Lipschitz. Since $u_n \to q$ as $n \to \infty$, $q \in F(W)$. Moreover, from Lemma 2.4.23 (1), we can conclude that $q \in F := \bigcap_{i=1}^{N} F(T_i)$.

Step 5. Show that $q \in MEP(f, \varphi)$.
Noting $u_n = S_{r_n}x_n$, we have

$$f(u_n, y) + \varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \quad \forall y \in C.$$

From (A2), we obtain

$$\varphi(y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n) + \varphi(u_n), \quad \forall y \in C.$$

It follows from (A5) and the weakly lower semi-continuity of φ, $\|x_n - u_n\|/r_n \to 0$, and $u_n \to q$ that

$$f(y, q) + \varphi(q) \leq \varphi(y), \quad \forall y \in C.$$

Put $y_t = ty + (1-t)q$ for each $t \in (0, 1]$ and $y \in C \cap dom\varphi$. Since $y \in C \cap dom\varphi$ and $q \in C \cap dom\varphi$, we obtain $y_t \in C \cap dom\varphi$. Hence $f(y_t, q) + \varphi(q) \leq \varphi(y_t)$. By (A1), (A4) and the convexity of φ, we have

$$\begin{aligned} 0 &= f(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tf(y_t, y) + (1-t)f(y_t, q) + t\varphi(y) + (1-t)\varphi(q) - \varphi(y_t) \\ &\leq t[f(y_t, y) + \varphi(y) - \varphi(y_t)]. \end{aligned}$$

Hence

$$f(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0.$$

By letting $t \to 0$, it follows from (A3) and the weakly semi-continuity of φ that

$$f(q, y) + \varphi(y) \geq \varphi(q)$$

for all $y \in C \cap dom\varphi$. Observe that if $y \in C \setminus dom\varphi$, then $f(q, y) + \varphi(y) \geq \varphi(q)$ holds. Hence $q \in MEP(f, \varphi)$.

Step 6. Show that $q \in I(A, M)$.

First observe that A is an $(1/\alpha)$-Lipschitz monotone mapping and $D(A) = H$. From Lemma 2.4.10, we know that $M + A$ is maximal monotone. Let $(v, g) \in G(M + A)$, that is, $g - Av \in M(v)$. Since $z_n = J_{M,\lambda_n}(y_n - \lambda_n Ay_n)$, we get $y_n - \lambda_n Ay_n \in (I + \lambda_n M)(z_n)$, that is,

$$\frac{1}{\lambda_n}(y_n - z_n - \lambda_n Ay_n) \in M(z_n).$$

By the maximal monotonicity of $M + A$, we have

$$\Big\langle v - z_n, g - Av - \frac{1}{\lambda_n}(y_n - z_n - \lambda_n Ay_n)\Big\rangle \geq 0,$$

and hence

$$\begin{aligned} \langle v - z_n, g\rangle &\geq \Big\langle v - z_n, Av + \frac{1}{\lambda_n}(y_n - z_n - \lambda_n Ay_n)\Big\rangle \\ &= \Big\langle v - z_n, Av - Az_n + Az_n - Ay_n + \frac{1}{\lambda_n}(y_n - z_n)\Big\rangle \\ &\geq 0 + \langle v - z_n, Az_n - Ay_n\rangle + \Big\langle v - z_n, \frac{1}{\lambda_n}(y_n - z_n)\Big\rangle. \end{aligned}$$

It follows from $\|y_n - z_n\| \to 0$, $\|Ay_n - Az_n\| \to 0$ and $z_n \to q$ that

$$\lim_{n\to\infty}\langle v - z_n, g\rangle = \langle v - q, g\rangle \geq 0.$$

By the maximal monotonicity of $M + A$, we have $\theta \in (M + A)(q)$; consequently, $q \in I(A, M)$.

Step 7. Show that $q = z_0 = P_\Omega x_0$.

Since $x_n = P_{C_n} x_0$ and $\Omega \subset C_n$, we obtain

$$\big\langle x_0 - x_n, x_n - p\big\rangle \geq 0 \ \ \forall p \in \Omega. \tag{4.1.10}$$

By taking the limit in (4.1.10), we obtain

$$\big\langle x_0 - q, q - p\big\rangle \geq 0 \ \ \forall p \in \Omega.$$

This shows that $q = P_\Omega x_0 = z_0$.

From Step 1 to Step 7, we can conclude that $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_\Omega x_0$. This completes the proof. □

As a direct consequence of Theorem 4.1.2, we obtain new results in a Hilbert space as follows:

Corollary 4.1.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H, $f : C \times C \to \mathbb{R}$ a bifunction satisfying* (A1) − (A5)*, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ a proper, lower semi-continuous and convex function, $A : C \to H$ an α-inverse strongly monotone mapping, and $\{T_i\}_{i=1}^N$ a finite family of quasi-nonexpansive and L_i-Lipschitz mappings on C. Assume that $\Omega := \bigcap_{i=1}^N F(T_i) \cap MEP(f, \varphi) \cap VI(A, C) \neq \emptyset$ and either* (B1) *or* (B2) *holds. Let W_n be the W-mapping generated by $T_1, T_2, \cdots, T_N$ and $\beta_{n,1}, \beta_{n,2}, \cdots, \beta_{n,N}$. For an initial point $x_0 \in H$ with $C_1 = C$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be defined by*

$$\begin{cases} f(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\big\langle y - u_n, u_n - x_n\big\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) W_n u_n, \\ z_n = P_C(y_n - \lambda_n A y_n), \\ C_{n+1} = \big\{z \in C_n : \|z_n - z\| \leq \|y_n - z\| \leq \|x_n - z\|\big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{r_n\} \subset [b, \infty)$ for some $b \in (0, \infty)$ and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$.

Then $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_\Omega x_0$.

Proof. In Theorem 4.1.2, take $M = \partial\delta_C : H \to 2^H$, where $\delta_C : H \to [0, \infty]$ is the indicator function of C. It is well-known that the subdifferential $\partial\delta_C$ is a maximal monotone operator. Then the resolvent operator $J_{M,\lambda_n} = P_C$ for all $n \in \mathbb{N}$. □

Corollary 4.1.4. *Let C be a nonempty, closed and convex subset of a Hilbert space H, $f : C \times C \to R$ a bifunction satisfying* (A1) − (A5)*, $A : H \to H$ an α-inverse strongly monotone mapping, $M : H \to 2^H$ a maximal monotone mapping and $\{T_i\}_{i=1}^N$ a finite family of quasi-nonexpansive and L_i-Lipschitz mappings on C. Assume that $\Omega := \bigcap_{i=1}^N F(T_i) \cap EP(f) \cap I(A, M) \neq \emptyset$ and either* (B1) *or* (B3) *holds. Let W_n be the W-mapping generated by $T_1, T_2, \cdots, T_N$ and $\beta_{n,1}, \beta_{n,2}, \cdots, \beta_{n,N}$. For $x_0 \in H$ with $C_1 = C$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be defined by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\big\langle y - u_n, u_n - x_n\big\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) W_n u_n, \\ z_n = J_{M,\lambda_n}(y_n - \lambda_n A y_n), \\ C_{n+1} = \big\{z \in C_n : \|z_n - z\| \leq \|y_n - z\| \leq \|x_n - z\|\big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, a]$ *for some* $a \in [0, 1)$, $\{r_n\} \subset [b, \infty)$ *for some* $b \in (0, \infty)$ *and* $\{\lambda_n\} \subset [c, d]$ *for some* $c, d \in (0, 2\alpha)$.

Then $\{x_n\}, \{y_n\}, \{z_n\}$ *and* $\{u_n\}$ *converge strongly to* $z_0 = P_\Omega x_0$.

Remark 4.1.5. Theorem 4.1.2 mainly improves and extends the main results obtained in [71, 99, 104].

4.2 Convergence Analysis for a System of Generalized Equilibrium Problems and a Countable Family of Strict Pseudocontractions

In this section, we study strong convergence of a system of generalized equilibrium problems and a countable family of strict pseudocontractions in Hilbert spaces.

Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping. Let $\{f_k\}_{k=1}^M : C \times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \to H$ be a family of α_k-inverse-strongly monotone mappings and let $r_k \in (0, 2\alpha_k)$. For each $k \in \{1, 2, \cdots, M\}$, we denote the mapping $T_{r_k}^{f_k, A_k} : C \to C$ by

$$T_{r_k}^{f_k, A_k} := T_{r_k}^{f_k}(I - r_k A_k) \tag{4.2.1}$$

where $T_{r_k}^{f_k}$ is the mapping defined as in Lemma 2.4.35. For each $t \in (0, 1)$, we define the mapping $S_t : C \to C$ as follows:

$$S_t x = S T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_1}^{f_1, A_1} P_C[(1 - t)x], \quad \forall x \in C.$$

We see that $T_{r_k}^{f_k}$ and $I - r_k A_k$ are nonexpansive for each $k \in \{1, 2, \cdots, M\}$. So the mapping $T_{r_k}^{f_k, A_k}$ is also nonexpansive for each $k \in \{1, 2, \cdots, M\}$. This implies that S_t is a contraction. Then the Banach contraction principle ensures that there exists a unique fixed point x_t of S_t in C, that is,

$$x_t = S T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_1}^{f_1, A_1} P_C[(1 - t)x_t], \; t \in (0, 1). \tag{4.2.2}$$

Proposition 4.2.1. *Let* C *be a nonempty, closed and convex subset of a Hilbert space* H. *Let* $S : C \to C$ *be a nonexpansive mapping. Let* $\{f_k\}_{k=1}^M : C \times C \to \mathbb{R}$ *be a family of bifunctions, let* $\{A_k\}_{k=1}^M : C \to H$ *be a family of* α_k*-inverse strongly monotone mappings and let* $r_k \in (0, 2\alpha_k)$. *For each* $k \in \{1, 2, \cdots, M\}$, *let the mapping* $T_{r_k}^{f_k, A_k}$ *be defined by* (4.2.1). *Assume that* $F := \left(\bigcap_{k=1}^M GEP(f_k, A_k)\right) \cap F(S) \neq \emptyset$. *For each* $t \in (0, 1)$, *let the net* $\{x_t\}$ *be generated by* (4.2.2). *Then, as* $t \to 0$, *the net* $\{x_t\}$ *converges strongly to an element in* F.

Proof. First, we show that $\{x_t\}$ is bounded. For each $t \in (0, 1)$, let $y_t = P_C[(1 - t)x_t]$ and $u_t = T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_1}^{f_1, A_1} y_t$. From (4.2.2) we have for each $p \in F$ that

$$\|x_t - p\| = \|Su_t - Sp\| \leq \|u_t - p\| \leq \|y_t - p\| \leq (1 - t)\|x_t - p\| + t\|p\|.$$

It follows that

$$\|x_t - p\| \leq \|p\|.$$

Hence $\{x_t\}$ is bounded and so are $\{y_t\}$ and $\{u_t\}$. Observe that

$$\|y_t - x_t\| \leq t\|x_t\| \to 0, \tag{4.2.3}$$

as $t \to 0$ since $\{x_t\}$ is bounded.

Next, we show that $\|u_t - x_t\| \to 0$ as $t \to 0$. Denote $\Theta^k = T_{r_k}^{f_k,A_k} T_{r_{k-1}}^{f_{k-1},A_{k-1}} \cdots T_{r_1}^{f_1,A_1}$ for any $k \in \{1, 2, \cdots, M\}$ and $\Theta^0 = I$. We note that $u_t = \Theta^M y_t$ for each $t \in (0, 1)$. For each $k \in \{1, 2, \cdots, M\}$ and $p \in F$, we see that

$$\begin{aligned}
\|\Theta^k y_t - p\|^2 &= \|T_{r_k}^{f_k,A_k}\Theta^{k-1}y_t - T_{r_k}^{f_k,A_k}\Theta^{k-1}p\|^2 \\
&= \|T_{r_k}^{f_k}(\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t) - T_{r_k}^{f_k}(\Theta^{k-1}p - r_kA_k\Theta^{k-1}p)\|^2 \\
&\leq \|(\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t) - (\Theta^{k-1}p - r_kA_k\Theta^{k-1}p)\|^2 \\
&\leq \|\Theta^{k-1}y_t - p\|^2 + r_k(r_k - 2\alpha_k)\|A_k\Theta^{k-1}y_t - A_kp\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_t - p\|^2 &= \|\Theta^M y_t - p\|^2 \\
&\leq \|y_t - p\|^2 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\|A_i\Theta^{i-1}y_t - A_ip\|^2 \\
&= \|P_C[(1-t)x_t] - p\|^2 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\|A_i\Theta^{i-1}y_t - A_ip\|^2 \\
&\leq \big(\|x_t - p\| + t\|x_t\|\big)^2 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\|A_i\Theta^{i-1}y_t - A_ip\|^2 \\
&\leq \|x_t - p\|^2 + tM_1 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\|A_i\Theta^{i-1}y_t - A_ip\|^2 \qquad (4.2.4)
\end{aligned}$$

where $M_1 = \sup_{0<t<1}\{2\|x_t - p\|\|x_t\| + t\|x_t\|^2\}$. So we have

$$\begin{aligned}
\|x_t - p\|^2 &\leq \|u_t - p\|^2 \\
&\leq \|x_t - p\|^2 + tM_1 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\|A_i\Theta^{i-1}y_t - A_ip\|^2,
\end{aligned}$$

which implies

$$\lim_{t\to 0}\|A_k\Theta^{k-1}y_t - A_kp\| = 0 \tag{4.2.5}$$

for each $k \in \{1, 2, \cdots, M\}$. Since $T_{r_k}^{f_k}$ is firmly nonexpansive for each $k \in \{1, 2, \cdots, M\}$, we have for each $p \in F$ and $k \in \{1, 2, \cdots, M\}$ that

$$\begin{aligned}
\|\Theta^k y_t - p\|^2 &= \|T_{r_k}^{f_k,A_k}\Theta^{k-1}y_t - T_{r_k}^{f_k,A_k}\Theta^{k-1}p\|^2 \\
&= \|T_{r_k}^{f_k}(\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t) - T_{r_k}^{f_k}(\Theta^{k-1}p - r_kA_k\Theta^{k-1}p)\|^2 \\
&\leq \big\langle \Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t - (p - r_kA_kp), \Theta^k y_t - p \big\rangle \\
&= \frac{1}{2}\Big(\|\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t - (p - r_kA_kp)\|^2 + \|\Theta^k y_t - p\|^2 \\
&\quad - \|\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t - (p - r_kA_kp) - (\Theta^k y_t - p)\|^2\Big)
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}\Big(\|\Theta^{k-1}y_t - p\|^2 + \|\Theta^k y_t - p\|^2 \\
&\quad - \|\Theta^{k-1}y_t - \Theta^k y_t - r_k(A_k\Theta^{k-1}y_t - A_k p)\|^2\Big) \\
&\leq \frac{1}{2}\Big(\|\Theta^{k-1}y_t - p\|^2 + \|\Theta^k y_t - p\|^2 - \|\Theta^{k-1}y_t - \Theta^k y_t\|^2 \\
&\quad + 2r_k\|\Theta^{k-1}y_t - \Theta^k y_t\|\|A_k\Theta^{k-1}y_t - A_k p\|\Big).
\end{aligned}$$

This implies

$$\begin{aligned}
\|\Theta^k y_t - p\|^2 &\leq \|\Theta^{k-1}y_t - p\|^2 - \|\Theta^{k-1}y_t - \Theta^k y_t\|^2 \\
&\quad + 2r_k\|\Theta^{k-1}y_t - \Theta^k y_t\|\|A_k\Theta^{k-1}y_t - A_k p\| \\
&\leq \|\Theta^{k-1}y_t - p\|^2 - \|\Theta^{k-1}y_t - \Theta^k y_t\|^2 + M_2\|A_k\Theta^{k-1}y_t - A_k p\|,
\end{aligned}$$

where $M_2 = \max_{1\leq k\leq M}\sup_{0<t<1}\left\{2r_k\|\Theta^{k-1}y_t - \Theta^k y_t\|\right\}$. This shows that

$$\begin{aligned}
\|u_t - p\|^2 &= \|\Theta^M y_t - p\|^2 \\
&\leq \|y_t - p\|^2 - \sum_{i=1}^{M}\|\Theta^{i-1}y_t - \Theta^i y_t\|^2 + M_2\sum_{i=1}^{M}\|A_i\Theta^{i-1}y_t - A_i p\| \\
&\leq \|x_t - p\|^2 + tM_1 - \sum_{i=1}^{M}\|\Theta^{i-1}y_t - \Theta^i y_t\|^2 + M_2\sum_{i=1}^{M}\|A_i\Theta^{i-1}y_t - A_i p\|.
\end{aligned}$$

Hence

$$\begin{aligned}
\|x_t - p\|^2 &\leq \|u_t - p\|^2 \\
&\leq \|x_t - p\|^2 + tM_1 - \sum_{i=1}^{M}\|\Theta^{i-1}y_t - \Theta^i y_t\|^2 + M_2\sum_{i=1}^{M}\|A_i\Theta^{i-1}y_t - A_i p\|.
\end{aligned}$$

From (4.2.5) we obtain

$$\sum_{i=1}^{M}\|\Theta^{i-1}y_t - \Theta^i y_t\| \to 0,$$

as $t \to 0$. So we can conclude that

$$\lim_{t\to 0}\|\Theta^{k-1}y_t - \Theta^k y_t\| = 0 \tag{4.2.6}$$

for each $k \in \{1, 2, \cdots, M\}$. Observing

$$\begin{aligned}
\|u_n - y_t\| &= \|\Theta^M y_t - y_t\| \\
&\leq \|\Theta^M y_t - \Theta^{M-1}y_t\| + \|\Theta^{M-1}y_t - \Theta^{M-2}y_t\| + \cdots + \|\Theta^1 y_t - y_t\|,
\end{aligned}$$

it follows by (4.2.6) that

$$\lim_{t\to 0}\|u_t - y_t\| = 0. \tag{4.2.7}$$

From (4.2.3) and (4.2.7) we have

$$\lim_{t\to 0}\|u_t - x_t\| = 0. \tag{4.2.8}$$

Hence

$$\|x_t - Sx_t\| = \|Su_t - Sx_t\| \leq \|u_t - x_t\| \to 0, \tag{4.2.9}$$

as $t \to 0$.

We next show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (4.2.9) we obtain

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{4.2.10}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to $x^* \in C$. Applying Lemma 2.4.3 to (4.2.10), we can conclude that $x^* \in F(S)$.

Next, we show that $x^* \in \bigcap_{k=1}^{M} GEP(f_k, A_k)$. Note that $\Theta^k y_n = T_{r_k}^{f_k,A_k}\Theta^{k-1}y_n = T_{r_k}^{f_k}(\Theta^{k-1}y_n - r_k A_k \Theta^{k-1}y_n)$ for each $k \in \{1, 2, \cdots, M\}$. Hence, for each $y \in C$ and $k \in \{1, 2, \cdots, M\}$, we obtain

$$f_k(\Theta^k y_n, y) + \frac{1}{r_k}\Big\langle y - \Theta^k y_n, \Theta^k y_n - (\Theta^{k-1}y_n - r_k A_k \Theta^{k-1} y_n)\Big\rangle \geq 0.$$

From (A2) we have

$$\frac{1}{r_k}\Big\langle y - \Theta^k y_n, \Theta^k y_n - (\Theta^{k-1}y_n - r_k A_k \Theta^{k-1} y_n)\Big\rangle \geq f_k(y, \Theta^k y_n), \quad \forall y \in C.$$

Therefore

$$\Big\langle y - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1}y_{n_j}}{r_k} + A_k\Theta^{k-1}y_{n_j}\Big\rangle \geq f_k(y, \Theta^k y_{n_j}), \quad \forall y \in C. \tag{4.2.11}$$

For each $t \in (0,1)$ and $y \in C$, put $z_t = ty + (1-t)x^*$. Then we have $z_t \in C$. From (4.2.11) we get

$$\begin{aligned}
\big\langle z_t - \Theta^k y_{n_j}, A_k z_t\big\rangle &\geq \big\langle z_t - \Theta^k y_{n_j}, A_k z_t\big\rangle \\
&\quad - \Big\langle z_t - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1}y_{n_j}}{r_k} + A_k\Theta^{k-1}y_{n_j}\Big\rangle \\
&\quad + f_k(z_t, \Theta^k y_{n_j}) \\
&= \big\langle z_t - \Theta^k y_{n_j}, A_k z_t - A_k\Theta^k y_{n_j}\big\rangle \qquad (4.2.12)\\
&\quad + \big\langle z_t - \Theta^k y_{n_j}, A_k\Theta^k y_{n_j} - A_k\Theta^{k-1}y_{n_j}\big\rangle \\
&\quad - \Big\langle z_t - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1}y_{n_j}}{r_k}\Big\rangle + f_k(z_t, \Theta^k y_{n_j}).
\end{aligned}$$

We note that $\|A_k\Theta^k y_{n_j} - A_k\Theta^{k-1}y_{n_j}\| \leq \frac{1}{\alpha_k}\|\Theta^k y_{n_j} - \Theta^{k-1}y_{n_j}\| \to 0$, $\Theta^k y_{n_j} \rightharpoonup x^*$ as $j \to \infty$, and $\{A_k\}_{k=1}^{M}$ is a family of monotone mappings. It follows from (4.2.12) that

$$\langle z_t - x^*, A_k z_t\rangle \geq f_k(z_t, x^*). \tag{4.2.13}$$

So by (A1), (A4) and (4.2.13), we have for each $y \in C$ and $k \in \{1, 2, \cdots, M\}$ that

$$\begin{aligned}
0 &= f_k(z_t, z_t) \leq t f_k(z_t, y) + (1-t) f_k(z_t, x^*) \\
&\leq t f_k(z_t, y) + (1-t)\langle z_t - x^*, A_k z_t\rangle \\
&= t f_k(z_t, y) + t(1-t)\langle y - x^*, A_k z_t\rangle.
\end{aligned}$$

This implies that

$$f_k(z_t, y) + (1-t)\langle y - x^*, A_k z_t\rangle \geq 0, \quad \forall y \in C. \tag{4.2.14}$$

Letting $t \to 0$ in (4.2.14), it follows from (A3) that

$$f_k(x^*, y) + \langle y - x^*, A_k x^*\rangle \geq 0, \quad \forall y \in C.$$

Hence $x^* \in \bigcap_{k=1}^{M} GEP(f_k, A_k)$; consequently, $x^* \in F$. Further, we see that

$$\begin{aligned} \|x_t - x^*\|^2 &= \|Su_t - x^*\|^2 \\ &\leq \|u_t - x^*\|^2 \\ &\leq \|y_t - x^*\|^2 \\ &\leq \|x_t - x^* - tx_t\|^2 \\ &= \|x_t - x^*\|^2 - 2t\langle x_t, x_t - x^*\rangle + t^2\|x_t\|^2 \\ &= \|x_t - x^*\|^2 - 2t\langle x_t - x^*, x_t - x^*\rangle - 2t\langle x^*, x_t - x^*\rangle + t^2\|x_t\|^2. \end{aligned}$$

So we have

$$\|x_t - x^*\|^2 \leq \langle x^*, x^* - x_t\rangle + \frac{t}{2}\|x_t\|^2.$$

In particular,

$$\|x_n - x^*\|^2 \leq \langle x^*, x^* - x_n\rangle + \frac{t_n}{2}\|x_n\|^2.$$

Since $x_n \rightharpoonup x^*$, $x_n \to x^*$ as $n \to \infty$. By using the same argument as in the proof of Theorem 3.1 in [115], we can show that $x_t \to x^* \in F$ as $t \to 0$. This completes the proof. □

Now we prove our main theorem.

Theorem 4.2.2. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{f_k\}_{k=1}^{M} : C \times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^{M} : C \to H$ be a family of α_k-inverse strongly monotone mappings and let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a countable family of κ-strict pseudocontractions for some $0 < \kappa < 1$ such that $F := \big(\bigcap_{k=1}^{M} GEP(f_k, A_k)\big) \cap \big(\bigcap_{n=1}^{\infty} F(T_n)\big) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, $\{\beta_n\}_{n=1}^{\infty} \subset (0,1)$, $\gamma \in (\kappa, 1)$ and $r_k \in (0, 2\alpha_k)$ for each $k \in \{1, 2, \cdots, M\}$ satisfy the following conditions:*

(1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Suppose that $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition. Let $\{x_n\}$ be generated by $x_1 \in C$,

$$\begin{aligned} y_n &= P_C\big[(1-\alpha_n)x_n\big], \\ u_n &= T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_2}^{f_2, A_2} T_{r_1}^{f_1, A_1} y_n, \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)\big[\gamma u_n + (1-\gamma) T_n u_n\big], \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ converges strongly to an element in F.

Proof. For each $n \in \mathbb{N}$, define $S_n : C \to C$ by $S_n x = \gamma x + (1-\gamma)T_n x$, $x \in C$. Then $F(S_n) = F(T_n) = F(T)$ since $\gamma \in (0,1)$. Moreover, we know that $\{S_n\}$ satisfies the AKTT-condition since $\{T_n\}$ satisfies the AKTT-condition. From Lemma 2.4.15, we can define the mapping $S : C \to C$ by $Sx = \lim_{n\to\infty} S_n x$ for $x \in C$. Hence $Sx = \gamma x + (1-\gamma)Tx$, $x \in C$ since $T_n x \to Tx$ for $x \in C$. Further, we know that S_n is nonexpansive for each $n \in \mathbb{N}$. Indeed, for each $x, y \in C$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|S_n x - S_n y\|^2 &= \|\gamma x + (1-\gamma)T_n x - \gamma y - (1-\gamma)T_n y\|^2 \\
&= \|\gamma(x-y) + (1-\gamma)(T_n x - T_n y)\|^2 \\
&= \gamma\|x-y\|^2 + (1-\gamma)\|T_n x - T_n y\|^2 \\
&\quad - \gamma(1-\gamma)\|(I-T_n)x - (I-T_n)y\|^2 \\
&\le \gamma\|x-y\|^2 + (1-\gamma)\|x-y\|^2 + (1-\gamma)\kappa\|(I-T_n)x - (I-T_n)y\|^2 \\
&\quad - \gamma(1-\gamma)\|(I-T_n)x - (I-T_n)y\|^2 \\
&= \|x-y\|^2 + (1-\gamma)(\kappa-\gamma)\|(I-T_n)x - (I-T_n)y\|^2 \\
&\le \|x-y\|^2.
\end{aligned}$$

Hence S_n is nonexpansive for each $n \in \mathbb{N}$ and so is S.

Next, we show that $\{x_n\}$ is bounded. Denote $\Theta^k = T_{r_k}^{f_k,A_k} T_{r_{k-1}}^{f_{k-1},A_{k-1}} \cdots T_{r_1}^{f_1,A_1}$ for any $k \in \{1, 2, \cdots, M\}$ and $\Theta^0 = I$. We note that $u_n = \Theta^M y_n$. Then for each $p \in F$,

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n x_n + (1-\beta_n)S_n u_n\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|S_n u_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|u_n - p\| \\
&= \beta_n\|x_n - p\| + (1-\beta_n)\|\Theta^M y_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\big[(1-\alpha_n)\|x_n - p\| + \alpha_n\|p\|\big] \\
&= \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - p\| + \alpha_n(1-\beta_n)\|p\| \\
&\le \max\{\|x_n - p\|, \|p\|\}.
\end{aligned}$$

Hence, by induction, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{u_n\}$.

Next, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since $u_n = \Theta^M y_n$ and $u_{n+1} = \Theta^M y_{n+1}$,

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|\Theta^M y_{n+1} - \Theta^M y_n\| \\
&\le \|y_{n+1} - y_n\|. \qquad (4.2.15)
\end{aligned}$$

Set $z_n = S_n u_n$, $n \in \mathbb{N}$. From (4.2.15) we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|S_{n+1}u_{n+1} - S_n u_n\| \\
&\le \|S_{n+1}u_{n+1} - S_{n+1}u_n\| + \|S_{n+1}u_n - S_n u_n\| \\
&\le \|u_{n+1} - u_n\| + \|S_{n+1}u_n - S_n u_n\| \\
&\le \|y_{n+1} - y_n\| + \|S_{n+1}u_n - S_n u_n\| \\
&\le \|(1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n\| + \sup_{z\in\{u_n\}} \|S_{n+1}z - S_n z\| \\
&\le \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| + \sup_{z\in\{u_n\}} \|S_{n+1}z - S_n z\|.
\end{aligned}$$

Since $\{S_n\}$ satisfies the AKTT-condition and $\lim_{n\to\infty} \alpha_n = 0$, it follows that

$$\limsup_{n\to\infty} \big(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\big) \le 0.$$

So from Lemma 2.4.7 we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{4.2.16}$$

Hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|z_n - x_n\| = 0. \tag{4.2.17}$$

Observe that

$$\|y_n - x_n\| = \|P_C[(1 - \alpha_n)x_n] - P_C x_n\| \le \alpha_n \|x_n\| \to 0, \tag{4.2.18}$$

as $n \to \infty$. Similar to the proof of Proposition 4.2.1, for each $p \in F$, we have

$$\|u_n - p\|^2 \;\le\; \|x_n - p\|^2 + \alpha_n M_1' + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\|A_i\Theta^{i-1}y_n - A_i p\|^2 \tag{4.2.19}$$

and

$$\begin{aligned} \|u_n - p\|^2 \;\le\;& \|x_n - p\|^2 + \alpha_n M_1' - \sum_{i=1}^{M} \|\Theta^{i-1}y_n - \Theta^i y_n\|^2 \\ &+ M_2' \sum_{i=1}^{M} \|A_i\Theta^{i-1}y_n - A_i p\| \end{aligned} \tag{4.2.20}$$

for some $M_1' > 0$ and $M_2' > 0$. Then from (4.2.19) we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|S_n u_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|u_n - p\|^2 \\ &\le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\Big(\|x_n - p\|^2 + \alpha_n M_1' \\ &\qquad + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\|A_i\Theta^{i-1}y_n - A_i p\|^2\Big) \\ &\le \|x_n - p\|^2 + \alpha_n M_1' + (1 - \beta_n)\sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\|A_i\Theta^{i-1}y_n - A_i p\|^2, \end{aligned}$$

which implies that

$$(1 - \beta_n)\sum_{i=1}^{M} r_i(2\alpha_i - r_i)\|A_i\Theta^{i-1}y_n - A_i p\|^2 \;\le\; \|x_{n+1} - p\|^2 - \|x_n - p\|^2 + \alpha_n M_1'.$$

So from (4.2.17), (1), (2) and $0 < r_k < 2\alpha_k$ for each $k = 1, 2, \cdots, M$ we have

$$\lim_{n\to\infty} \|A_k\Theta^{k-1}y_n - A_k p\| = 0 \tag{4.2.21}$$

for each $k \in \{1, 2, \cdots, M\}$. Similarly, from (4.2.20), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|S_n u_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|u_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n)\Big(\|x_n - p\|^2 + \alpha_n M_1' \\
&\quad - \sum_{i=1}^{M} \|\Theta^{i-1} y_n - \Theta^i y_n\|^2 + M_2' \sum_{i=1}^{M} \|A_i \Theta^{i-1} y_n - A_i p\|\Big) \\
&\leq \|x_n - p\|^2 + \alpha_n M_1' - (1 - \beta_n) \sum_{i=1}^{M} \|\Theta^{i-1} y_n - \Theta^i y_n\|^2 \\
&\quad + M_2' \sum_{i=1}^{M} \|A_i \Theta^{i-1} y_n - A_i p\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
(1 - \beta_n) \sum_{i=1}^{M} \|\Theta^{i-1} y_n - \Theta^i y_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n M_1' \\
&\quad + M_2' \sum_{i=1}^{M} \|A_i \Theta^{i-1} y_n - A_i p\|.
\end{aligned}$$

From (1), (2), (4.2.17) and (4.2.21), it follows that

$$\lim_{n\to\infty} \|\Theta^{k-1} y_n - \Theta^k y_n\| = 0 \tag{4.2.22}$$

for each $k \in \{1, 2, \cdots, M\}$.

Next, we show that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

Observing

$$\begin{aligned}
\|u_n - y_n\| &= \|\Theta^M y_n - y_n\| \\
&\leq \|\Theta^M y_n - \Theta^{M-1} y_n\| + \|\Theta^{M-1} y_n - \Theta^{M-2} y_n\| + \cdots + \|\Theta^1 y_n - y_n\|,
\end{aligned}$$

it follows, by (4.2.22), that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{4.2.23}$$

From (4.2.18) and (4.2.23), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{4.2.24}$$

We see that

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - S_n u_n\| + \|S_n u_n - S_n x_n\| + \|S_n x_n - Sx_n\| \\
&\leq \|x_n - S_n u_n\| + \|u_n - x_n\| + \sup_{z \in \{x_n\}} \|S_n z - Sz\|.
\end{aligned}$$

So, by (4.2.16), (4.2.24) and Lemma 2.4.15, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{4.2.25}$$

Let the net $\{x_t\}$ be defined by (4.2.2). From Proposition 4.2.1, we have $x_t \to x^* \in F$ as $t \to 0$. Moreover, by proving in the same manner as in Theorem 3.2 of [115], we can show that

$$\limsup_{n\to\infty} \langle x^*, x^* - x_n\rangle \leq 0. \tag{4.2.26}$$

Finally, we show that $x_n \to x^* \in F$ as $n \to \infty$. We see that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|S_n u_n - x^*\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|(1-\alpha_n)(x_n - x^*) - \alpha_n x^*\|^2 \\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\Big((1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad - 2\alpha_n(1-\alpha_n)\langle x^*, x_n - x^*\rangle + \alpha_n^2\|x^*\|^2\Big) \\
&= \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - x^*\|^2 \\
&\quad + \alpha_n(1-\beta_n)\Big(2(1-\alpha_n)\langle x^*, x^* - x_n\rangle + \alpha_n\|x^*\|^2\Big).
\end{aligned}$$

By (1) and (4.2.26), it follows that $x_n \to x^* \in F$. This completes the proof. □

Corollary 4.2.3. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{f_k\}_{k=1}^M : C \times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \to H$ be a family of α_k-inverse strongly monotone mappings and let $\{S_i\}_{i=1}^\infty$ be a sequence of κ_i-strict pseudocontractions of C into itself such that $F := \big(\bigcap_{k=1}^M GEP(f_k, A_k)\big) \cap \big(\bigcap_{i=1}^\infty F(S_i)\big) \neq \emptyset$ and $\sup\{\kappa_i : i \in \mathbb{N}\} = \kappa > 0$. Assume that $\gamma \in (\kappa, 1)$ and $r_k \in (0, 2\alpha_k)$ for each $k \in \{1, 2, \cdots, M\}$. Define the sequence $\{x_n\}$ by $x_1 \in C$,*

$$\begin{aligned}
y_n &= P_C\big[(1-\alpha_n)x_n\big], \\
u_n &= T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_2}^{f_2, A_2} T_{r_1}^{f_1, A_1} y_n, \\
x_{n+1} &= \beta_n x_n + (1-\beta_n)\Big[\gamma u_n + (1-\gamma)\sum_{i=1}^n \mu_n^i S_i u_n\Big], \quad n \geq 1,
\end{aligned}$$

where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ which satisfy (1)-(2) of Theorem 4.2.2, and $\{\mu_n^i\}$ is a real sequence which satisfies (1)-(3) of Lemma 2.4.19. Then $\{x_n\}$ converges strongly to an element in F.

Remark 4.2.4. Proposition 4.2.1 and Theorem 4.2.2 extend the main results in [115] from a nonexpansive mapping to an infinite family of strict pseudocontractions and a system of generalized equilibrium problems.

Remark 4.2.5. If we take $A_k \equiv 0$ and $f_k \equiv 0$ for each $k = 1, 2, \cdots, M$, then Proposition 4.2.1, Theorem 4.2.2 and Corollary 4.2.3 can be applied to a system of equilibrium problems and to a system of variational inequality problems, respectively.

Remark 4.2.6. Let $S_1, S_2, \cdots,$ be an infinite family of nonexpansive mappings of C into itself and let $\xi_1, \xi_2, \cdots$ be real numbers such that $0 < \xi_i < 1$ for all $i \in \mathbb{N}$. Moreover, let W_n and W be the W-mappings [94] generated by $S_1, S_2, \cdots, S_n$ and $\xi_1, \xi_2, \cdots, \xi_n$, and $S_1, S_2, \cdots$ and $\xi_1, \xi_2, \cdots$. Then, we know from [79, 94] that $\big(\{W_n\}, W\big)$ satisfies the AKTT-condition. Therefore, in Theorem 4.2.2, the mapping T_n can be also replaced by W_n.

4.3 Existence and Iteration for a Mixed Equilibrium Problem and a Countable Family of Nonexpansive Mappings in Banach Spaces

In this section, we first prove the existence and the convergence theorems concerning the mixed equilibrium problem and the fixed points of nonexpansive mappings in Banach spaces. For solving the mixed equilibrium problem, let us assume the following conditions for a bifunction f:

(A1) $f(x,x)=0$ for all $x\in C$;
(A2) f is monotone, *i.e.* $f(x,y)+f(y,x)\le 0$ for all $x,y\in C$;
(A3) for all $y\in C$, $f(\cdot,y)$ is weakly upper semi-continuous;
(A4) for all $x\in C$, $f(x,\cdot)$ is convex.

Proposition 4.3.1. *Let X be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, bounded, closed and convex subset of X, let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let φ be a lower semi-continuous and convex function from C to $\mathbb{R}$. For all $r>0$ and $x\in X$, define the mapping $S_r : X\to 2^C$ as follows:*

$$S_r(x)=\big\{z\in C: f(z,y)+\varphi(y)+\frac{1}{r}\big\langle y-z, J(z-x)\big\rangle \ge \varphi(z),\quad \forall y\in C\big\}.$$

Then the following statements hold:

(1) *for each $x\in X$, $S_r(x)\neq\emptyset$;*

(2) *S_r is single-valued;*

(3) *$\big\langle S_rx-S_ry, J(S_rx-x)\big\rangle \le \big\langle S_rx-S_ry, J(S_ry-y)\big\rangle$ for all $x,y\in X$;*

(4) *$F\big(S_r\big)=MEP(f,\varphi)$;*

(5) *$MEP(f,\varphi)$ is nonempty, closed and convex.*

Proof. (1) Let x_0 be any given point in X. For each $y\in C$, we define

$$G(y)=\big\{z\in C: f(z,y)+\varphi(y)+\frac{1}{r}\big\langle y-z, J(z-x_0)\big\rangle \ge \varphi(z)\big\}.$$

Since $y\in G(y)$, we have $G(y)\neq\emptyset$. First, we will show that G is a KKM mapping. Suppose that there exists a finite subset $\{y_1,y_2,\cdots,y_m\}$ of C and $\beta_i\ge 0$ with $\sum_{i=1}^m\beta_i=1$ such that $\hat{x}=\sum_{i=1}^m\beta_iy_i\notin G(y_i)$ for all $i=1,2,\cdots,m$. It follows that

$$f(\hat{x},y_i)+\varphi(y_i)-\varphi(\hat{x})+\frac{1}{r}\big\langle y_i-\hat{x}, J(\hat{x}-x_0)\big\rangle<0,\quad i=1,2,\cdots,m.$$

By (A1), (A4) and the convexity of φ, we have

$$\begin{aligned}0 &= f(\hat{x},\hat{x})+\varphi(\hat{x})-\varphi(\hat{x})+\frac{1}{r}\big\langle \hat{x}-\hat{x}, J(\hat{x}-x_0)\big\rangle\\ &\le \sum_{i=1}^m\beta_i\Big(f(\hat{x},y_i)+\varphi(y_i)-\varphi(\hat{x})+\frac{1}{r}\big\langle y_i-\hat{x}, J(\hat{x}-x_0)\big\rangle\Big)<0,\end{aligned}$$

which is a contradiction. Thus G is a KKM mapping on C.

Next, we show that $G(y)$ is closed for all $y \in C$. Let $\{z_n\}$ be a sequence in $G(y)$ such that $z_n \to z$ as $n \to \infty$. Hence $z_n - x_0 \to z - x_0$ as $n \to \infty$. Since $z_n \in G(y)$,

$$f(z_n, y) + \varphi(y) + \frac{1}{r}\big\langle y - z_n, J(z_n - x_0)\big\rangle \geq \varphi(z_n). \tag{4.3.1}$$

By (A3), the norm to weak* continuity of J, and the lower semi-continuity of φ and $\|\cdot\|^2$, it follows from (4.3.1) that

$$\begin{aligned}
\varphi(z) &\leq \liminf_{n\to\infty} \varphi(z_n) \\
&\leq \limsup_{n\to\infty} \Big(f(z_n, y) + \varphi(y) + \frac{1}{r}\big\langle y - z_n, J(z_n - x_0)\big\rangle \Big) \\
&= \limsup_{n\to\infty} \Big(f(z_n, y) + \varphi(y) + \frac{1}{r}\big\langle y - x_0, J(z_n - x_0)\big\rangle + \frac{1}{r}\big\langle x_0 - z_n, J(z_n - x_0)\big\rangle \Big) \\
&= \limsup_{n\to\infty} \Big(f(z_n, y) + \varphi(y) + \frac{1}{r}\big\langle y - x_0, J(z_n - x_0)\big\rangle - \frac{1}{r}\|z_n - x_0\|^2 \Big) \\
&\leq \limsup_{n\to\infty} f(z_n, y) + \varphi(y) + \frac{1}{r}\limsup_{n\to\infty}\big\langle y - x_0, J(z_n - x_0)\big\rangle - \frac{1}{r}\liminf_{n\to\infty}\|z_n - x_0\|^2 \\
&\leq f(z, y) + \varphi(y) + \frac{1}{r}\big\langle y - x_0, J(z - x_0)\big\rangle - \frac{1}{r}\|z - x_0\|^2 \\
&= f(z, y) + \varphi(y) + \frac{1}{r}\big\langle y - x_0, J(z - x_0)\big\rangle - \frac{1}{r}\big\langle z - x_0, J(z - x_0)\big\rangle \\
&= f(z, y) + \varphi(y) + \frac{1}{r}\big\langle y - z, J(z - x_0)\big\rangle.
\end{aligned}$$

This shows that $z \in G(y)$ and hence $G(y)$ is closed for all $y \in C$.

We now equip X with the weak topology. Then C, as a closed bounded convex subset in a reflexive space, is weakly compact. Hence $G(y)$ is also weakly compact. Then by Lemma 2.4.2, we have $S_r(x_0) = \bigcap_{y\in C} G(y) \neq \emptyset$. From the arbitrariness of x_0, we can conclude that $S_r(x) \neq \emptyset$ for all $x \in X$.

(2) For $x \in C$ and $r > 0$, let $z_1, z_2 \in S_r(x)$. Then,

$$f(z_1, z_2) + \varphi(z_2) - \varphi(z_1) + \frac{1}{r}\big\langle z_2 - z_1, J(z_1 - x)\big\rangle \geq 0$$

and

$$f(z_2, z_1) + \varphi(z_1) - \varphi(z_2) + \frac{1}{r}\langle z_1 - z_2, J(z_2 - x)\rangle \geq 0.$$

Adding the two inequalities, we have

$$f(z_2, z_1) + f(z_1, z_2) + \frac{1}{r}\big\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\big\rangle \geq 0.$$

It follows from (A2) that

$$\big\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\big\rangle \geq 0.$$

Hence

$$0 \leq \big\langle z_2 - z_1, J(z_1 - x) - J(z_2 - x)\big\rangle = \big\langle (z_2 - x) - (z_1 - x), J(z_1 - x) - J(z_2 - x)\big\rangle.$$

Since J is monotone and X is strictly convex, we obtain that $z_1 - x = z_2 - x$ and hence $z_1 = z_2$. Therefore S_r is single-valued.

(3) For $x, y \in C$, we have

$$f(S_r x, S_r y) + \varphi(S_r y) - \varphi(S_r x) + \frac{1}{r}\big\langle S_r y - S_r x, J(S_r x - x)\big\rangle \geq 0$$

and

$$f(S_r y, S_r x) + \varphi(S_r x) - \varphi(S_r y) + \frac{1}{r}\big\langle S_r x - S_r y, J(S_r y - y)\big\rangle \geq 0.$$

Again, adding the two inequalities, we also have

$$\big\langle S_r y - S_r x, J(S_r x - x) - J(S_r y - y)\big\rangle \geq 0.$$

Hence

$$\big\langle S_r x - S_r y, J(S_r x - x)\big\rangle \leq \big\langle S_r x - S_r y, J(S_r y - y)\big\rangle.$$

(4) It is easy to see that

$$\begin{aligned} z \in F(S_r) \quad &\Leftrightarrow \quad z = S_r z \\ &\Leftrightarrow \quad f(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r}\langle y - z, J(z - z)\rangle \geq 0, \ \forall y \in C \\ &\Leftrightarrow \quad f(z, y) + \varphi(y) - \varphi(z) \geq 0, \ \forall y \in C \\ &\Leftrightarrow \quad z \in MEP(f, \varphi). \end{aligned}$$

This implies that $F(S_r) = MEP(f, \varphi)$.

(5) Finally, we claim that $MEP(f, \varphi)$ is nonempty, closed and convex. For each $y \in C$, we define

$$H(y) = \big\{x \in C : f(x, y) + \varphi(y) \geq \varphi(x)\big\}.$$

Since $y \in H(y)$, we have $H(y) \neq \emptyset$. Suppose that there exists a finite subset $\{z_1, z_2, ..., z_m\}$ of C and $\alpha_i \geq 0$ with $\sum_{i=1}^m \alpha_i = 1$ such that $\hat{z} = \sum_{i=1}^m \alpha_i z_i \notin H(z_i)$ for all $i = 1, 2, ..., m$. Then

$$f(\hat{z}, z_i) + \varphi(z_i) - \varphi(\hat{z}) < 0, \quad i = 1, 2, \cdots, m.$$

From (A1), (A4) and the convexity of φ, we have

$$0 = f(\hat{z}, \hat{z}) + \varphi(\hat{z}) - \varphi(\hat{z}) \leq \sum_{i=1}^m \alpha_i \Big(f(\hat{z}, z_i) + \varphi(z_i) - \varphi(\hat{z})\Big) < 0,$$

which is a contradiction. Thus H is a KKM mapping on C.

Let $u_n \in H(y)$ such that $u_n \to u$. Then, for each $y \in C$, we have

$$f(u_n, y) + \varphi(y) \geq \varphi(u_n).$$

By (A3) and the lower semi-continuity of φ, we see that

$$f(u, y) + \varphi(y) \geq \limsup_{n\to\infty} f(u_n, y) + \varphi(y) \geq \liminf_{n\to\infty} \varphi(u_n) \geq \varphi(u).$$

This shows that $u \in H(y)$ and $H(y)$ is closed for each $y \in C$. Thus $\bigcap_{y\in C} H(y) = MEP(f,\varphi)$ is also closed. Since C is bounded, closed and convex, we also have $H(y)$ is weakly compact in the weak topology. By Lemma 2.4.2, we get that $\bigcap_{y\in C} H(y) = MEP(f,\varphi) \neq \emptyset$.

Let $u, v \in F(S_r)$ and $z_t = tu + (1-t)v$ for $t \in (0,1)$. From (3), we know that

$$\big\langle S_r u - S_r z_t, J(S_r z_t - z_t) - J(S_r u - u)\big\rangle \geq 0.$$

This yields that

$$\big\langle u - S_r z_t, J(S_r z_t - z_t)\big\rangle \geq 0. \tag{4.3.2}$$

Similarly, we also have

$$\big\langle v - S_r z_t, J(S_r z_t - z_t)\big\rangle \geq 0. \tag{4.3.3}$$

It follows from (4.3.2) and (4.3.3) that

$$\begin{aligned}\|z_t - S_r z_t\|^2 &= \big\langle z_t - S_r z_t, J(z_t - S_r z_t)\big\rangle \\ &= t\big\langle u - S_r z_t, J(z_t - S_r z_t)\big\rangle + (1-t)\big\langle v - S_r z_t, J(z_t - S_r z_t)\big\rangle \\ &\leq 0.\end{aligned}$$

Hence $z_t \in F(S_r) = MEP(f,\varphi)$ and $MEP(f,\varphi)$ is convex. This completes the proof. □

Before proving the main result, we consider the following condition introduced in [69, 70]: let C be a nonempty, closed and convex subset of a Banach space X and let $\{T_n\}$ be sequence of mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}$ is said to satisfy the *NST-condition* if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$$

implies $\omega_w(z_n) \subset \bigcap_{n=1}^{\infty} F(T_n)$, where $\omega_w(z_n)$ is the set of all weak cluster points of $\{z_n\}$.

Remark 4.3.2. It is remarked that if $(\{T_n\}, T)$ satisies the AKTT-condition, then $\{T_n\}$ satisfies the NST-condition (see [72]) and also satisfies the $(*)$-condition.

Theorem 4.3.3. *Let X be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of X. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4)*, let φ be a lower semi-continuous and convex function from C to $\mathbb{R}$ and let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $F := \bigcap_{n=0}^{\infty} F(T_n) \cap MEP(f,\varphi) \neq \emptyset$ and suppose that $\{T_n\}_{n=0}^{\infty}$ satisfy the NST-condition. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, \quad D_0 = C, \\ C_n = \overline{co}\{z \in C : \|z - T_n z\| \leq t_n \|x_n - T_n x_n\|\}, \quad n \geq 0, \\ D_n = \{z \in D_{n-1} : \langle S_{r_n} x_n - z, J(x_n - S_{r_n} x_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases}$$

where $\{t_n\}$ and $\{r_n\}$ are real sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} t_n = 0$*;*
(C2) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $P_F x_0$, where P_F is the metric projection from C onto F.

Proof. We first show that $\{x_n\}$ is well-defined. We see that $C_n \cap D_n$ is closed and convex and $F \subset C_n$ for all $n \geq 0$. Since $D_0 = C$, we have $F \subset C_0 \cap D_0$. Suppose that $F \subset C_{k-1} \cap D_{k-1}$ for $k \geq 2$. From Proposition 4.3.1 (3) we have

$$\big\langle S_{r_k}x_k - S_{r_k}u, J(S_{r_k}u - u) - J(S_{r_k}x_k - x_k)\big\rangle \geq 0,$$

for all $u \in F$. This implies that

$$\big\langle S_{r_k}x_k - u, J(x_k - S_{r_k}x_k)\big\rangle \geq 0,$$

for all $u \in F$. Hence $F \subset D_k$. By induction, we get $F \subset C_n \cap D_n$ for each $n \geq 0$ and hence $\{x_n\}$ is well-defined. Put $w = P_F x_0$. Since $F \subset C_n \cap D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|w - x_0\|, \quad n \geq 0. \tag{4.3.4}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v \in C$. Since $x_{n+2} \in D_{n+1} \subset D_n$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, we have

$$\|x_{n+1} - x_0\| \leq \|x_{n+2} - x_0\|. \tag{4.3.5}$$

Combining (4.3.4) and (4.3.5), we have $\lim_{n\to\infty} \|x_n - x_0\| = d$. Moreover, by the convexity of D_n, we also have $\frac{1}{2}(x_{n+1} + x_{n+2}) \in D_n$ and hence

$$\|x_0 - x_{n+1}\| \leq \Big\|x_0 - \frac{x_{n+1} + x_{n+2}}{2}\Big\| \leq \frac{1}{2}\Big(\|x_0 - x_{n+1}\| + \|x_0 - x_{n+2}\|\Big).$$

This implies that

$$\lim_{n\to\infty} \Big\|x_0 - \frac{x_{n+1} + x_{n+2}}{2}\Big\| = d.$$

From Lemma 2.4.6 we have $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Next, we show that $v \in \bigcap_{n=0}^{\infty} F(T_n)$. Since $x_{n+1} \in C_n$ and $t_n > 0$, there exists $m \in \mathbb{N}$, $\{\lambda_0, \lambda_1, \cdots, \lambda_m\} \subset [0,1]$ and $\{y_0, y_1, \cdots, y_m\} \subset C$ such that

$$\sum_{i=0}^{m} \lambda_i = 1, \ \Big\|x_{n+1} - \sum_{i=0}^{m} \lambda_i y_i\Big\| < t_n, \text{ and } \|y_i - T_n y_i\| \leq t_n \|x_n - T_n x_n\|$$

for each $i = 0, 1, \cdots, m$. Since C is bounded, by Lemma 2.3.7, we have

$$\begin{aligned}
\|x_n - T_n x_n\| &\leq \|x_n - x_{n+1}\| + \Big\|x_{n+1} - \sum_{i=0}^{m} \lambda_i y_i\Big\| + \Big\|\sum_{i=0}^{m} \lambda_i y_i - \sum_{i=0}^{m} \lambda_i T_n y_i\Big\| \\
&\quad + \Big\|\sum_{i=0}^{m} \lambda_i T_n y_i - T_n\big(\sum_{i=0}^{m} \lambda_i y_i\big)\Big\| + \Big\|T_n\big(\sum_{i=0}^{m} \lambda_i y_i\big) - T_n x_n\Big\| \\
&\leq 2\|x_n - x_{n+1}\| + (2 + 2M)t_n \\
&\quad + \gamma^{-1}\Big(\max_{0 \leq i \leq j \leq m} \big(\|y_i - y_j\| - \|T_n y_i - T_n y_j\|\big)\Big) \\
&\leq 2\|x_n - x_{n+1}\| + (2 + 2M)t_n \\
&\quad + \gamma^{-1}\Big(\max_{0 \leq i \leq j \leq m} \big(\|y_i - T_n y_i\| + \|y_j - T_n y_j\|\big)\Big) \\
&\leq 2\|x_n - x_{n+1}\| + (2 + 2M)t_n + \gamma^{-1}(4M t_n),
\end{aligned}$$

where $M = \sup_{n\geq 0} \|x_n - w\|$. It follows from (C1) that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.$$

Since $\{T_n\}$ satisfies the NST-condition, we have $v \in \bigcap_{n=0}^{\infty} F(T_n)$.

Next, we show that $v \in MEP(f,\varphi)$. By construction of the set D_n, we see that $S_{r_n}x_n = P_{D_n}x_n$. Since $x_{n+1} \in D_n$, we obtain

$$\|x_n - S_{r_n}x_n\| \leq \|x_n - x_{n+1}\| \to 0,$$

as $n \to \infty$. From (C2), we also have

$$\frac{1}{r_n}\big\|J(x_n - S_{r_n}x_n)\big\| = \frac{1}{r_n}\big\|x_n - S_{r_n}x_n\big\| \to 0, \tag{4.3.6}$$

as $n \to \infty$. Since $x_{n_i} \rightharpoonup v$, we also have $S_{r_{n_i}}x_{n_i} \rightharpoonup v$. By definition of $S_{r_{n_i}}$, for each $y \in C$, we obtain

$$f(S_{r_{n_i}}x_{n_i}, y) + \varphi(y) + \frac{1}{r_{n_i}}\big\langle y - S_{r_{n_i}}x_{n_i}, J(S_{r_{n_i}}x_{n_i} - x_{n_i})\big\rangle \geq \varphi(S_{r_{n_i}}x_{n_i}).$$

By (A3), (4.3.6) and the weak lower semi-continuity of φ, we have

$$f(v,y) + \varphi(y) \geq \varphi(v), \quad \forall y \in C.$$

This shows that $v \in MEP(f,\varphi)$ and hence $v \in F := \bigcap_{n=0}^{\infty} F(T_n) \cap MEP(f,\varphi)$.

Note that $w = P_F x_0$. Finally, we show that $x_n \to w$ as $n \to \infty$. By the weak lower semi-continuity of the norm, it follows from (4.3.4) that

$$\|x_0 - w\| \leq \|x_0 - v\| \leq \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \leq \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \leq \|x_0 - w\|.$$

This shows that

$$\lim_{i\to\infty} \|x_0 - x_{n_i}\| = \|x_0 - w\| = \|x_0 - v\|$$

and $v = w$. Since X is uniformly convex, $x_0 - x_{n_i} \to x_0 - w$ by the Kadec-Klee property. It follows that $x_{n_i} \to w$. So we have $x_n \to w$ as $n \to \infty$. This completes the proof. □

Corollary 4.3.4. *Let X be a uniformly convex and smooth Banach space and C a nonempty, bounded, closed and convex subset of X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings of C into itself such that $F := \bigcap_{n=0}^{\infty} F(T_n) \neq \emptyset$ and suppose that $\{T_n\}_{n=0}^{\infty}$ satisfy the NST-condition. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, \\ C_n = \overline{co}\{z \in C : \|z - T_n z\| \leq t_n\|x_n - T_n x_n\|\}, \\ x_{n+1} = P_{C_n}x_0, \quad n \geq 0. \end{cases}$$

If $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$, then $\{x_n\}$ converges strongly to $P_F x_0$, where P_F is the metric projection from C onto F.

Remark 4.3.5. From [52], if we define $T_n = \alpha_n I + (1-\alpha_n)\sum_{k=0}^{n} \beta_n^k S_k$ for all $n \geq 0$ in Theorem 4.3.3 and Corollary 4.3.4, then the results also hold. Moreover, the mapping T_n can be replaced by the W-mapping W_n studied in [52].

If we take $T_n = I$ for all $n \geq 0$ in Theorem 4.3.3, then we obtain the following result.

Corollary 4.3.6. *Let X be a uniformly convex and smooth Banach space and C a nonempty, bounded, closed and convex subset of X. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let φ be a lower semi-continuous and convex function from C to $\mathbb{R}$. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, \quad D_0 = C, \\ D_n = \{z \in D_{n-1} : \langle S_{r_n}x_n - z, J(x_n - S_{r_n}x_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{D_n}x_0, \quad n \geq 0. \end{cases}$$

If $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$, then $\{x_n\}$ converges strongly to $P_{MEP(f,\varphi)}x_0$, where $P_{MEP(f,\varphi)}$ is the metric projection from C onto $MEP(f, \varphi)$.

If we take $\varphi \equiv 0$ in Corollary 4.3.6, then we obtain the following result concerning an equilibrium problem in a Banach space setting.

Corollary 4.3.7. *Let X be a uniformly convex and smooth Banach space and C a nonempty, bounded, closed and convex subset of X. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, \quad D_0 = C, \\ y_n \in C \quad \text{such that} \quad f(y_n, y) + \frac{1}{r_n}\langle y - y_n, J(y_n - x_n)\rangle \geq 0 \quad \forall y \in C, \; n \geq 1, \\ D_n = \{z \in D_{n-1} : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{D_n}x_0, \quad n \geq 0. \end{cases}$$

If $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$, then $\{x_n\}$ converges strongly to $P_{EP(f)}x_0$, where $P_{EP(f)}$ is the metric projection from C onto $EP(f)$.

If we take $f \equiv 0$ in Corollary 4.3.6, then we obtain the following result concerning a convex minimization problem in a Banach space setting.

Corollary 4.3.8. *Let X be a uniformly convex and smooth Banach space and C a nonempty, bounded, closed and convex subset of X. Let φ be a lower semi-continuous and convex function from C to $\mathbb{R}$. Let $\{x_n\}$ be the sequence in C generated by*

$$\begin{cases} x_0 \in C, \quad D_0 = C, \\ y_n \in C \quad \text{such that} \quad \varphi(y) + \frac{1}{r_n}\langle y - y_n, J(y_n - x_n)\rangle \geq \varphi(y_n), \quad \forall y \in C, \; n \geq 1, \\ D_n = \{z \in D_{n-1} : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \quad n \geq 1, \\ x_{n+1} = P_{D_n}x_0, \quad n \geq 0. \end{cases}$$

If $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$, then $\{x_n\}$ converges strongly to $P_{CMP(\varphi)}x_0$, where $P_{CMP(\varphi)}$ is the metric projection from C onto $CMP(\varphi)$.

Remark 4.3.9. The main result obtained in this section generalizes that of Matsushita-Takahashi [63] from a nonexpansive mapping to a countable family of nonexpansive mappings and a mixed equilibrium problem in Banach spaces.