# Chapter 5
# Hybrid Methods for Relatively Quasi-nonexpansive Mappings and Equilibrium Problems

In this chapter, we study strong convergence of the sequences generated by hybrid projection methods of relatively quasi-nonexpasive mappings and equilibrium problems in the framework of Banach spaces.

## 5.1 Convergence Analysis for a System of Equilibrium Problems and a Countable Family of Relatively Quasi - nonexpansive Mappings in Banach Spaces

**Theorem 5.1.1.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $\{f_j\}_{j=1}^{M}$ be bifunctions from $C\times C$ to $\mathbb{R}$ which satisfies conditions* (A1)-(A4) *and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from $C$ into itself. Assume that $F := \Big(\bigcap_{i=1}^{\infty} F(T_i)\Big) \cap \Big(\bigcap_{j=1}^{M} EP(f_j)\Big) \neq \emptyset$. For any $x_0 \in X$ with $x_1 = \Pi_{C_1}x_0$ and $C_1 = C$, define $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JT_i x_n\big), \\ u_{n,i} = T^{f_M}_{r_{M,n}} T^{f_{M-1}}_{r_{M-1,n}} \cdots T^{f_1}_{r_{1,n}} y_{n,i}, \\ C_{n+1} = \big\{z \in C_n : \sup_{i\geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{cases} \tag{5.1.1}$$

*Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \cdots, M$ are sequences satisfying the following:*

(B1) $\limsup_{n\to\infty} \alpha_n < 1$;
(B2) $\liminf_{n\to\infty} r_{j,n} > 0$.

*Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

*Proof.* We divide our proof into six steps.

**Step 1**. $F \subset C_n$ for all $n \geq 1$.

From Lemma 2.4.31 we know that $F(T_i)$ is closed and convex for all $i \geq 1$. From Lemma 2.4.35 (4), we also know that $EP(f_j)$ is closed and convex for each $j = 1, 2, \cdots, M$. Hence $F := \Big(\bigcap_{i=1}^{\infty} F(T_i)\Big) \cap \Big(\bigcap_{j=1}^{M} EP(f_j)\Big)$ is a nonempty, closed and convex subset of $C$. Clearly $C_1 = C$ is closed and convex. Suppose that $C_k$ is closed and convex for some $k \in \mathbb{N}$. For each $z \in C_k$ and $i \geq 1$, we see that $\phi(z, u_{k,i}) \leq \phi(z, x_k)$ is equivalent to

$$2\langle z, Jx_k\rangle - 2\langle z, Ju_{k,i}\rangle \leq \|x_k\|^2 - \|u_{k,i}\|^2.$$

By the construction of the set $C_{k+1}$, we see that

$$\begin{aligned} C_{k+1} &= \big\{z \in C_k : \sup_{i\geq 1}\phi(z, u_{k,i}) \leq \phi(z, x_k)\big\} \\ &= \bigcap_{i=1}^{\infty}\big\{z \in C_k : \phi(z, u_{k,i}) \leq \phi(z, x_k)\big\}. \end{aligned}$$

Hence $C_{k+1}$ is also closed and convex.

It is obvious that $F \subset C_1 = C$. Now, suppose that $F \subset C_k$ for some $k \in \mathbb{N}$ and let $p \in F := \Big(\bigcap_{i=1}^{\infty} F(T_i)\Big) \cap \Big(\bigcap_{j=1}^{M} EP(f_j)\Big)$. Then

$$\begin{aligned} \phi(p, u_{k,i}) &= \phi(p, T^{f_M}_{r_{M,n}} T^{f_{M-1}}_{r_{M-1,n}} \cdots T^{f_1}_{r_{1,n}} y_{k,i}) \\ &\leq \phi(p, T^{f_{M-1}}_{r_{M-1,n}} T^{f_{M-2}}_{r_{M-2,n}} \cdots T^{f_1}_{r_{1,n}} y_{k,i}) \\ &\ \ \vdots \\ &\leq \phi(p, T^{f_1}_{r_{1,n}} y_{k,i}) \\ &\leq \phi(p, y_{k,i}) \\ &= \phi\Big(p, J^{-1}\big(\alpha_k Jx_k + (1-\alpha_k)JT_i x_k\big)\Big) \\ &= \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1-\alpha_k)JT_i x_k\rangle \\ &\quad + \|\alpha_k Jx_k + (1-\alpha_k)JT_i x_k\|^2 \\ &\leq \|p\|^2 - 2\alpha_k\langle p, Jx_k\rangle - 2(1-\alpha_k)\langle p, JT_i x_k\rangle \\ &\quad + \alpha_k\|x_k\|^2 + (1-\alpha_k)\|T_i x_k\|^2 \\ &= \alpha_k\phi(p, x_k) + (1-\alpha_k)\phi(p, T_i x_k) \\ &\leq \phi(p, x_k). \end{aligned} \tag{5.1.2}$$

Hence $F \subset C_{k+1}$. By induction, we can conclude that $F \subset C_n$ for all $n \geq 1$.

**Step 2.** $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

From $x_n = \Pi_{C_n}x_0$ and $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad n \geq 1. \tag{5.1.3}$$

From Lemma 2.4.29 we get that

$$\phi(x_n, x_0) = \phi(\Pi_{C_n}x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0). \tag{5.1.4}$$

Combining (5.1.3) and (5.1.4), we get that $\lim_{n\to\infty}\phi(x_n, x_0)$ exists.

**Step 3.** $\{x_n\}$ is a Cauchy sequence in $C$.

Since $x_m = \Pi_{C_m}x_0 \in C_m \subset C_n$ for $m > n$, we obtain from Lemma 2.4.29 that

$$\begin{aligned} \phi(x_m, x_n) = \phi(x_m, \Pi_{C_n}x_0) &\leq \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0). \end{aligned}$$

We see that $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$ which implies with Lemma 2.4.26 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence. By the completeness of the space $X$ and the closedness of the set $C$, we can assume that $x_n \to q \in C$ as $n \to \infty$. Moreover, we get

$$\lim_{n\to\infty}\phi(x_{n+1}, x_n) = 0. \tag{5.1.5}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have for all $i \geq 1$ that

$$\phi(x_{n+1}, u_{n,i}) \leq \phi(x_{n+1}, x_n) \to 0. \tag{5.1.6}$$

Applying Lemma 2.4.26 to (5.1.5) and (5.1.6), we derive

$$\lim_{n\to\infty} \|u_{n,i} - x_n\| = 0, \quad \forall i \geq 1. \tag{5.1.7}$$

This shows that $u_{n,i} \to q$ as $n \to \infty$ for all $i \geq 1$. Since $J$ is uniformly norm-to-norm continuous on bounded subsets of $E$, we obtain that

$$\lim_{n\to\infty} \|Ju_{n,i} - Jx_n\| = 0, \quad \forall i \geq 1. \tag{5.1.8}$$

**Step 4.** $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

Denote $\Theta_n^j = T_{r_{j,n}}^{f_j} T_{r_{j-1,n}}^{f_{j-1}} \cdots T_{r_{1,n}}^{f_1}$ for any $j \in \{1, 2, \cdots, M\}$ and $\Theta_n^0 = I$ for all $n \geq 1$. We note that $u_{n,i} = \Theta_n^M y_{n,i}$ for all $i \geq 1$. From (5.1.2) we observe that

$$\phi(p, \Theta_n^{M-1} y_{n,i}) \leq \phi(p, \Theta_n^{M-2} y_{n,i}) \leq \cdots \leq \phi(p, y_{n,i}) \leq \phi(p, x_n), \quad \forall i \geq 1. \tag{5.1.9}$$

Since $p \in EP(f_M) = F(T_{r_{M,n}}^{f_M})$ for all $n \geq 1$, it follows from (5.1.9) and Lemma 2.4.36 that

$$\begin{aligned} \phi(u_{n,i}, \Theta_n^{M-1} y_{n,i}) &\leq \phi(p, \Theta_n^{M-1} y_{n,i}) - \phi(p, u_{n,i}) \\ &\leq \phi(p, x_n) - \phi(p, u_{n,i}). \end{aligned}$$

From (5.1.7) and (5.1.8), we get that $\lim_{n\to\infty} \phi(u_{n,i}, \Theta_n^{M-1} y_{n,i}) = 0$ for all $i \geq 1$. From Lemma 2.4.26, we have

$$\lim_{n\to\infty} \|u_{n,i} - \Theta_n^{M-1} y_{n,i}\| = 0, \quad \forall i \geq 1. \tag{5.1.10}$$

From (5.1.7) and (5.1.10), we have

$$\lim_{n\to\infty} \|x_n - \Theta_n^{M-1} y_{n,i}\| = 0, \quad \forall i \geq 1 \tag{5.1.11}$$

and hence,

$$\lim_{n\to\infty} \|Jx_n - J\Theta_n^{M-1} y_{n,i}\| = 0, \quad \forall i \geq 1 \tag{5.1.12}$$

Again, since $p \in EP(f_{M-1}) = F(T_{r_{M-1,n}}^{f_{M-1}})$ for all $n \geq 1$, it follows from (5.1.9) and Lemma 2.4.36 that

$$\begin{aligned} \phi(\Theta_n^{M-1} y_{n,i}, \Theta_n^{M-2} y_{n,i}) &\leq \phi(p, \Theta_n^{M-2} y_{n,i}) - \phi(p, \Theta_n^{M-1} y_{n,i}) \\ &\leq \phi(p, x_n) - \phi(p, \Theta_n^{M-1} y_{n,i}). \end{aligned}$$

From (5.1.11) and (5.1.12), we also have

$$\lim_{n\to\infty} \|\Theta_n^{M-1} y_{n,i} - \Theta_n^{M-2} y_{n,i}\| = 0, \quad \forall i \geq 1. \tag{5.1.13}$$

Hence, from (5.1.11) and (5.1.13), we get

$$\lim_{n\to\infty} \|x_n - \Theta_n^{M-2} y_{n,i}\| = 0, \quad \forall i \geq 1 \tag{5.1.14}$$

and

$$\lim_{n\to\infty} \|Jx_n - J\Theta_n^{M-2} y_{n,i}\| = 0, \quad \forall i \geq 1. \tag{5.1.15}$$

In a similar way, we can verify that

$$\lim_{n\to\infty} \|\Theta_n^{M-2} y_{n,i} - \Theta_n^{M-3} y_{n,i}\| = \cdots = \lim_{n\to\infty} \|\Theta_n^1 y_{n,i} - y_{n,i}\| = 0$$

for all $i \geq 1$ and

$$\lim_{n\to\infty} \|x_n - \Theta_n^{M-3} y_{n,i}\| = \cdots = \lim_{n\to\infty} \|x_n - y_{n,i}\| = 0$$

for all $i \geq 1$ and

$$\lim_{n\to\infty} \|Jx_n - J\Theta_n^{M-3} y_{n,i}\| = \cdots = \lim_{n\to\infty} \|Jx_n - Jy_{n,i}\| = 0 \tag{5.1.16}$$

for all $i \geq 1$. Hence, we can conclude that

$$\lim_{n\to\infty} \|\Theta_n^j y_{n,i} - \Theta_n^{j-1} y_{n,i}\| = 0 \tag{5.1.17}$$

for each $j = 1, 2, \cdots, M$ and $i \geq 1$. Observe

$$\begin{aligned} \|Jy_{n,i} - Jx_n\| &= \|\alpha_n Jx_n + (1-\alpha_n) JT_i x_n - Jx_n\| \\ &= (1-\alpha_n)\|JT_i x_n - Jx_n\| \end{aligned}$$

then we obtain from (B1) and (5.1.16) that

$$\lim_{n\to\infty} \|JT_i x_n - Jx_n\| = 0, \quad \forall i \geq 1.$$

Since $J^{-1}$ is also uniformly norm-to-norm continuous on bounded subsets, we get that

$$\lim_{n\to\infty} \|T_i x_n - x_n\| = 0, \quad \forall i \geq 1.$$

Since $T_i$ is closed for all $i \geq 1$ and $x_n \to q$, we conclude that $q \in \bigcap_{i=1}^{\infty} F(T_i)$.

**Step 5.** $q \in \bigcap_{j=1}^{M} EP(f_j)$.

From (5.1.17) and (B2), we have that $\frac{\|J\Theta_n^j y_{n,i} - J\Theta_n^{j-1} y_{n,i}\|}{r_{j,n}} \to 0$ as $n \to \infty$. Then, for each $j = 1, 2, \cdots, M$, we obtain that

$$f_j(\Theta_n^j y_{n,i}, y) + \frac{1}{r_{j,n}} \langle y - \Theta_n^j y_{n,i}, J\Theta_n^j y_{n,i} - J\Theta_n^{j-1} y_{n,i} \rangle \geq 0, \quad \forall y \in C.$$

From (A2) we have that

$$\begin{aligned} \|y - \Theta_n^j y_{n,i}\| \frac{\|J\Theta_n^j y_{n,i} - J\Theta_n^{j-1} y_{n,i}\|}{r_{j,n}} &\geq \frac{1}{r_{j,n}} \langle y - \Theta_n^j y_{n,i}, J\Theta_n^j y_{n,i} - J\Theta_n^{j-1} y_{n,i} \rangle \\ &\geq -f_j(\Theta_n^j y_{n,i}, y) \geq f_j(y, \Theta_n^j y_{n,i}), \quad \forall y \in C. \end{aligned}$$

From (A4) and the fact that $\Theta_n^j y_{n,i} \to q$ for $i \geq 1$, we get $f_j(y, q) \leq 0$ for all $y \in C$. For each $0 < t < 1$ and $y \in C$, denote $y_t = ty + (1-t)q$. Then $y_t \in C$, which implies that $f_j(y_t, q) \leq 0$. From (A1) and (A4), we obtain that $0 = f_j(y_t, y_t) \leq$

$tf_j(y_t, y) + (1-t)f_j(y_t, q) \leq tf_j(y_t, y)$. Thus, $f_j(y_t, y) \geq 0$. From (A3), we have $f_j(q, y) \geq 0$ for all $y \in C$ and $j = 1, 2, \cdots, M$. Hence $q \in \bigcap_{j=1}^{M} EP(f_j)$.

**Step 6.** $q = \Pi_F x_0$.
From $x_n = \Pi_{C_n} x_0$, we have

$$\left\langle Jx_0 - Jx_n, x_n - z \right\rangle \geq 0 \quad \forall z \in C_n.$$

Since $F \subset C_n$, we also have

$$\left\langle Jx_0 - Jx_n, x_n - p \right\rangle \geq 0 \quad \forall p \in F. \tag{5.1.18}$$

Letting $n \to \infty$ in (5.1.18), we obtain that

$$\left\langle Jx_0 - Jq, q - p \right\rangle \geq 0 \quad \forall p \in F.$$

From Lemma 2.4.28 we conclude that $q = \Pi_F x_0$. This completes the proof. □

**Remark 5.1.2.** Theorem 5.1.1 improves and extends Theorem 3.1 of Takahashi-Zembayashi [105] in the following senses:

(1) from the case of an equilibrium problem to a finite family of equilibrium problems;

(2) from a single relatively nonexpansive mapping to an infinitely countable family of relatively quasi-nonexpansive mappings;

(3) if $M = 1$ and $T_i = T$ for all $i \geq 1$, then our restriction on $\{\alpha_n\}$ is weaker than Theorem 3.1 of [105].

**Remark 5.1.3.** The iteration defined by (5.1.1) can be viewed as a modification of [105] in the following ways:

(1) We use the composition of mappings $\{T_{r_{j,n}}^{f_j}\}_{j=1}^{M}$ in the second step.

(2) We construct the set $C_{n+1}$ by using the concept of supremum concerning an infinitely countable family of closed and relatively quasi-nonexpansive mappings $\{T_i\}_{i=1}^{\infty}$.

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 5.1.1, then we have the following corollary.

**Corollary 5.1.4.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $\{f_j\}_{j=1}^{M}$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions* (A1)-(A4) *and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from $C$ into itself. Assume that $F := \left(\bigcap_{i=1}^{\infty} F(T_i)\right) \cap \left(\bigcap_{j=1}^{M} EP(f_j)\right) \neq \emptyset$. For any $x_0 \in X$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = T_i x_n, \\ u_{n,i} = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} = \left\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{cases}$$

*If $\liminf_{n\to\infty} r_{j,n} > 0$ for each $j = 1, 2, \cdots, M$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

We next give several applications of Theorem 5.1.1 in the framework of Banach spaces and Hilbert spaces.

Let $A : C \to X^*$ be a nonlinear mapping. The variational inequality problem is to find $\hat{x} \in C$ such that

$$\langle A\hat{x}, y - \hat{x}\rangle \geq 0, \quad \forall y \in C. \tag{5.1.19}$$

The solutions set of (5.1.19) is denoted by $VI(C, A)$. For each $r > 0$ and $x \in X$, define the mapping $T_r^A : X \to C$ as follows:

$$T_r^A(x) = \big\{z \in C : \langle Az, y - z\rangle + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C\big\}.$$

**Theorem 5.1.5.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $\{A_j\}_{j=1}^M$ be continuous and monotone operators from $C$ to $X^*$ and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from $C$ into itself such that $F := \Big(\bigcap_{i=1}^\infty F(T_i)\Big) \cap \Big(\bigcap_{j=1}^M VI(C, A_j)\Big) \neq \emptyset$. For any $x_0 \in X$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = J^{-1}\big(\alpha_n Jx_n + (1 - \alpha_n)JT_i x_n\big), \\ u_{n,i} = T_{r_{M,n}}^{A_M} T_{r_{M-1,n}}^{A_{M-1}} \cdots T_{r_{1,n}}^{A_1} y_{n,i}, \\ C_{n+1} = \big\{z \in C_n : \sup_{i\geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{cases}$$

*Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \cdots, M$ are sequences which satisfy conditions* (B1) *and* (B2) *of Theorem* 5.1.1. *Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

*Proof.* Define $f_j(x, y) = \langle A_j x, y - x\rangle$ for all $x, y \in C$ and $j = 1, 2, \cdots, M$. First, we see that $F(T_{r_j}^{f_j}) = EP(f_j) = VI(C, A_j) = F(T_{r_j}^{A_j})$ for each $j = 1, 2, \cdots, M$.

Next, we show that $\{f_j\}_{j=1}^M$ satisfy conditions (A1)-(A4).

(A1) $f_j(x, x) = \langle A_j x, x - x\rangle = 0$ for all $x \in C$ and $j = 1, 2, \cdots, M$.

(A2) For each $x, y \in C$ and $j = 1, 2, \cdots, M$, we observe that

$$\begin{aligned} f_j(x, y) + f_j(y, x) &= \langle A_j x, y - x\rangle + \langle A_j y, x - y\rangle \\ &= \langle A_j x - A_j y, y - x\rangle. \end{aligned}$$

By the monotonicity of $A_j$, we obtain that $f_j$ is monotone. Thus $\{f_j\}_{j=1}^M$ satisfy condition (A2).

(A3) For each $x, y, z \in C$ and $j = 1, 2, \cdots, M$, we have by the continuity of $A_j$ that

$$\begin{aligned} \limsup_{t\downarrow 0} f_j\big(tz + (1 - t)x, y\big) &= \limsup_{t\downarrow 0} \big\langle A_j(tz + (1 - t)x), y - (tz + (1 - t)x)\big\rangle \\ &= \langle A_j x, y - x\rangle \\ &= f_j(x, y). \end{aligned}$$

This shows that $\{f_j\}_{j=1}^M$ satisfy condition (A3).

(A4) Let $u, v \in C$ and $s \in (0,1)$. Then, for each $x \in C$ and $j = 1, 2, \cdots, M$, we have

$$\begin{aligned} f_j\big(x, su + (1-s)v\big) &= \big\langle A_j x, su + (1-s)v - x\big\rangle \\ &= s\langle A_j x, u - x\rangle + (1-s)\langle A_j x, v - x\rangle \\ &= s f_j(x,u) + (1-s) f_j(x,v). \end{aligned}$$

Thus $f_j$ is convex in the second variable. Let $u_n \in C$ and $\lim_{n\to\infty} u_n = u$. Then

$$\begin{aligned} f_j(x,u) &= \langle A_j x, u - x\rangle \\ &= \lim_{n\to\infty} \langle A_j x, u_n - x\rangle \\ &= \lim_{n\to\infty} f_j(x, u_n). \end{aligned}$$

This shows that $f_j$ is lower semi-continuous in the second variable. Hence $\{f_j\}_{j=1}^M$ satisfy condition (A4). From Theorem 5.1.1 we obtain the desired result. □

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 5.1.5, we have the following corollary.

**Corollary 5.1.6.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $\{A_j\}_{j=1}^M$ be continuous and monotone operators from $C$ to $X^*$ and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from $C$ into itself such that $F := \Big(\bigcap_{i=1}^\infty F(T_i)\Big) \cap \Big(\bigcap_{j=1}^M VI(C, A_j)\Big) \neq \emptyset$. For any $x_0 \in X$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = T_i x_n, \\ u_{n,i} = T_{r_{M,n}}^{A_M} T_{r_{M-1,n}}^{A_{M-1}} \cdots T_{r_{1,n}}^{A_1} y_{n,i}, \\ C_{n+1} = \big\{z \in C_n : \sup_{i\geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{cases}$$

*If $\liminf_{n\to\infty} r_{j,n} > 0$ for each $j = 1, 2, \cdots, M$, then $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

Let $\varphi : C \to \mathbb{R}$ be a real-valued function. The convex minimization problem is to find $\hat{x} \in C$ such that

$$\varphi(\hat{x}) \leq \varphi(y), \quad \forall y \in C. \tag{5.1.20}$$

The solutions set of (5.1.20) is denoted by $CMP(\varphi)$. For each $r > 0$ and $x \in X$, define the mapping $T_r^\varphi : X \to C$ as follows:

$$T_r^\varphi(x) = \Big\{z \in C : \varphi(y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq \varphi(z), \quad \forall y \in C\Big\}.$$

**Theorem 5.1.7.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $\{\varphi_j\}_{j=1}^M$ be lower semi-continuous and convex functions from $C$ to $\mathbb{R}$ and let $\{T_i\}_{i=1}^\infty$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from $C$ into itself such that $F := \Big(\bigcap_{i=1}^\infty F(T_i)\Big) \cap \Big(\bigcap_{j=1}^M CMP(\varphi_j)\Big) \neq \emptyset$. For any $x_0 \in X$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T_i x_n\big), \\ u_{n,i} = T_{r_{M,n}}^{\varphi_M} T_{r_{M-1,n}}^{\varphi_{M-1}} \cdots T_{r_{1,n}}^{\varphi_1} y_{n,i}, \\ C_{n+1} = \big\{z \in C_n : \sup_{i\geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{cases}$$

*Assume that* $\{\alpha_n\}$ *and* $\{r_{j,n}\}$ *for* $j = 1, 2, \cdots, M$ *are sequences which satisfy conditions* (B1) *and* (B2) *of Theorem* 5.1.1. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

*Proof.* Define $f_j(x, y) = \varphi_j(y) - \varphi_j(x)$ for all $x, y \in C$ and $j = 1, 2, \cdots, M$. Then $F(T_{r_j}^{f_j}) = EP(f_j) = CMP(\varphi_j) = F(T_{r_j}^{\varphi_j})$ for each $j = 1, 2, \cdots, M$ and therefore $\{f_j\}_{j=1}^M$ satisfy conditions (A1) and (A2).

Next, we show that $\{f_j\}_{j=1}^M$ satisfy conditions (A3) and (A4). For each $x, y, z \in C$, we have by the lower semi-continuity of $\varphi_j$ that

$$\begin{aligned}\limsup_{t\downarrow 0} f_j\big(tz + (1-t)x, y\big) &= \limsup_{t\downarrow 0} \big(\varphi_j(y) - \varphi_j(tz + (1-t)x)\big)\\ &\le \varphi_j(y) - \varphi_j(x)\\ &= f_j(x, y).\end{aligned}$$

This implies that $\{f_j\}_{j=1}^M$ satisfy condition (A3).

Let $u, v \in C$ and $s \in (0, 1)$. For each $x \in C$, we have by the convexity of $\varphi_j$ that

$$\begin{aligned}f_j\big(x, su + (1-s)v\big) &= \varphi_j(su + (1-s)v) - \varphi_j(x)\\ &\le s\varphi_j(u) + (1-s)\varphi_j(v) - \varphi_j(x)\\ &= s\big(\varphi_j(u) - \varphi_j(x)\big) + (1-s)\big(\varphi_j(v) - \varphi_j(x)\big)\\ &= sf_j(x, u) + (1-s)f_j(x, v).\end{aligned}$$

On the other hand, let $u_n \in C$ and $\lim_{n\to\infty} u_n = u$. By the lower semi-continuity of $\varphi_j$ we have

$$\begin{aligned}f_j(x, u) &= \varphi_j(u) - \varphi_j(x)\\ &\le \liminf_{n\to\infty} \big(\varphi_j(u_n) - \varphi_j(x)\big)\\ &= \liminf_{n\to\infty} f_j(x, u_n).\end{aligned}$$

Thus $\{f_j\}_{j=1}^M$ satisfy condition (A4). From Theorem 5.1.1 we obtain the desired result. □

If we take $\alpha_n = 0$ for all $n \in \mathbb{N}$ in Theorem 5.1.7, we have the following corollary.

**Corollary 5.1.8.** *Let* $X$ *be a uniformly convex and uniformly smooth Banach space and let* $C$ *be a nonempty, closed and convex subset of* $X$. *Let* $\{\varphi_j\}_{j=1}^M$ *be lower semi-continuous and convex functions from* $C$ *to* $\mathbb{R}$ *and let* $\{T_i\}_{i=1}^\infty$ *be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from* $C$ *into itself such that* $F := \left(\bigcap_{i=1}^\infty F(T_i)\right) \cap \left(\bigcap_{j=1}^M CMP(\varphi_j)\right) \neq \emptyset$. *For any* $x_0 \in X$ *with* $x_1 = \Pi_{C_1} x_0$ *and* $C_1 = C$, *define* $\{x_n\}$ *by*

$$\begin{cases} y_{n,i} = T_i x_n,\\ u_{n,i} = T_{r_{M,n}}^{\varphi_M} T_{r_{M-1,n}}^{\varphi_{M-1}} \cdots T_{r_{1,n}}^{\varphi_1} y_{n,i},\\ C_{n+1} = \big\{z \in C_n : \sup_{i\ge 1} \phi(z, u_{n,i}) \le \phi(z, x_n)\big\},\\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \ge 1.\end{cases}$$

*If* $\liminf_{n\to\infty} r_{j,n} > 0$ *for each* $j = 1, 2, \cdots, M$, *then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$.

As a direct consequence of Theorem 5.1.1, we obtain the following application in a Hilbert space.

**Theorem 5.1.9.** *Let $C$ be a nonempty, closed and convex subset of a Hilbert space $H$. Let $\{f_j\}_{j=1}^{M}$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions* (A1)-(A4) *and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and quasi-nonexpansive mappings from $C$ into itself such that $F := \left(\bigcap_{i=1}^{\infty} F(T_i)\right) \cap \left(\bigcap_{j=1}^{M} EP(f_j)\right) \neq \emptyset$. For any $x_0 \in H$ with $x_1 = P_{C_1}x_0$ and $C_1 = C$, define $\{x_n\}$ by*

$$\begin{cases} y_{n,i} = \alpha_n x_n + (1-\alpha_n)T_i x_n, \\ u_{n,i} = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} = \left\{ z \in C_n : \sup_{i\geq 1} \|z - u_{n,i}\| \leq \|z - x_n\| \right\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \geq 1, \end{cases}$$

*where $P$ is the metric projection. Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \cdots, M$ are sequences which satisfy conditions* (B1) *and* (B2) *of Theorem* 5.1.1. *Then $\{x_n\}$ converges strongly to $P_F x_0$.*

*Proof.* Taking $X = H$ a Hilbert space in Theorem 5.1.1, the result is obtained. □

**Remark 5.1.10.** Theorem 5.1.9 improves and extends the main results of [71, 99, 104] in the following senses:

(1) from the case of an equilibrium problem to a system of equilibrium problems;

(2) from the class of nonexpansive mappings to the class of an infinitely countable family of quasi-nonexpansive mappings.

## 5.2 A Hybrid Method for a Family of Relatively Quasi-nonexpansive Mappings and an Equilibrium Problem in Banach Spaces

In this section, we introduce a new hybrid algorithm for a family of relative quasi-nonexpansive mappings and equilibrium problems in Banach spaces. Using the concept of Mosco convergence, we prove strong convergence theorems.

**Theorem 5.2.1.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)-(A4). *Let $\{T_n\}_{n=1}^{\infty}$ be a family of relatively quasi-nonexpansive mappings of $C$ into itself which satisfies the $(*)$-condition such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(f) \neq \emptyset$. For any $x \in X$, define $\{x_n\}$ by $x_1 \in C$, $C_1 = C$ and*

$$\begin{cases} y_n = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T_n x_n\big), \\ u_n \in C \quad \text{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \left\{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \right\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \quad n \geq 1, \end{cases}$$

*where $\{\alpha_n\} \subset [0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\{r_n\} \subset (0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.*

*Proof.* From Lemma 2.4.30, we see that $C_n$ is closed and convex for all $n \geq 1$. From Lemma 2.4.31 and Lemma 2.4.35 (4), we get $F := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(f)$ is closed and convex. We next show that $F \subset C_n$ for all $n \geq 1$. Note that $u_n = T_{r_n} y_n$ for all $n \geq 1$. Let $u \in F$. Then for each $n \geq 1$,

$$\begin{aligned} \phi(u, u_n) = \phi(u, T_{r_n} y_n) \leq \phi(u, y_n) &= \phi\Big(u, J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T_n x_n\big)\Big) \\ &\leq \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, T_n x_n) \\ &\leq \phi(u, x_n). \end{aligned} \tag{5.2.1}$$

Thus $u \in C_n$ for all $n \geq 1$ and hence $F \subset C_n$ for all $n \geq 1$. Since $F$ is nonempty, $C_n$ is a nonempty, closed and convex subset of $E$. Thus $\{x_n\}$ is well defined. By the construction of the set $C_n$, we see that $\{C_n\}$ is a decreasing sequence of closed and convex subsets of $E$ such that $C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset$. It follows by Lemma 2.2.8 that

$$M - \lim_{n\to\infty} C_n = C_0 = \bigcap_{n=1}^{\infty} C_n \neq \emptyset.$$

By Lemma 2.4.33, we get that $\{x_n\} = \{\Pi_{C_n} x\}$ converges strongly to $x_0 = \Pi_{C_0} x$.

Next, we show that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. Since $x_0 \in C_n$ for all $n \geq 1$, $\phi(x_0, u_n) \leq \phi(x_0, x_n)$ for all $n \geq 1$. From Remark 2.4.25 (2) we see that

$$\begin{aligned} \phi(x_n, u_n) &= \phi(x_n, x_0) + \phi(x_0, u_n) + 2\langle x_n - x_0, Jx_0 - Ju_n\rangle \\ &\leq \phi(x_n, x_0) + \phi(x_0, x_n) + 2\langle x_n - x_0, Jx_0 - Ju_n\rangle \\ &= \Big(\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2\Big) + \Big(\|x_0\|^2 - 2\langle x_0, Jx_n\rangle + \|x_n\|^2\Big) \\ &\quad + 2\langle x_n - x_0, Jx_0 - Ju_n\rangle \\ &= 2\langle x_n - x_0, Jx_n - Jx_0\rangle + 2\langle x_n - x_0, Jx_0 - Ju_n\rangle \\ &= 2\langle x_n - x_0, Jx_n - Ju_n\rangle \\ &\leq 2\|x_n - x_0\|\|Jx_n - Ju_n\|. \end{aligned}$$

Since $x_n \to x_0$ as $n \to \infty$, $\lim_{n\to\infty} \phi(x_n, u_n) = 0$. From Lemma 2.4.26, we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{5.2.2}$$

This implies that

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{5.2.3}$$

From (5.2.1), we know that $\phi(u, y_n) \leq \phi(u, x_n)$ for all $n \geq 1$. From Lemma 2.4.36 we have

$$\phi(u_n, y_n) = \phi(T_{r_n} y_n, y_n) \leq \phi(u, y_n) - \phi(u, T_{r_n} y_n) \leq \phi(u, x_n) - \phi(u, u_n).$$

From (5.2.2) and (5.2.3) we have $\lim_{n\to\infty} \phi(u_n, y_n) = 0$; consequently, Lemma 2.4.26 asserts that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{5.2.4}$$

It also follows from (5.2.2) and (5.2.4) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{5.2.5}$$

On the other hand, we see that

$$(1-\alpha_n)\|JT_nx_n - Jx_n\| = \|Jy_n - Jx_n\|,$$

which implies by $\limsup_{n\to\infty}\alpha_n < 1$, (5.2.5) and the uniform norm-to-norm continuity of $J^{-1}$ that

$$\lim_{n\to\infty}\|JT_nx_n - Jx_n\| = \lim_{n\to\infty}\|T_nx_n - x_n\| = 0. \tag{5.2.6}$$

Since $T_n$ satisfies the $(*)$-condition, we have $x_0 \in \bigcap_{n=1}^{\infty} F(T_n)$.

Next, we will show that $x_0 \in EP(f)$. From (5.2.4) and $\liminf_{n\to\infty} r_n > 0$, we have $\frac{\|Ju_n - Jy_n\|}{r_n} \to 0$. From $u_n = T_{r_n}y_n$, we get that

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C.$$

By (A2), we have

$$\begin{aligned}
\|y - u_n\|\frac{\|Ju_n - Jy_n\|}{r_n} &\geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \\
&\geq -f(u_n, y) \geq f(y, u_n), \quad \forall y \in C.
\end{aligned}$$

From (A4) and $u_n \to x_0$, we get that $f(y, x_0) \leq 0$ for all $y \in C$. For $0 < t < 1$ and $y \in C$, Define $y_t = ty + (1-t)x_0$. Then $y_t \in C$, which implies that $f(y_t, x_0) \leq 0$. From (A1), we obtain that $0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, x_0) \leq tf(y_t, y)$. Thus, $f(y_t, y) \geq 0$. From (A3), we have $f(x_0, y) \geq 0$ for all $y \in C$. Hence $x_0 \in EP(f)$ and $x_0 \in F$.

Finally, we show that $x_0 = \Pi_F x$. From $x_n = \Pi_{C_n}x$ and $F \subset C_n$ for all $n \geq 1$, we have

$$\big\langle Jx - Jx_n, x_n - p\big\rangle \geq 0 \quad \forall p \in F. \tag{5.2.7}$$

By taking limit in (5.2.7), we obtain that

$$\big\langle Jx - Jx_0, x_0 - p\big\rangle \geq 0 \quad \forall p \in F.$$

By Lemma 2.4.28, we conclude that $x_0 = \Pi_F x$. This completes the proof. □

As a direct consequence of Theorem 5.2.1, Lemma 2.4.39, Lemma 2.4.42 and Lemma 2.4.45, we obtain the following results concerning the approximating fixed point of a family of relatively quasi-nonexpansive mappings in Banach spaces.

**Corollary 5.2.2.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $f : C\times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)-(A4). *Let $V_n$ be as in Lemma* 2.4.39 *such that $F := \bigcap_{i=1}^{N} F(T_i) \cap EP(f) \neq \emptyset$. For any $x \in X$, define $\{x_n\}$ by $x_1 \in C$, $C_1 = C$ and*

$$\begin{cases}
y_n = J^{-1}\big(\alpha_nJx_n + (1-\alpha_n)JV_nx_n\big),\\
u_n \in C \;\; \text{such that} \;\; f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,\\
C_{n+1} = \big\{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\big\},\\
x_{n+1} = \Pi_{C_{n+1}}x, \quad n \geq 1,
\end{cases}$$

*where $\{\alpha_n\} \subset [0,1]$ with $\limsup_{n\to\infty}\alpha_n < 1$ and $\{r_n\} \subset (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.*

**Remark 5.2.3.** Corollary 5.2.2 improves and extends Theorem 3.1 of [84] from two relatively quasi-nonexpansive mappings to a family of relative quasi-nonexpansive mappings.

**Corollary 5.2.4.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)-(A4). *Let $W_n$ be as in Lemma* 2.4.42 *such that $F := \bigcap_{i=1}^{N} F(T_i) \cap EP(f) \neq \emptyset$. For any $x \in X$, define $\{x_n\}$ by $x_1 \in C$, $C_1 = C$ and*

$$\begin{cases} y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JW_n x_n\big), \\ u_n \in C \quad \textit{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \big\{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \quad n \geq 1, \end{cases}$$

*where $\{\alpha_n\} \subset [0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\{r_n\} \subset (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.*

**Corollary 5.2.5.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions* (A1)-(A4). *Let $K_n$ be as in Lemma* 2.4.45 *such that $F := \bigcap_{i=1}^{N} F(T_i) \cap EP(f) \neq \emptyset$. For any $x \in X$, define $\{x_n\}$ by $x_1 \in C$, $C_1 = C$ and*

$$\begin{cases} y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JK_n x_n\big), \\ u_n \in C \quad \textit{such that} \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \big\{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \quad n \geq 1, \end{cases}$$

*where $\{\alpha_n\} \subset [0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\{r_n\} \subset (0,\infty)$ with$\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.*

We next give applications of Theorem 5.2.1.

**Theorem 5.2.6.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $A : C \to X^*$ be a continuous and monotone mapping. Let $\{T_n\}_{n=1}^{\infty}$ be a family of relatively quasi-nonexpansive mappings of $C$ into itself such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap VI(C, A) \neq \emptyset$. For any $x \in X$, define $\{x_n\}$ by $x_1 \in C$, $C_1 = C$ and*

$$\begin{cases} y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n\big), \\ u_n \in C \quad \textit{such that} \quad \langle Au_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \big\{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \quad n \geq 1, \end{cases}$$

*where $\{\alpha_n\} \subset [0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\{r_n\} \subset (0,\infty)$ with$\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.*

*Proof.* Define $f(x, y) = \langle Ax, y - x\rangle$ for all $x, y \in C$. Then $f$ satisfies the conditions (A1)-(A4). From Theorem 5.2.1 we obtain the desired result. □

**Theorem 5.2.7.** *Let $X$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty, closed and convex subset of $X$. Let $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function. Let $\{T_n\}_{n=1}^{\infty}$ be a family of relatively quasi-nonexpansive mappings of $C$ into itself such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap CMP(\varphi) \neq \emptyset$. For any $x \in X$, define $\{x_n\}$ by $x_1 \in C$, $C_1 = C$ and*

$$\begin{cases} y_n = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n\big), \\ u_n \in C \quad \text{such that} \quad \varphi(y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq \varphi(u_n), \quad \forall y \in C, \\ C_{n+1} = \big\{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \quad n \geq 1, \end{cases}$$

*where $\{\alpha_n\} \subset [0,1]$ with $\limsup_{n\to\infty} \alpha_n < 1$ and $\{r_n\} \subset (0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x$.*

*Proof.* Define $f(x,y) = \varphi(y) - \varphi(x)$ for all $x, y \in C$. Then $f$ satisfies the conditions (A1)-(A4). So the result follows from Theorem 5.2.1. □

**Remark 5.2.8.** Theorem 5.2.1 mainly extends the main result announced by Takahashi-Zembayashi [106].