Chapter 6
Conclusion

In this chapter, we conclude all main results obtained in the thesis. It is organized by dividing into 3 sections.

6.1 Approximation Methods for Common Fixed Points of Strict Pseudocontractions in Banach Spaces

(1) Let X be a uniformly convex Banach space with the Fréchet differentiable norm and let C be a nonempty, closed and convex subset of X. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that $\beta^*(t) \leq 2t,\ t \in [0, \infty)$ where β^* is a function appearing in Lemma 2.4.14. Define the sequence $\{x_n\}$ by $x_1 \in C$,

$$x_{n+1} \;=\; (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0, \lambda]$ satisfying $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$. If $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

(2) Let C be a nonempty, closed and convex subset of a uniformly smooth Banach space X that either is uniformly convex or satisfies Opial's condition. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudo-contractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that $\Phi(t) \leq 2t^2,\ t \in [0, \infty)$ where Φ is a function appearing in (2.3.1). Let $\{\alpha_n\}$ be a real sequence in $(0, \lambda]$ which satisfies the conditions (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$,

$$x_{n+1} \;=\; (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1.$$

If $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

(3) Let C be a nonempty, bounded, closed and convex subset of a uniformly smooth Banach space X that either is uniformly convex or satisfies Opial's condition. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudocontractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that $\Phi(t) \leq 2t^2,\ t \in [0, \infty)$ where Φ is a function appearing in (2.3.1). Let $\{\alpha_n\}$ be a real sequence in $(0, \lambda]$ which satisfies $\sum_{n=1}^{\infty} \alpha_n(\lambda - \alpha_n) = \infty$. Let $\{x_n\}$ be the sequence generated by $x_1 \in C$,

$$x_{n+1} \;=\; (1-\alpha_n)x_n + \alpha_n T_n x_n, \quad n \geq 1.$$

If $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_n\}_{n=1}^{\infty}$.

(4) Let C be a nonempty, bounded, closed and convex subset of a real uniformly smooth Banach space X. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudo-contractions for some $0 < \lambda < 1$ such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that $\Phi(t) \leq 2t^2,\ t \in [0, \infty)$ where Φ is a function appearing in (2.3.1). Given $u, x_1 \in C$ and sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0, 1)$, the following control conditions are satisfied:

(C1) $a \leq \alpha_n \leq \lambda$ for some $a > 0$ and for all $n \geq 1$;

(C2) $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 1$;

(C3) $\beta_n \to 0$ as $n \to \infty,\ \sum_{n=1}^{\infty} \beta_n = +\infty$;

(C4) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$;

(C5) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Suppose that $\big(\{T_n\}, T\big)$ satisfies the AKTT-condition. Define the sequence $\{x_n\}$ by $x_1 \in C$,

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n T_n x_n, \\ x_{n+1} = \beta_n u + \gamma_n x_n + \delta_n y_n, \quad n \geq 1, \end{cases}$$

Then $\{x_n\}$ converges strongly to a common fixed point z of $\{T_n\}_{n=1}^{\infty}$, where $z = Q_F u$ and $Q_F : C \to F$ is the unique sunny nonexpansive retraction from C onto F.

6.2 Equilibrium Problems and Fixed Points of Some Generalized Nonexpansive Mappings

6.2.1 Equilibrium Problems, Variational Inclusions and Fixed Points of Quasi-nonexpansive Mappings

(1) Let C be a nonempty, closed and convex subset of a Hilbert space H, $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A1) – (A5), $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ a proper, lower semi-continuous and convex function, $A : H \to H$ an α-inverse strongly monotone mapping, $M : H \to 2^H$ a maximal monotone mapping and $\{T_i\}_{i=1}^{N}$ a finite family of quasi-nonexpansive and L_i-Lipschitz mappings of C into itself. Assume that $\Omega := \bigcap_{i=1}^{N} F(T_i) \cap MEP(f, \varphi) \cap I(A, M) \neq \emptyset$ and either (B1) or (B2) holds. Let W_n be the W-mapping generated by $T_1, T_2, \cdots, T_N$ and $\beta_{n,1}, \beta_{n,2}, \cdots, \beta_{n,N}$. For $x_0 \in H$ with $C_1 = C$ and $x_1 = P_{C_1} x_0$, let $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ be defined by

$$\begin{cases} f(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) W_n u_n, \\ z_n = J_{M,\lambda_n}(y_n - \lambda_n A y_n), \\ C_{n+1} = \big\{ z \in C_n : \|z_n - z\| \leq \|y_n - z\| \leq \|x_n - z\| \big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{r_n\} \subset [b, \infty)$ for some $b \in (0, \infty)$ and $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 2\alpha)$.

Then $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{u_n\}$ converge strongly to $z_0 = P_\Omega x_0$.

6.2.2 Generalized Equilibrium Problems and Fixed Points of Strict Pseudocontractions

(1) Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $\{f_k\}_{k=1}^M : C \times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \to H$ be a family of α_k-inverse-strongly monotone mappings and let $\{T_n\}_{n=1}^\infty : C \to C$ be a countable family of κ-strict pseudocontractions for some $0 < \kappa < 1$ such that $F := \big(\bigcap_{k=1}^M GEP(f_k, A_k)\big) \cap \big(\bigcap_{n=1}^\infty F(T_n)\big) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$, $\{\beta_n\}_{n=1}^\infty \subset (0,1)$, $\gamma \in (\kappa, 1)$ and $r_k \in (0, 2\alpha_k)$ for each $k \in \{1, 2, \cdots, M\}$ satisfy the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = +\infty$;

(C2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Suppose that $(\{T_n\}, T)$ satisfies the AKTT-condition. Define the sequence $\{x_n\}$ by $x_1 \in C$ and

$$\begin{aligned} y_n &= P_C\big[(1-\alpha_n)x_n\big], \\ u_n &= T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_2}^{f_2, A_2} T_{r_1}^{f_1, A_1} y_n, \\ x_{n+1} &= \beta_n x_n + (1-\beta_n)\big[\gamma u_n + (1-\gamma) T_n u_n\big], \quad n \geq 1. \end{aligned}$$

Then $\{x_n\}$ converges strongly to an element in F.

6.2.3 Mixed Equilibrium Problems and Fixed Points of Nonexpansive Mappings

(1) Let X be a uniformly convex and smooth Banach space and let C be a nonempty, bounded, closed and convex subset of X. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4), let φ be a lower semi-continuous and convex function from C to $\mathbb{R}$ and let $\{T_n\}_{n=0}^\infty$ be a sequence of nonexpansive mappings of C into itself such that $F := \bigcap_{n=0}^\infty F(T_n) \cap MEP(f, \varphi) \neq \emptyset$ and suppose that $\{T_n\}_{n=0}^\infty$ satisfy the NST-condition. Let $\{x_n\}$ be the sequence in C generated by

$$\begin{cases} x_0 \in C, \quad D_0 = C, \\ C_n = \overline{co}\big\{z \in C : \|z - T_n z\| \leq t_n \|x_n - T_n x_n\|\big\}, \quad n \geq 0, \\ D_n = \big\{z \in D_{n-1} : \big\langle S_{r_n} x_n - z, J(x_n - S_{r_n} x_n)\big\rangle \geq 0\big\}, \quad n \geq 1, \\ x_{n+1} = P_{C_n \cap D_n} x_0, \quad n \geq 0, \end{cases}$$

where $\{t_n\}$ and $\{r_n\}$ are sequences which satisfy the conditions:

(C1) $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$;

(C2) $\{r_n\} \subset (0, \infty)$ and $\liminf_{n\to\infty} r_n > 0$.

Then $\{x_n\}$ converges strongly to $P_F x_0$, where P_F is the metric projection from C onto F.

6.3 Hybrid Methods for Relatively Quasi-nonexpansive Mappings and Equilibrium Problems

(1) Let X be a uniformly convex and uniformly smooth Banach space and let C be a nonempty, closed and convex subset of X. Let $\{f_j\}_{j=1}^{M}$ be bifunctions from $C \times C$ to $\mathbb{R}$ which satisfies conditions (A1)-(A4) and let $\{T_i\}_{i=1}^{\infty}$ be an infinitely countable family of closed and relatively quasi-nonexpansive mappings from C into itself. Assume that $F := \Big(\bigcap_{i=1}^{\infty} F(T_i)\Big) \cap \Big(\bigcap_{j=1}^{M} EP(f_j)\Big) \neq \emptyset$. For $x_0 \in X$ with $x_1 = \Pi_{C_1} x_0$ and $C_1 = C$, define the sequence $\{x_n\}$ by

$$\begin{cases} y_{n,i} = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T_i x_n\big), \\ u_{n,i} = T_{r_{M,n}}^{f_M} T_{r_{M-1,n}}^{f_{M-1}} \cdots T_{r_{1,n}}^{f_1} y_{n,i}, \\ C_{n+1} = \big\{ z \in C_n : \sup_{i \geq 1} \phi(z, u_{n,i}) \leq \phi(z, x_n) \big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1. \end{cases}$$

Assume that $\{\alpha_n\}$ and $\{r_{j,n}\}$ for $j = 1, 2, \cdots, M$ are sequences which satisfy the following conditions:

(C1) $\limsup_{n\to\infty} \alpha_n < 1$;

(C2) $\liminf_{n\to\infty} r_{j,n} > 0$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$.

(2) Let X be a uniformly convex and uniformly smooth Banach space and let C be a nonempty, closed and convex subset of X. Let $f : C \times C \to \mathbb{R}$ be a bifunction which satisfies conditions (A1)-(A4). Let $\{T_n\}_{n=1}^{\infty}$ be a family of relatively quasi-nonexpansive mappings of C into itself which satisfies the $(*)$-condition such that $F := \bigcap_{n=1}^{\infty} F(T_n) \cap EP(f) \neq \emptyset$. For any $x \in X$, define the sequence $\{x_n\}$ by $x_1 \in C$, $C_1 = C$ and

$$\begin{cases} y_n = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T_n x_n\big), \\ u_n \in C \ \ \textit{such that} \ \ f(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \big\{ z \in C_n : \phi(z, u_n) \leq \phi(z, x_n) \big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x, \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ satisfying $\limsup_{n\to\infty} \alpha_n < 1$ and $\{r_n\} \subset (0, \infty)$ satisfying $\liminf_{n\to\infty} r_n > 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x$.