Chapter 1
Introduction

Various problems in science, applied science, optimization and economics can be formulated in forms of equations, inequalities, a system of equations and a system of inequalities. Those equations, inequalities, system of equation and system of inequalities are concerned with nonlinear operators. Two main important questions have to be considered for solving those problems. The first one is about the existence of solutions of those problems and the second one is how we can find or approximate solutions of such problems. For most important nonlinear problems in science and applied science, solving a given equation or inequality can be reduced to finding the fixed points of a certain operator. Therefore, fixed point theory plays an important role in solving solutions of those problems. The metric fixed point theory has developed significantly in the second part of the 20th century. Currently, there is a big group of mathematicians who are interested in studying and investigating those two problems. They have been discovering many new theorems which are very useful for solving many problems in science, applied science, optimization and economics.

The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator $T : X \to X$, in order to obtain results on:

- the existence (and uniqueness) of fixed points;
- the data dependence of fixed points;
- the construction of fixed points.

The ambient spaces X involved in fixed point theory cover a variety of spaces: lattice, metric space, normed linear space, generalized metric space, uniform space, linear topological space, etc., while the conditions imposed on an operator are generally metric or compactness type conditions.

A plethora of metric fixed point theorems have been obtained. They usually establish the existence, or the existence and uniqueness of fixed points for a certain nonlinear mapping. Among these fixed point theorems, only a small number are important from a practical point of view since they offer a constructive method for finding the fixed points. However, it is important not only to know that a fixed point exists and the fixed point is possibly unique but also to be able to construct a fixed point. As the constructive methods used in the metric fixed point theory are prevailingly iterative procedures, i.e., approximation methods, it is also crucial importance to have a priori or/and a posteriori error estimates or rate of convergence for such methods. For example, the Banach fixed point theorem concerns certain mappings, i.e., contractions, of a completed metric space into itself. It states sufficient conditions for the existence and uniqueness of a fixed point and it also gives a constructive procedure for obtaining better approximations to a fixed point which is the solution of the practical problem. By definition, that is a method such that we choose an arbitrary element in a given set and calculate a sequence $\{x_0, x_1, x_2, ...\}$ from a relation of the form

$$x_n = Tx_{n-1} = T^n x_0, \forall n = 1, 2, 3, ...$$

That is, we choose an arbitrary x_0 and determine successively $x_1 = Tx_0, x_2 = Tx_1, ...$ It is also known as the *Picard iteration* starting at x_0.

Iteration procedures are used in almost every branch of applied mathematics. Convergence proofs and error estimates are very often obtained by an application of the Banach fixed point theorem (or more difficult fixed point theorems).

Many researchers are interested in obtaining conditions on T and X as general as possible, and which should guarantee the (strong) convergence of the Picard iteration to a fixed point of T. They are also be interested in evaluating the error estimate or the rate of convergence of the method, and then get stopping criterions for the sequence of successive approximations. However, the Picard iteration may not converge even in the weak topology.

Construction of fixed point iteration processes of nonlinear mappings is an important subject in the theory of nonlinear mappings. Its application is found in a number of applied areas. In the present, fixed point iteration processes for approximating fixed points of nonexpansive mappings, relatively nonexpansive mappings and maximal monotone operators in various spaces have been studied by many mathematicians. Let $(X, \|\cdot\|)$ be a real normed space and $C \subset X$ be closed and convex. Three classical iteration processes are often used to approximate a fixed point of a nonlinear mapping $T : C \to C$ and have been studied by many researchers.

There are many new nonlinear mappings arising from various problems in science, applied science, optimization and economics. So the fixed point problems concerning those mappings are interesting for investigating. Therefore, we aim to study the structure of fixed point sets and approximation methods for finding fixed points of new mappings. The mappings we are interested in this study are nonexpansive mappings and other generalizations of these mappings.

There are many problems in economics, finance, transportation, network and structure analysis, elasticity and optimization which can be solved by using equilibrium theory. The equilibrium problems cover many important problems such as minimization problems and variational inequality problems. In 1997, Combettes and Hirstoaga first studied and constructed an iterative method under some sufficient conditions. Their main results can be applied for solving variational inequality problems and they showed that there is a close relation between fixed point problems and equilibrium problems. Therefore, many techniques in fixed point theory can be applied for the equilibrium problems. Many problems in science and applied science, economics and other areas are more complicated. The solutions of those problems concern intersection of fixed point problems, variational inequality and equilibrium problems. Therefore, the problem of finding common solutions of those problems is very useful for applications and widely studied by many mathematicians in the present.

Motivated by many previous works concerning the equilibrium problems, fixed point problems and other important problems in science, applied science and economics, we are interested in studying and constructing new methods for solving those problems.

The purpose of this thesis can be separated to be 2 parts. First, we construct a new iterative scheme to approximate a common fixed point for a finite family of asymptotically quasi-nonexpansive mappings. We prove several strong and weak convergence results of the proposed iteration in Banach spaces. These results generalize and refine many known results in the current literature. For the second purpose, we use a hybrid iterative method to find a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in real Hilbert spaces. We also show that our main strong convergence theorem for finding that common element can be deduced for nonexpansive mappings and applied for strict pseudo-contraction mappings. Our results extend the work by Cho et al. (2009) [14].

This thesis is divided into 5 chapters. Chapter 1 is the introduction of research problems. Chapter 2 is devoted to basic definitions and background information needed throughout the thesis. Moreover, some useful results on fixed points are also presented in this chapter. Chapters 3 and 4 are the main results of this thesis and the conclusion is in Chapter 5.

Precisely, we give more details about our main results here. In Chapter 3, we study a new iteration for finding common fixed points of asymptotically quasi-expansive mappings in Banach spaces. In Section 3.1, we introduce an iterative method with some known results and then introduce a new iterative process motivated by the previous iteration. In Section 3.2, we prove a strong convergence theorem of a new iterative scheme under some appropriate conditions in Banach spaces. In Section 3.3, we prove some strong and weak convergence results for the new iterative scheme on uniformly convex Banach spaces without using the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ appearing in Section 3.2. Instead, we consider $(L-\gamma)$ uniform Lipshitz mappings, condition (A''), semi-compact mappings, Opial property and demiclosed mappings at 0.

In Chapter 4, we study a hybrid iterative method for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in real Hilbert spaces. In Section 4.1, we prove our main strong convergence theorem for finding that common element. In Section 4.2, we show that the main result in Section 4.2 can be deduced for nonexpansive mappings and applied for strict pseudo-contraction mappings.