Chapter 2
Basic Concepts and Preliminaries

The purpose of this chapter is to explain certain notations, terminologies and elementary results used throughout the thesis. Although details are included in some cases, many of the fundamental principles of functional analysis are merely stated without proof.

2.1 Metric Spaces

Definition 2.1.1. ([30]) A *metric space* is a pair (X, d), where X is a set and d is a metric on X(or distance function on X),that is, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) $d(x, y) = d(y, x)$ (symmetry),

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Example 2.1.2. ([2])

(1) $X = \mathbb{R}$; $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$, where $|.|$ denotes the absolute value, is a metric (a distance) on $\mathbb{R}$;

(2) $X = \mathbb{R}^n$; $d(x, y) = [\sum_{i=1}^{n}(x_i - y_i)^2]^{1/2}$, for all $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, is a metric on $\mathbb{R}^n$, called the *euclidean metric*. The next two mappings

$$\delta(x, y) = \sum_{i=1}^{n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n$$

and

$$\rho(x, y) = \max_{1 \leq 1 \leq n} |x_i - y_i|, \quad x, y \in \mathbb{R}^n$$

are also metrics on $\mathbb{R}^n$;

(3) Let $X = \{f : [a, b] \to \mathbb{R}|$ f is continuous$\}$. We define $d : X \times X \to \mathbb{R}^+$ by

$$d(f, g) = \max_{x \in [a,b]} |f(x) - g(x)|, \quad \text{for all } f, g \in X.$$

Then d is a metric on X (called the *Chebyshev metric*); the metric space (X,d) is usually denoted by $C[a, b]$;

(4) Let X be as (3) and $\delta : X \times X \to \mathbb{R}^+$ be given by

$$\delta(f,g) = \max_{x\in[a,b]} (|f(x) - g(x)|e^{-\tau|x-x_0|}),$$

for all $f, g \in X$, where $\tau > 0$ is a constant and $x_0 \in [a,b]$ is fixed. Then δ is a metric on X, called the *Bielecki metric*, and the metric space (X, δ) is usually denoted by $B[a,b]$. □

Definition 2.1.3. ([2]) Let (X, d) be a metric space. The topology having basis as the family of all open balls, $B(x,r)$, $x \in X$, $r > 0$, is called the *topology induced* by the metric d

Definition 2.1.4. ([2]) Two metrics d_1 and d_2 defined on the set X are called *equivalent* if they induce the same topology on X.

Remark 2.1.5. ([2])

(1) Two metrics d_1 and d_2 are metrically equivalent if there exist two constants $m > 0$ and $M > 0$ such that

$$md_1(x,y) \le d_2(x,y) \le Md_1(x,y), \quad \text{for all } x, y \in X;$$

(2) In Example 2.1.2, the metrics d, δ and ρ from (2) are equivalent; the metrics d from (3) and ρ from (4) are also equivalent.

Definition 2.1.6. ([30]) A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *convergent* if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

x is called the *limit* of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x.$$

We say that $\{x_n\}$ *converges to* x. In the case that $\{x_n\}$ is not convergent, it is said to be *divergent.*

Definition 2.1.7. ([30])A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be *Cauchy* if for every $\epsilon > 0$ there is an $N(\epsilon) \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for every $m, n \ge N(\epsilon)$.

Definition 2.1.8. ([30]) A metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges.

Example 2.1.9. [52] Let S be a nonempty set and let $B(S)$ be the function space of bounded real valued functions defined on S. For any $f, g \in B(S)$, define their metric $d(f, g)$ by

$$d(f,g) = \sup_{t\in S} |f(t) - g(t)|.$$

Then $B(S)$ is a complete metric space. □

Example 2.1.10. [52] The N-dimensional Euclidean space $\mathbb{R}^N$ is complete, that is, if for any $x = (x_1, x_2, ..., x_N)$, $y = (y_1, y_2, ..., y_N) \in \mathbb{R}^N$,

$$d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2},$$

$(\mathbb{R}^N, d)$ is complete. □

Definition 2.1.11. ([2]) Let (X, d) be a metric space. A mapping $T : X \to X$ is called

- Lipschitzian (or L-Lipschitzian) if there exists $L > 0$ such that

  $$d(Tx, Ty) \leq Ld(x, y), \text{for all } x, y \in X;$$

- (strict) contraction (or a-contraction) if T is a-Lipschitzian, with $a \in [0, 1)$;
- nonexpansive if T is 1-Lipschitzian;
- contractive if $d(Tx, Ty) < d(x, y)$, for all $x, y \in X$, $x \neq y$;
- isometry if $d(Tx, Ty) = d(x, y)$, for all $x, y \in X$.

Example 2.1.12. ([2])

(1) The function $T : \mathbb{R} \to \mathbb{R}$, $T(x) = \frac{x}{2} + 3$, $x \in \mathbb{R}$, is a strict contraction;

(2) The function $T : [1/2, 2] \to [1/2, 2]$, $Tx = 1/x$, is 4-Lipschitzian;

(3) The function $T : \mathbb{R} \to \mathbb{R}$, $T(x) = x + 2$, is isometry;

(4) The function $T : [1, +\infty] \to [1, +\infty]$, $Tx = x + \frac{1}{x}$, is contractive.

Theorem 2.1.13. ([30]) Every convergent sequence in a metric space is a Cauchy sequence.

Theorem 2.1.14. ([33]) Let $\{x_n\}$ be a sequence in $\mathbb{R}$. If every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a convergent subsequence, then $\{x_n\}$ is convergent.

Definition 2.1.15. ([33]) Let X be a metric space and A be any nonempty subset of X. For each x in X, the *distance* $d(x, A)$ from x to A is $\inf\{d(x, y) | \ y \in A\}$.

Let X be a metric space with a metric d. A subset F of X is called a *closed set* if $\{x_n\} \subset F$ and $x_n \to x$ imply $x \in F$.

Theorem 2.1.16. ([52]) (**The fundamental properties of closed sets**) Let X be a metric space. Then the following hold:

(1) X and $\emptyset$ are closed sets;

(2) any intersection of closed sets in X is closed, that is,

$$F_\mu \ (\mu \in M) \text{ are closed } \Rightarrow \bigcap_{\mu \in M} F_\mu \text{ is closed;}$$

(3) any finite union of closed sets in X is closed, that is,

$$F_i \ (i = 1, 2, \ldots, m) \text{ are closed } \Rightarrow \bigcup_{i=1}^{m} F_i \text{ is closed.}$$

Definition 2.1.17. ([52]) Let X and Y be metric spaces and let f be a mapping of X into Y. Then f is said to be *continuous* at x_0 in X if

$$x_n \to x_0 \Rightarrow f(x_n) \to f(x_0).$$

A mapping f of X into Y is said to be *continuous* if it is *continuous* at each x in X, that is

$$x_n \to x \Rightarrow f(x_n) \to f(x).$$

Definition 2.1.18. [53] Let X be a nonempty set. A class $\mathbf{G}$ of subsets of X is called a *topology* on X if it satisfies the following conditions:

(1) $X \in \mathbf{G}$ and $\emptyset \in \mathbf{G}$;

(2) the union of every class of sets in $\mathbf{G}$ is a set in $\mathbf{G}$;

(3) the intersection of every finite class of sets in $\mathbf{G}$ is a set in $\mathbf{G}$.

A topological space consists of two objects: A nonempty set X and a topology $\mathbf{G}$ on X. The sets in the class $\mathbf{G}$ are called the *open sets* of the topological space $(X, \mathbf{G})$.

Definition 2.1.19. [2] Let (X, d) be a metric space. The topology having as basis the family of all open balls, $B_r(x)$, $x \in X$, $r > 0$, is called the *topology induced* by the metrics d

Definition 2.1.20. ([53]) Let X be a topological space and let f be a function of X into $(-\infty, \infty]$. Then f is said to be *lower semicontinuous* on X if for any real number a, the set $\{x \in X : f(x) \leq a\}$ is closed in X.

Definition 2.1.21. Let X be a topological space and let f be a function of X into $(-\infty, \infty]$. Then f is said to be a *proper lower semicontinuous* function on X if f is lower semi-continuous and there is $x \in X$ such that $f(x) \in (-\infty, \infty)$.

Theorem 2.1.22. ([53]) Let X be a topological space and let f be a function of X into $(-\infty, \infty]$. Then, f is lower semicontinuous on X if and only if, for any $x_0 \in X$,

$$x_\alpha \to x_0 \Longrightarrow f(x_0) \leq \liminf_\alpha f(x_\alpha). \tag{2.1.1}$$

Definition 2.1.23. ([53]) Let X be a topological space and let f be a function of X into $[-\infty, \infty)$. Then f is called *upper semicontinuous* if for any real number a, the set $\{x \in X : f(x) \geq a\}$ is closed in X.

Theorem 2.1.24. Let X be a topological space and let f be a function of X into $[-\infty, \infty)$. Then, f is upper semicontinuous on X if and only if, for any $x_0 \in X$,

$$x_\alpha \to x_0 \Longrightarrow \limsup_\alpha f(x_\alpha) \leq f(x_0). \tag{2.1.2}$$

Definition 2.1.25. Let X be a topological space and let f be a function of X into $[-\infty, \infty)$. Then f is called *weakly upper semicontinuous* if for any $x_0 \in X$,

$$x_\alpha \rightharpoonup x_0 \Longrightarrow \limsup_\alpha f(x_\alpha) \leq f(x_0). \tag{2.1.3}$$

Definition 2.1.26. [30] A *linear space* or *vector space* X over the field $\mathbb{K}$(the real field $\mathbb{R}$ or the complex field $\mathbb{C}$) is a set X together with an internal binary operation "+" called *addition* and a *scalar multiplication* carrying (α, x) in $\mathbb{K} \times X$ to αx in X satisfying the following for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

(1) $x + y = y + x$;

(2) $(x + y) + z = x + (y + z)$;

(3) there exists an element $0 \in X$ called the *zero vector* of X such that $x + 0 = x$ for all $x \in X$;

(4) for every element $x \in X$, there exists an element $-x \in X$ called the *addition inverse* or the *negative* of x such $x + (-x) = 0$;

(5) $\alpha(x + y) = \alpha x + \alpha y$;

(6) $(\alpha + \beta)x = \alpha x + \beta X$;

(7) $(\alpha\beta)x = \alpha(\beta x)$;

(8) $1 \cdot x = x$.

The elements of a vector space X are called *vector*, and the elements of $\mathbb{K}$ called *scalars*. In the sequel, unless otherwise stated, X denotes a linear space over field $\mathbb{R}$.

Example 2.1.27. [52] Consider the set $\mathbb{R}^N = \{x = (a_1, a_2, ..., a_N) : a_1, a_2, ..., a_N \in \mathbb{R}\}$. For any $x = (a_1, a_2, ..., a_N)$, $y = (b_1, b_2, ..., b_N) \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$, we define their *addition* $x + y$ and their *scalar multiplication* αx by

$$\begin{aligned} x + y &= (a_1 + b_1, a_2 + b_2, ..., a_N + b_N), \\ \alpha x &= (\alpha a_1, \alpha a_2, ..., \alpha a_N). \end{aligned}$$

Then $x + y \in \mathbb{R}^N$ and $\alpha x \in \mathbb{R}^N$. Thus $\mathbb{R}^N$ is a linear space with this operations. $\square$

Example 2.1.28. [52] Let L be the set of all real valued functions f defined on $[0,1]$. For any $f, g \in L$ and $\alpha \in \mathbb{R}$, we define their *addition* $f+g$ and their *scalar multiplication* αf by

$$\begin{aligned} f+g &: (f+g)(t) = f(t)+g(t), \\ \alpha f &: (\alpha f)(t) = \alpha f(t), \\ 0 &: 0(t) = 0, \\ -f &: (-f)(t) = -f(t). \end{aligned}$$

Then L is a linear space with these operations. □

Definition 2.1.29. ([30]) A subset C of a vector space X is said to be *convex* if $x, y \in C$ implies $M = \{z \in X |\ z = tx + (1-t)y,\ 0 \le t \le 1\} \subseteq C$.

Definition 2.1.30. [53] Let X be a vector space and let C be a convex subset of X. A function $F : C \to (-\infty, \infty]$ is *convex* on C if for any $x_1, x_2 \in C$ and $t \in [0,1]$,

$$F(tx_1 + (1-t)x_2) \le tF(x_1) + (1-t)F(x_2).$$

2.2 Banach Spaces and Hilbert spaces

Definition 2.2.1. ([33]) Let X be a linear space (or vector space). A *norm* on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:

(1) $\|x\| \ge 0$ and $\|x\| = 0$ if and only if $x = 0$,

(2) $\|\alpha x\| = |\alpha| \|x\|$,

(3) $\|x+y\| \le \|x\| + \|y\|$ (triangle inequality).

The ordered pair $(X, \|\cdot\|)$ is called a *normed space* or *normed vector space* or *normed linear space*.

Definition 2.2.2. ([33]) Let X be a normed space. The *metric induced by the norm* of X is the metric d on X defined by the formula $d(x,y) = \|x-y\|$ for all $x, y \in X$. The *norm topology* of X is the topology obtained from this metric.

Definition 2.2.3. ([1]) The space of all bounded linear functionals on a normed space X is called the *dual space* of X and is denoted by X^*.

Definition 2.2.4. ([30]) Let x be an element and $\{x_n\}$ be a sequence in a normed space X. Then $\{x_n\}$ *converges strongly* to x written by $x_n \to x$, if $\lim_{n\to\infty} \|x_n - x\| = 0$.

Definition 2.2.5. ([30]) Let x be an element and $\{x_n\}$ be a sequence in a normed space X. Then $\{x_n\}$ *converges weakly* to x written by $x_n \rightharpoonup x$, if $f(x_n) \to f(x)$ wherever $f \in X^*$.

Theorem 2.2.6. ([53]) A normed space X is *reflexive* if and only if each of its bounded sequence has a weakly convergent subsequence.

Definition 2.2.7. ([2]) A linear normed space X is called *strictly convex* if $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|(1-\lambda)x + \lambda y\| = 1$ for a $\lambda \in (0,1)$ holds if and only if $x = y$.

Definition 2.2.8. ([33]) A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a *Banach space*, *B-space* or *complete normed space* if its norm is a Banach norm.

Example 2.2.9. [52] The real line $\mathbb{R}$ is a Banach space with the norm $\|x\| = |x|$. The complex plan $\mathbb{C}$ is also a Banach space. From Example 2.1.10, the N-dimensional Euclidean space $\mathbb{R}^N$ is a Banach space. □

Example 2.2.10. [52] The space ℓ^∞ of all bounded sequences $x = (x_1, x_2, ..., x_n, ...)$ of real numbers is a Banach space with the norm defined by

$$\|x\| = \sup_n |x_n|.$$ □

Example 2.2.11. [52] Let p be a real number such that $1 \leq p < \infty$. We denote by ℓ^p the space of all sequences $x = (x_1, x_2, ..., x_n, ...)$ of real numbers such that $\sum_{n=1}^\infty |x_n|^p < \infty$ with the norm defined by

$$\|x\|_p = \Big(\sum_{n=1}^\infty |x_n|^p\Big)^{\frac{1}{p}}.$$

Then ℓ^p is a Banach space. □

Example 2.2.12. [52] Let $C[a,b]$ be the set of all continuous real valued functions f on $[a,b]$, with the norm defined by

$$\|x\| = \max_{a \leq x \leq b} |x(t)|.$$

Then $C[a,b]$ is a Banach space. □

Definition 2.2.13. ([53]) A Banach space X is *uniformly convex* if for any two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$, imply $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Theorem 2.2.14. ([53]) Let X be a Banach space. Then the following conditions are equivalent:

(1) X is uniformly convex;

(2) if for any two sequences $\{x_n\}$, $\{y_n\}$ in X,

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

then $\lim_{n\to\infty} \|x_n - y_n\| = 0$;

(3) for any ϵ with $0 < \epsilon \leq 2$, there exists $\delta > 0$ depending only on $\epsilon > 0$ such that

$$\|\frac{x+y}{2}\| \leq 1 - \delta$$

for any $x, y \in X$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$.

Example 2.2.15. [16, 18] The L^p and ℓ^p spaces are uniformly convex for $p \in (1, \infty)$.

Theorem 2.2.16. ([12]) Every uniformly convex space is strictly convex.

Lemma 2.2.17. ([60]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that

$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 1,$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2.18. ([51]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{b_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1$. Suppose $x_{n+1} = (1 - b_n)y_n + b_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Definition 2.2.19. ([37]) A Banach space X is said to satisfy *Opial's condition* if $x_n \rightharpoonup x$ as $n \to \infty$ and $x \neq y$ imply that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|.$$

Lemma 2.2.20. ([49, Lemma 2.7]) Let X be a Banach space which satisfies Opial's condition and let $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim\limits_{n\to\infty} \|x_n - u\|$ and $\lim\limits_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.

Definition 2.2.21. ([12]) A Banach space E is said to have *Kadec-Klee property* if, for every sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $\|x_n - x\| \to 0$.

Definition 2.2.22. ([30]) An *inner product space* is a vector space X with an inner product defined on X. A *Hilbert space* is a complete inner product space. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$; that is, with every pair of vectors x and y there is an associated scalar which is written and is called the inner product of x and y, such that for all vectors x, y, z and scalar $\alpha \in \mathbb{F}$ we have:

(1) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$,

(2) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$,

(3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$,

(4) $\langle x+y,z\rangle = \langle x,z\rangle + \langle y,z\rangle$.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{\langle x,x\rangle}$.

Remark 2.2.23. ([52])

(1) An inner product space is called a *real inner product space* if the scalars are real numbers and $\langle x,y\rangle$ is a real number. Therefore, the equality (3) in Definition (2.2.22) is equivalent to

$$\langle x,y\rangle = \langle y,x\rangle.$$

(2) Using (2), (3) and (4) in Definition (2.2.22), we obtain that for $x,y\in X$ and $\alpha,\beta\in C$,

$$\langle x,\alpha y+\beta z\rangle = \bar{\alpha}\langle x,y\rangle + \bar{\beta}\langle x,z\rangle.$$

Example 2.2.24. [52] The 3-dimensional Euclidean space $\mathbb{R}^3$ is Hilbert spaces, that is, if for any $x=(a_1,a_2,a_3)$, $y=(b_1,b_2,b_3)\in\mathbb{R}^3$,

$$\langle x,y\rangle = a_1b_1+a_2b_2+a_3b_3, \text{ and } \|x\| = \sqrt{a_1^2+a_2^2+a_3^2}.$$

Then $\mathbb{R}^3$ is a Hilbert space. □

Example 2.2.25. [52] The 3-dimensional Euclidean space $\mathbb{C}^3$ is Hilbert spaces, that is, if for any $x=(u_1,u_2,u_3)$, $y=(v_1,v_2,v_3)\in\mathbb{C}^3$,

$$\langle x,y\rangle = u_1\overline{v_1}+u_2\overline{v_2}+u_3\overline{v_3}, \text{ and } \|x\| = \sqrt{u_1^2+u_2^2+u_3^2}.$$

Then $\mathbb{C}^3$ is a Hilbert space. □

Theorem 2.2.26. ([53])(**The Schwarz inequality**) If x and y are any two vectors in an inner product space X, then $|\langle x,y\rangle| \le \|x\|\|y\|$.

Theorem 2.2.27. ([52]) For any inner product space H, the following holds:

$$\|x+y\|^2+\|x-y\|^2 = 2\|x\|^2+2\|y\|^2.$$

Theorem 2.2.28. ([52]) **(Opial's Theorem)** Any Hilbert space satisfies Opial's condition.

Theorem 2.2.29. ([52]) The inner product in an inner product space H is jointly continuous:

$$x_n\to x \text{ and } y_n\to y \Rightarrow \langle x_n,y_n\rangle \to \langle x,y\rangle.$$

Remark 2.2.30. ([52]) We of course obtain from Theorem 2.2.29 that $x_n\to x$, then for a fixed $y\in H$,

$$\langle x_n,y\rangle\to\langle x,y\rangle \text{ and } \langle y,x_n\rangle\to\langle y,x\rangle.$$

Definition 2.2.31. ([52]) Let H be an inner product space, $\{x_n\}$ be a sequence of H and x be an element of H. Then $\{x_n\}$ is said to *converge weakly* to x, denoted by $x_n\rightharpoonup x$, if for any $y\in H$, $\langle x_n,y\rangle\to\langle x,y\rangle$.

Remark 2.2.32. ([52]) If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$, then $x = y$. In fact, we have

$$\begin{aligned}\|x-y\|^2 &= \langle x-y, x-y\rangle = \langle x-x_n+x_n-y, x-y\rangle \\ &= \langle x-x_n, x-y\rangle + \langle x_n-y, x-y\rangle \to 0.\end{aligned}$$

So, we get $x = y$.

Lemma 2.2.33. ([52]) Let H be an inner product space and $\{x_n\}$ be a bounded sequence of H such that $x_n \rightharpoonup x$. Then following inequality holds:

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|.$$

Lemma 2.2.34. ([52]) Let $\{x_n\}$ be a Cauchy sequence of an inner product space H such that $x_n \rightharpoonup x$. Then $x_n \to x$.

Let C be a nonempty closed convex subset of a real Hilbert space H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x-y\|, \quad \forall y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2.1}$$

Obviously, this immediately implies that

$$\|(x-y)-(P_C x - P_C y)\|^2 \leq \|x-y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.2.2}$$

Recall that, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned}&\langle x - P_C x, y - P_C x\rangle \leq 0, \\ &\|x-y\|^2 \geq \|x - P_C x\|^2 + \|P_C x - y\|^2,\end{aligned} \tag{2.2.3}$$

for all $x \in H$ and $y \in C$; see Goebel and Kirk [25] for more details.

Lemma 2.2.35. *([5]) Given $x \in H$ and $y \in C$. Then $P_C x = y$ if and only if there holds the inequality*

$$\langle x-y, y-z\rangle \geq 0, \quad \forall z \in C.$$

2.3 The Background of Fixed Point Theory

Let X be a nonempty set and $T : X \to X$ be a self-map. We say that $x \in X$ is a fixed point of T if

$$Tx = x$$

and we denote the set of all fixed points of T by $F(T)$.

Example 2.3.1. ([2])

(1) If $X = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $F(T) = \{-2\}$;

(2) If $X = \mathbb{R}$ and $T(x) = x^2 - x$, then $F(T) = \{0, 2\}$;

(3) If $X = \mathbb{R}$ and $T(x) = x + 2$, then $F(T) = \emptyset$;

(4) If $X = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$.

Let X be any set and and $T : X \to X$ be a self-map. For any given $x \in X$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we call $T^n(x)$ the *iterate of* x *under* T. In order to simplify the notations we will often use Tx instead of $T(x)$.

The mapping $T^n (n \geq 1)$ is called the n^{th} *iterate* of T. For any $x_0 \in X$, the sequence $\{x_n\}_{n\geq 0}$, given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \ldots,$$

is called *the sequence of successive approximations with the initial value* x_0. It is also known as the *Picard iteration* starting at x_0.

For a given self-map the following properties obviously hold:

(1) $F(T) \subset F(T^n)$, for each $n \in \mathbb{N}$;

(2) $F(T^n) = \{x\}$, for some $n \in \mathbb{N} \Rightarrow F(T) = \{x\}$.

The reverse of (2) is not true, in general, as shown by the next example.

Example 2.3.2. ([2]) Let $T : \{1, 2, 3\} \to \{1, 2, 3\}$, $T(1) = 3$, $T(2) = 2$ and $T(3) = 1$. Then $F(T^2) = \{1, 2, 3\}$ but $F(T) = \{2\}$. □

Theorem 2.3.3. ([2]) (**Contraction mapping principle**) Let (X, d) be a complete metric space and $T : X \to X$ be a given contraction. Then T has a unique fixed point p and

$$T^n(x) \to p \ \ (\text{as } n \to \infty), \quad \text{for each } x \in X.$$

We state several definitions of mappings in Banach spaces (or Hilbert spaces) at the same time since it is easy to be read and compared.

Definition 2.3.4. Let C be a nonempty subset of a real Banach space (or Hilbert space) X and T be a self-mapping of C. The fixed point set of T is denoted by $F(T) = \{x \in C : Tx = x\}$. If $F(T)$ is not empty, then T is called

(1) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \tag{2.3.1}$$

for all $x, y \in C$;

(2) *quasi-nonexpansive* [19] if

$$\|Tx - p\| \leq \|x - p\|, \tag{2.3.2}$$

for all $x \in C$ and $p \in F(T)$;

(3) *asymptotically nonexpansive* [24] if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with
$\lim_{n\to\infty} r_n = 0$ and

$$\|T^n x - T^n y\| \le (1 + r_n)\|x - y\|, \tag{2.3.3}$$

for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(4) *asymptotically quasi-nonexpansive* if there exists a sequence $\{r_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} r_n = 0$ and

$$\|T^n x - p\| \le (1 + r_n)\|x - p\|, \tag{2.3.4}$$

for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(5) *generalized quasi-nonexpansive* [48] if there exists a sequence $\{s_n\}$ in $[0, \infty)$ with $s_n \to 0$ as $n \to \infty$ such that

$$\|T^n x - p\| \le \|x - p\| + s_n, \tag{2.3.5}$$

for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(6) *generalized asymptotically quasi-nonexpansive* [48] if there exist two sequences $\{r_n\}$ and $\{s_n\}$ in $[0, \infty)$ with $r_n \to 0$ and $s_n \to$ as $n \to \infty$ such that

$$\|T^n x - p\| \le (1 + r_n)\|x - p\| + s_n, \tag{2.3.6}$$

for all $x \in C$, $p \in F(T)$ and $n = 1, 2, 3, \ldots$;

(7) *uniformly L-Lipschitzian* if there exists constant $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \tag{2.3.7}$$

for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(8) $(L - \gamma)$ *uniform Lipschitz* if there are constants $L > 0$ and $\gamma > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|^{\gamma}, \tag{2.3.8}$$

for all $x, y \in C$ and $n = 1, 2, 3, \ldots$;

(9) *semi-compact* if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

Remark 2.3.5. It is easy to see that,

(1) a nonexpansive mapping is also quasi-nonexpansive and asymptotically nonexpansive;

(2) an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive;

(3) a quasi-nonexpansive mapping is generalized quasi-nonexpansive;

(4) an asymptotically quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive;

(5) a quasi-nonexpansive mapping is asymptotically quasi-nonexpansive;

(6) a generalized quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive;

(7) a uniformly L-Lipschitzian mapping is $(L-1)$ uniform Lipschitz.

The following diagram shows the relationship between nonexpansive, quasi-nonexpansive, asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings.

$$\begin{tikzcd}
\text{Nonexpansive Mappings} & \text{Asymp. nonexpansive Mappings} \arrow[l] \\
\text{Quasi-nonexpansive Mappings} \arrow[u] & \text{Asymp. Quasi-nonexpansive Mappings} \arrow[l] \arrow[u]
\end{tikzcd}$$

Example 2.3.6. Let $C = [1, 20]$ and $T : C \to C$, $Tx = \dfrac{1}{x}$, for all $x \in C$. Then T is nonexpansive, T has unique fixed point, $F(T) = \{1\}$. $\square$

Now we would like to show that the class of quasi-nonexpansive mappings properly includes that of nonexpansive maps with fixed points.

Example 2.3.7. ([12]) Let $X = l_\infty$ and $K := \{x \in l_\infty : \|x\| \le 1\}$. Define $f : K \to K$ by $f(x) = (0, x_1^2, x_2^2, x_3^3, \ldots)$ for $x = (x_1, x_2, x_3, \ldots)$ in K. Then it is clear that f is continuous and maps K into K. Moreover $f(p) = p$ if and only if $p = 0$. Furthermore,

$$\begin{aligned}
\|f(x) - p\|_\infty &= \|f(x)\|_\infty = \|(0, x_1^2, x_2^2, x_3^2, \ldots)\|_\infty \\
&\le \|(0, x_1, x_2, x_3, \ldots)\|_\infty \\
&= \|x\|_\infty = \|x - p\|_\infty
\end{aligned}$$

for all x in K. Therefore, f is quasi-nonexpansive. However, f is not nonexpansive.

For $x = (\frac{3}{4}, \frac{3}{4}, \ldots)$ and $y = (\frac{1}{2}, \frac{1}{2}, \ldots)$, it is clear that x and y belong to K. Furthermore, $\|x-y\|_\infty = \|(\frac{1}{4}, \frac{1}{4}, \ldots)\| = \frac{1}{4}$, and $\|f(x)-f(y)\|_\infty = \|(0, \frac{5}{16}, \frac{5}{16}, \ldots)\|_\infty = \frac{5}{16} > \frac{1}{4} = \|x - y\|_\infty$. $\square$

The next example shows that there is an asymptotically quasi-nonexpansive mapping which is not quasi-nonexpansive.

Example 2.3.8. ([26]) Let $X = l^2$ with the norm $\|\cdot\|$ defined by

$$\|x\| = \sqrt{\sum_{i=1}^{\infty} x_i^2}, \quad \text{for all } x = (x_1, x_2, \ldots, x_n, \ldots) \in X,$$

and a mapping $T : X \to X$ defined by

$$Tx = (0, 2x_1, 0, 0, \ldots, 0, \ldots).$$

By letting $Tx = x$ for any $x = (x_1, x_2, \ldots, x_n, ...) \in X$, we have

$$(0, 2x_1, 0, 0, \ldots, 0, \ldots) = (x_1, x_2, \ldots, x_n, ...),$$

i.e., $F(T) = \{0\}$. Moreover,

$$T^n x = (0, 0, 0, \ldots, 0, \ldots)$$

for all $n = 2, 3, 4, \ldots$.

For the sequence $\{r_n\}$, where $r_n = \frac{1}{n}$, and $p \in F(T)$, we have

$$\|Tx \quad - \quad p\| = 2\|x_1\| \leq (1 + r_1)\|x - p\| \quad \text{and} \quad \|T^n x - p\| \leq (1 + r_n)\|x - p\|,$$

for all $n = 2, 3, 4, \ldots$. This implies that T is an asymptotically quasi-nonexpansive mapping. However, T is not a quasi-nonexpansive mapping since, for $x^0 = (1, 0, 0, \ldots, 0, \ldots)$ in X,

$$\|Tx^0 - p\| = \|(0, 2, 0, 0, \ldots, 0, \ldots)\| = 2 > 1 = \|x^0 - p\|. \qquad \square$$

Now we would like to show an example of Lipschitzian mappings which are not nonexpansive.

Example 2.3.9. [24] Let B denote the unit ball in the Hilbert space l^2 and let U defined as follows:

$$U : (x_1, x_2, x_3, \ldots) \to (0, x_1^2, a_2 x_2, a_3 x_3, \ldots),$$

where $\{a_i\}$ is a sequence of numbers such that $0 < a_i < 1$ and $\prod_{i=2}^{\infty} a_i = \frac{1}{2}$. Then U is Lipschitzian and $\|Ux - Uy\| \leq 2\|x - y\|$, $x, y \in B$; moreover,

$$\|U^i x - U^i y\| \leq 2 \prod_{j=2}^{i} a_j \|x - y\| \quad \forall\ i = 1, 2, \ldots.$$

Thus $\lim_{i \to \infty} k_i = \lim_{i \to \infty} 2 \prod_{j=2}^{i} a_j = 1$. Clearly, U is not nonexpansive. $\square$

Example 2.3.10. ([48]) Let $K = [\frac{-1}{\pi}, \frac{1}{\pi}]$ and define $Tx = \frac{x}{2}\sin(\frac{1}{x})$ if $x \neq 0$ and $Tx = 0$ if $x = 0$. Then $T^n x \to 0$ uniformly but T is not Lipschitz. Notice that $F(T) = \{0\}$. For each fixed n, define $f_n(x)$=$\|T^n x\| - \|x\|$ for $x \in K$. Set $c_n = \sup_{x \in K} f_n(x) \vee 0 = \sup_{x \in K}(\|T^n x\| - \|x\|) \vee 0$. Then $\lim_{n \to \infty} c_n = 0$ and

$$\|T^n x\| \leq \|x\| + c_n.$$

This show that T is a generalized asymptotically quasi-nonexpansive but it is not asymptotically quasi-nonexpansive and asymptotically nonexpansive because it is not Lipschitz. $\square$

Definition 2.3.11. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Let $B : C \to H$ be a mapping.

(1) A mapping $T : C \to C$ is said to be $\kappa-$*strictly pseudo-contrative* [7] if there exists $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \tag{2.3.9}$$

for all $x, y \in C$;

(2) B is said to be *monotone* if

$$\langle Bx - By, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

(3) B is said to be α*-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in C.$$

(4) B is said to be α*-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Bx - By, x - y\rangle \geq \alpha\|Bx - By\|^2, \quad \forall x, y \in C.$$

Definition 2.3.12. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H.

(1) A set-valued mapping $T : H \to 2^H$ is said to be *monotone* if, for all $x, y \in H, f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.

(2) A monotone mapping $T : H \to 2^H$ is said to be *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

In the other words, a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H, \langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$.

We know that $\kappa-$strict pseudo-contraction includes class of nonexpansive mapping. If $\kappa = 1$, T is said to be a *pseudo-contraction mapping*. T is *strong pseudo-contraction* if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contraction. In a real Hilbert space H, the inequality (2.3.9) is equivalent to

$$\langle Tx-Ty, x-y\rangle \leq \|x-y\|^2 - \frac{1-\kappa}{2}\|(I-T)x-(I-T)y\|^2 \ \forall\, x, y \in D(T). \tag{2.3.10}$$

The class of $\kappa-$strict pseudo-contraction falls into the one between classes of nonexpansive mappings and pseudo-contraction mappings. The class of strong pseudo-contraction mappings is independent of the class of $\kappa-$strict pseudo-contraction.

Example 2.3.13. [52] Let T be a nonexpansive mapping of C into itself and set $A = I - T$. Then A is $\frac{1}{2}$-inverse strongly monotone.

The metric projection P_C of H onto C is an important example of inverse strongly monotone operators, see Problem 5.2.1 [52].

Lemma 2.3.14. [52] *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be α-inverse-strongly monotone. If $0 < \lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H.*

Lemma 2.3.15. ([44]) Let $B : C \to H$ be a monotone mapping and $N_c v$ be the normal cone to C at $v \in C$, i.e., $N_c v = \{w \in H : \langle v - u, w\rangle \geq 0, \quad \forall u \in C\}$. Define a mapping T on C by

$$Tv = \begin{cases} Bv + N_c v & \text{if } v \in C \\ \emptyset & \text{if } v \notin C. \end{cases} \tag{2.3.11}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Bv, u - v\rangle \geq 0$ for all $u \in C$.

Let B be a β-inverse-strongly monotone mapping of C into H. It is easy to show that B is $\frac{1}{\beta}$-Lipschitz.

Definition 2.3.16. ([35]) Let C be a nonempty closed subset of a Hilbert space H. Let $\{T_n\}$ and Γ be two families of nonlinear mappings of C into itself such that $F(\Gamma) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, where $F(\Gamma) = \bigcap_{T\in\Gamma} F(T)$. $\{T_n\}$ is said to satisfy the *NST-condition* with Γ if for each bounded sequence $\{z_n\} \subset C$,

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \|z_n - T z_n\| = 0, \quad \text{for all } T \in \Gamma.$$

In the case $\Gamma \in \{T\}$, i.e., Γ consists of one mapping T, $\{T_n\}$ is said to satisfy the NST-condition with T.

We now define a new condition (A$''$).

Definition 2.3.17. Let C be a subset of a normed space X.

(1) A selfmapping T of C is said to have *condition (A)* [56] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x,F))$ for all $x \in C$ where $d(x,F) = \inf\{\|x - p\| : p \in F = F(T)\}$.

(2) A family of selfmappings $\{T_1, T_2\}$ of C is said to have *condition (A$'$)* [23] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - T_i x\| \geq f(d(x,F))$ for some $1 \leq i \leq 2$ and for all $x \in C$ where $d(x,F) = \inf\{\|x-p\| : p \in F = F(T_1)\bigcap F(T_2)\}$.

(3) A family of selfmappings $\{T_i : i = 1,2,\ldots,k\}$ of C is said to have *condition (A$''$)* [26] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - T_i x\| \geq f(d(x,F))$ for some $1 \leq i \leq k$ and for all $x \in C$ where $d(x,F) = \inf\{\|x-p\| : p \in F = \bigcap_{i=1}^{k} F(T_1)\}$.

Lemma 2.3.18. *([1]) Let C be a nonempty closed bounded subset of a Banach space X and $T : C \to C$ a mapping with $F(T) \neq \emptyset$. If $I - T$ maps closed bounded subsets of C onto closed subsets of X, then T satisfies Condition (A) on C.*

Definition 2.3.19. The map $T : C \to X$ is said to be *demiclosed at 0* if for each sequence $\{x_n\}$ in C converging weakly to $x \in C$ and Tx_n converging strongly to 0, we get $Tx = 0$.

2.4 Fixed Point Theory of Nonexpansive Mappings and $\kappa-$Strictly Pseudo-contractive Mappings

Let C be a closed bounded convex subset of a real Hilbert space H. In 1966 Petryshyn [42] studied the set of fixed points of T of the Krasnoselsij iteration defined by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \qquad n = 1,\ 2,\ \ldots \tag{2.4.1}$$

in Hilbert space, where $\lambda \in (0,1)$ and T is a nonexpansive and demicompact operator. It was found that the set of fixed points is nonempty and $\{x_n\}$ converges strongly.

In 1974, Senter and Dotson [46] studied the convergence of the Mann iteration scheme defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n,\ \forall n \geq 1, \tag{2.4.2}$$

in a uniformly convex Banach space, where $\{\alpha_n\}$ is a sequence satisfying $0 < a \leq \alpha_n \leq b < 1\ \forall n \geq 1$ and T is a nonexpansive (or a quasi-nonexpansive) mapping.

In 1993, Tan and Xu [56] proved weak convergence of the Ishikawa iteration scheme defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n T\big(\beta_n T x_n + (1-\beta_n)x_n\big) + (1-\alpha_n)x_n, \tag{2.4.3}$$

in uniformly convex Banach space which satisfies Opial's condition, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1\ \forall n \geq 1$ and T is nonexpansive mapping.

Lemma 2.4.1. ([52]) Let H be a real Hilbert space, C be a nonempty closed convex subset of H and a mapping T of H into itself be nonexpansive. Then $F(T)$ is always closed convex and also nonempty provided T has a bounded trajectory.

Lemma 2.4.2. [15, Lemma 1.6] Let X be a uniformly convex Banach space, C be a nonempty closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e., if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$.

Lemma 2.4.3. ([32]) Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C. If S is a κ-strict pseudo-contraction mapping, then S satisfies the Lipschitz condition

$$\|Sx - Sy\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|,\ \forall x,\ y \in C.$$

2.5 Equilibrium Problems and Generalized Mixed Equilibrium Problems in Hilbert Spaces

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Let $B : C \to H$ be a nonlinear mapping, $\varphi : C \to \mathbb{R} \bigcup \{+\infty\}$ be a function and $f : C \times C \to \mathbb{R}$ be a bifunction. Peng and Yao [38] considered the following *generalized mixed equilibrium problem*:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \varphi(y) + \langle Bu, y - u \rangle \geq \varphi(u), \quad \forall y \in C. \quad (2.5.1)$$

In this thesis, we denote the set of solutions of (2.5.1) by $GMEP\ (f, \varphi, B)$. It is obvious that if u is a solution of (2.5.1), it implies that $u \in$ dom $\varphi = \{u \in C : \varphi(u) < +\infty\}$.

If $B = 0$ in (2.5.1), we obtain the following *mixed equilibrium problem* [13]:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \varphi(y) \geq \varphi(u), \quad \forall y \in C. \quad (2.5.2)$$

We denote the set of solutions of (2.5.2) by $MEP\ (f, \varphi)$.

If $\varphi = 0$ in (2.5.1), we obtain the following *generalized equilibrium problem* [55]:

$$\text{Finding } u \in C \text{ such that } f(u, y) + \langle Bu, y - u \rangle \geq 0, \quad \forall y \in C. \quad (2.5.3)$$

We denote the set of solutions of (2.5.3) by $GEP\ (f, B)$.

If $\varphi = 0$ and $B = 0$ in (2.5.1), we obtain the following *equilibrium problem* [6]:

$$\text{Finding } u \in C \text{ such that } f(u, y) \geq 0, \quad \forall y \in C. \quad (2.5.4)$$

We denote the set of solutions of (2.5.4) by $EP\ (f)$.

If $f(x, y) = 0$ for all $x, y \in C$ in (2.5.1), we obtain the following *generalized variational inequality problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) + \langle Bu, y - u \rangle \geq \varphi(u), \quad \forall y \in C. \quad (2.5.5)$$

We denote the set of solutions of (2.5.5) by $GVI\ (C, \varphi, B)$.

If $\varphi = 0$ and $f(x, y) = 0$ for all $x, y \in C$ in (2.5.1), we obtain the following *variational inequality problem* (see also [3, 17]):

$$\text{Finding } u \in C \text{ such that } \langle Bu, y - u \rangle \geq 0, \quad \forall y \in C. \quad (2.5.6)$$

We denote the set of solutions of (2.5.6) by $VI\ (C, B)$.

If $B = 0$ and $f(x, y) = 0$ for all $x, y \in C$ in (2.5.1), we obtain the following *minimization problem*:

$$\text{Finding } u \in C \text{ such that } \varphi(y) \geq \varphi(u), \quad \forall y \in C. \tag{2.5.7}$$

We denote the set of solutions of (2.5.7) by $MP(C, \varphi)$.

Lemma 2.5.1. [52] *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then for* $\lambda > 0$,

$$u \in VI(C, A) \Leftrightarrow u = P_C(I - \lambda A)u,$$

where P_C is the metric projection of H onto C.

Theorem 2.5.2. [52] *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\alpha > 0$ and let $A : C \to H$ be α-inverse-strongly monotone. Then $VI(C, A) \neq \emptyset$.*

In 1994, Blum and Oettli [3] showed that the formulation of (2.5.4) covered monotone inclusive problems, saddle point problems, variational inequality problems, minimization problems, optimization problems, variational inequality problems, vector equilibrium problems and Nash equilibria in noncooperative games. Several problems in physics, optimization and economics can be reduced to find a solution of (2.5.4). The existence of equilibrium problems has been discovered by many authors (see, for example, [3, 21, 34, 54] and the references therein). Also, some solution methods have been studied by some authors (see, for example, [21, 43, 54] and the references therein).

In 2003, Takahashi and Toyoda [55] introduced the method for finding an element of $F(S) \cap VI(C, A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to H$ is a nonexpansive mapping and $A : C \to H$ is an inverse-strongly monotone mapping. Their iteration is the following:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), n \geq 0,$$

where $x_0 \in C$, $\{\alpha_n\}$ is a sequence in (0,1), $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ and P_C is the metric projection from H onto C. They proved that, if $F(S) \cap VI(C, A) \neq \emptyset$, $\{x_n\}$ converges weakly to a point $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

Later, Takahashi and Takahashi [54] studied the contraction method for finding $F(S) \cap EP(f)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to H$ is a nonexpansive mapping, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions. Their algorithm is the following:

$$\begin{aligned}
&x_1 \in H,\\
&f(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall u \in C,\\
&x_{n+1} = \alpha_n f_1(x_n) + (1 - \alpha_n) S y_n \quad \forall n \geq 1,
\end{aligned}$$

where $\{\alpha_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and some appropriate conditions. They proved that, if $F(S) \cap EP(f) \neq \emptyset$, $\{x_n\}$ and $\{y_n\}$ converge strongly to a point $z \in F(S) \cap EP(f)$, where $z = P_{F(S)\cap EP(f)}f(z)$.

Recently, Cho et al. [14] introduced a hybrid projection method for finding $F := F(S) \cap VI(C,B) \cap GEP(f,A)$ in real Hilbert spaces, where $C \subset H$ is closed and convex, $S : C \to C$ is a k-strict pseudo-contraction with a fixed point, f is a bifunction from $C \times C$ to $\mathbb{R}$ with some specific conditions, $A : C \to H$ is an α-inverse-strongly monotone mapping and $B : C \to H$ is an β-inverse-strongly monotone mapping. Their iterative scheme is the following:

$$
\begin{aligned}
& x_1 \in C, \\
& C_1 = C, \\
& f(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
& z_n = P_C(u_n - \lambda_n Bu_n), \\
& y_n = \alpha_n x_n + (1-\alpha_n)S_k z_n, \\
& C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\
& x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \geq 1,
\end{aligned}
$$

where $S_k x = kx + (1-k)Sx$ for all $x \in C$, $\{\alpha_n\} \subset [0,1)$, $\{\lambda_n\} \subset (0, 2\beta)$ and $\{r_n\} \subset (0, 2\alpha)$ and some appropriate conditions. They proved that, if $F \neq \emptyset$, $\{x_n\}$ converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

For solving the generalized mixed equilibrium problem, we may assume the following conditions for the bifunction f, the function φ and the set C:

(A1) $f(x,x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x \in C, y \mapsto f(x,y)$ is convex and lower semi-continuous;
(A4) for each $x \in C, y \mapsto f(x,y)$ is weakly upper semicontinuous;
(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C \bigcap \mathrm{dom}(\varphi)$ such that for any $z \in C \backslash D_x$,

$$f(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z);$$

(B2) C is a bounded set.

2.6 Some Useful Lemmas

Lemma 2.6.1. *(Cf. [50, Lemma 2.2].) Let the sequences $\{a_n\}$ and $\{\delta_n\}$ of real numbers satisfy:*

$$a_{n+1} \leq (1+\delta_n)a_n, \ \textit{where } a_n \geq 0, \ \delta_n \geq 0, \ \textit{for all } n = 1, 2, 3, \ldots$$

and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then

(1) $\lim_{n\to\infty} a_n$ *exists;*

(2) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.6.2. *(See [45, Lemma 1.3].) Let X be a uniformly convex Banach space. Assume that $0 < b \le t_n \le c < 1, n = 1, 2, 3, \ldots$. Let the sequences $\{x_n\}$ and $\{y_n\}$ in X be such that*

$$limsup_{n\to\infty}\|x_n\| \le a,\ limsup_{n\to\infty}\|y_n\| \le a$$

and

$$lim_{n\to\infty}\|t_n x_n + (1-t_n)y_n\| = a,$$

where $a \ge 0$. Then $lim_{n\to\infty}\|x_n - y_n\| = 0$.

Lemma 2.6.3. *([38, 39, 40]). Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\varphi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z,y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \ge \varphi(z), \quad \forall y \in C\}$$

for all $x \in H$. Then the following conclusions hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \le \langle T_r(x) - T_r(y), x - y\rangle;$$

(3) *$F(T_r) = MEP\ (F, \varphi)$;*

(4) *$MEP\ (F, \varphi)$ is closed and convex.*

Lemma 2.6.4. *([67]). Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ be a k-strict pseudo-contraction. Define a mapping $S : C \to C$ by $Sx = \alpha x + (1-\alpha)Tx$ for all $x \in C$ and $\alpha \in [k, 1)$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

We would like to mention the following remark since our result is very interesting. It shows that a monotone mapping maps all points in a generalized mixed equlibrium problem to the same point.

Remark 2.6.5. ([27]). Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying *(A2)* and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a function. Let A be a monotone mapping of C into H. Then $Au = Av$ for all $u, v \in GMEP(f, \varphi, A)$.

Proof. Let $u, v \in GMEP(f, \varphi, A)$. We then get

$$f(u, y) + \varphi(y) + \langle Au, y - u\rangle \geq \varphi(u), \quad \forall y \in C \tag{2.6.1}$$

and

$$f(v, y) + \varphi(y) + \langle Av, y - v\rangle \geq \varphi(v), \quad \forall y \in C. \tag{2.6.2}$$

By letting $y = v$ in (2.6.1) and $y = u$ in (2.6.2), we get

$$f(u, v) + \varphi(v) + \langle Au, v - u\rangle \geq \varphi(u) \tag{2.6.3}$$

and

$$f(v, u) + \varphi(u) + \langle Av, u - v\rangle \geq \varphi(v). \tag{2.6.4}$$

By (2.6.3), (2.6.4) and the condition (A2), we have

$$\langle Av - Au, u - v\rangle \geq f(u, v) + f(v, u) + \langle Au, v - u\rangle + \langle Av, u - v\rangle \geq 0. \tag{2.6.5}$$

From A is a α-inverse-strongly monotone mapping,

$$0 \leq \alpha\|Au - Av\|^2 \leq \langle Au - Av, u - v\rangle \leq 0.$$

That is $Au = Av$. □

By letting $f = 0$ and $\varphi = 0$ in Lemma 2.6.5, we obtain the following remark.

Remark 2.6.6. ([27]). Let C be a closed convex subset of a real Hilbert space H and A be a monotone mapping of C into H. Then $Au = Av$ for all $u, v \in VI(C, A)$.