Chapter 3
Common Fixed Points of Asymptotically Quasi-nonexpansive Mappings

3.1 A New Approximation Method for Common Fixed Points of a Finite Family of Asymptotically Quasi-nonexpansive Mappings in Banach Spaces

In 2008, Khan et al. [28] introduced an iterative process for a finite family of mappings as follows:

Let C be a convex subset of a Banach space X and let $\{T_i : i = 1, 2, \ldots, k\}$ be a family of selfmappings of C. Suppose that $\alpha_{in} \in [0, 1]$, for all $n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k$.

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned}
x_{n+1} &= (1-\alpha_{kn})x_n + \alpha_{kn}T_k^n y_{(k-1)n},\\
y_{(k-1)n} &= (1-\alpha_{(k-1)n})x_n + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n},\\
y_{(k-2)n} &= (1-\alpha_{(k-2)n})x_n + \alpha_{(k-2)n}T_{k-2}^n y_{(k-3)n},\\
&\ \vdots \\
y_{2n} &= (1-\alpha_{2n})x_n + \alpha_{2n}T_2^n y_{1n},\\
y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n},
\end{aligned} \tag{3.1.1}$$

where $y_{0n} = x_n$ for all n. The iterative process (3.1.1) is the generalized form of the modified Mann (one-step) iterative process by Schu [45], the modified Ishikawa (two-step) iterative process by Tan and Xu [57], and the three-step iterative process by Xu and Noor [59].

In 2007, Thianwan and Suantai [58] introduced the following implicit iteration for a finite family of nonexpansive self-mappings of C $\{T_i : i \in J\}$ with α_n and β_n are two real sequences in [0,1] and an initial point $x_0 \in C$:

$$x_n = \alpha_n x_{n-1} + \beta_n T_n x_{n-1} + (1-\alpha_n-\beta_n)T_n x_n, \quad \forall n \geq 1,$$

where $T_n = T_{n(mod\ N)}$ (here the *mod* N function takes values in J). We notice that x_n is calculated from x_{n-1} for each $n \in N$.

Goebel and Kirk [24], in 1972, introduced the new concept of asymptotically nonexpansive and proved that every asymptotically nonexpansive self-mapping of a nonempty closed bounded and convex subset of a uniformly convex Banach space has a fixed point. In 1978, Bose [4] studied an iterative scheme for fixed points of asymptotically nonexpansive mappings. In 2001, Khan and Takahashi [29] used the modified Ishikawa process to approximate common fixed points of two asymptotically nonexpansive mappings.

Common fixed points of nonlinear mappings play an important role in solving systems of equations and inequalities. Many researchers ([31, 65, 66]) are interested in studying approximation method for finding common fixed points of nonlinear mapping. Also, approximation methods for finding fixed points for nonexpansive mappings can be seen in [8, 9, 10, 11, 41, 62, 64].

In 2003, Sun [50] studied an implicit iterative scheme initiated by Xu and Ori [61] for a finite family of asymptotically quasi-nonexpansive mappings. Zhou et al. [68] introduced a new concept of generalized asymptotially nonexpansive mappings and provided a sufficient and necessary condition for the modified Ishikawa and Mann iterative process to fixed points for the class of mappings. Shahzand and Udomene [47], in 2006, proved some convergence theorems for the modified Ishikawa iterative process of two asymptotically quasi-nonexpence mappings to a common fixed point. Nammanee et al. [36] introduced a three-step iteration scheme for asymptotically nonexpansive mappings and proved weak and strong convergence theorems of that iteration scheme under some control conditions. In 2007, Fukhar-ud-din and Khan [22] studied a new three-step iteration scheme for approximating a common fixed point of asymptotically nonexpansive mappings in uniformly convex Banach spaces. Recently, Khan et al. [28] introduced the iterative sequence (3.1.1) for a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces.

Motivated by Khan et al. [28] and [58], we introduce a new iterative scheme for finding a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings as follow:

For $x_1 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} x_{n+1} &= (1-\alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1-\alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ &\vdots \\ y_{2n} &= (1-\alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \tag{3.1.2}$$

where $y_{0n} = x_n$ for all n.

The aim of this chapter is to obtain some strong and weak convergence results for the iterative process (3.1.2) of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces.

3.2 Strong Convergence Theorems for a Finite Family of Asymptotically Quasi-nonexpansive Mappings in Banach Spaces

The aim of this section is to establish the strong convergence of the iterative scheme (3.1.2) to converge to a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in a Banach space under some appropriate conditions.

Lemma 3.2.1. *Let C be a nonempty closed convex subset of a real Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of asymtotically quasi-nonexpansive self-mappings of C, i.e., $\|T_i^n x - p_i\| \le (1 + r_{in})\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i), i = 1, 2, \ldots, k$. Suppose that $F = \bigcap_{i=1}^{k} F(T_i) \ne \emptyset$, $x_1 \in C$, and the iterative sequence $\{x_n\}$, are defined by (3.1.2). For $p \in F$, we get*

(1) $\|x_n - T_i^n x_n\| \le (2 + r_n)\|x_n - p\|$, *for all* $i = 1, 2, \ldots, k$;

(2) $\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \le (2 + r_n)\|y_{(i-1)n} - p\|$, *for all* $i = 1, 2, \ldots, k$;

(3) $\|T_i^n y_{(i-1)n} - p\| \le (1 + r_n)\|y_{(i-1)n} - p\|$, *for all* $i = 1, 2, \ldots, k$;

(4) $\|y_{in} - p\| \le (1 + r_n)^i \|x_n - p\|$, *for* $i = 1, 2, \ldots, k-1$;

(5) $\|x_{n+1} - p\| \le (1 + \delta_n)\|x_n - p\|$;

(6) *if* $\sum_{n=1}^{\infty} r_n < \infty$, $\lim_{n\to\infty} \|x_n - p\|$ *exists,*

where $r_n = \max_{1 \le i \le k}\{r_{in}\}$ *and* $\delta_n = \binom{k}{1} r_n + \binom{k}{2} r_n^2 + \ldots + \binom{k}{k} r_n^k$.

Proof. Let $p \in F$.

(1) For $i = 1, 2, 3, \ldots, k$, we have

$$\begin{aligned} \|x_n - T_i^n x_n\| &\le \|x_n - p\| + \|T_i^n x_n - p\| \\ &\le \|x_n - p\| + (1 + r_n)\|x_n - p\| \\ &= (2 + r_n)\|x_n - p\|. \end{aligned} \tag{3.2.1}$$

(2) Similarly to part (1), we have

$$\|y_{(i-1)n} - T_i^n y_{(i-1)n}\| \le (2 + r_n)\|y_{(i-1)n} - p\|, \text{ for all } i = 1, 2, \ldots, k.$$

(3) For $i = 1, 2, \ldots, k$, we have

$$\begin{aligned} \|T_i^n y_{(i-1)n} - p\| &\le (1 + r_{in})\|y_{(i-1)n} - p\| \\ &\le (1 + r_n)\|y_{(i-1)n} - p\|. \end{aligned}$$

(4) By part (1) and $\alpha_{1n} \le 1$, we obtain

$$\begin{aligned} \|y_{1n} - p\| &= \|(1 - \alpha_{1n})(x_n - p) + \alpha_{1n}(T_1^n x_n - p)\| \\ &\le (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}\|T_1^n x_n - p\| \\ &\le (1 - \alpha_{1n})\|x_n - p\| + \alpha_{1n}(1 + r_n)\|x_n - p\| \\ &\le (1 + r_n)\|x_n - p\|. \end{aligned}$$

We assume that $\|y_{jn} - p\| \leq (1 + r_n)^j \|x_n - p\|$ holds for some $1 \leq j \leq k - 2$. From part (3) and $\alpha_{(j+1)n} \leq 1$, we then have

$$\begin{aligned}
\|y_{(j+1)n} - p\| &= \|(1 - \alpha_{(j+1)n})(y_{jn} - p) + \alpha_{(j+1)n}(T_{j+1}^n y_{jn} - p)\| \\
&\leq (1 - \alpha_{(j+1)n})\|y_{jn} - p\| + \alpha_{(j+1)n}\|T_{j+1}^n y_{jn} - p\| \\
&\leq (1 - \alpha_{(j+1)n})\|y_{jn} - p\| + \alpha_{(j+1)n}(1 + r_n)\|y_{jn} - p\| \\
&\leq (1 + r_n)\|y_{jn} - p\| \\
&\leq (1 + r_n)(1 + r_n)^j \|x_n - p\| \\
&= (1 + r_n)^{j+1} \|x_n - p\|.
\end{aligned}$$

Therefore, by mathematical induction, we obtain

$$\|y_{in} - p\| \leq (1 + r_n)^i \|x_n - p\|, \text{ for } i = 1, 2, \ldots, k - 1.$$

(5) By part (2), part (4), and $\alpha_{kn} \leq 1$, we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| \\
&\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - p\| \\
&\leq (1 - \alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}(1 + r_n)\|y_{(k-1)n} - p\| \\
&\leq (1 + r_n)\|y_{(k-1)n} - p\| \\
&\leq (1 + r_n)(1 + r_n)^{k-1}\|x_n - p\| \\
&= (1 + r_n)^k \|x_n - p\| \\
&\leq (1 + \delta_n)\|x_n - p\|,
\end{aligned}$$

where $\delta_n = \binom{k}{1} r_n + \binom{k}{2} r_n^2 + \ldots + \binom{k}{k} r_n^k$.

(6) By (5), we have $\|x_{n+1} - p\| \leq (1 + \delta_n)\|x_n - p\|$ for all $n \in \mathrm{N}$. From $\Sigma_{n=1}^{\infty} r_n < \infty$, we also have $\Sigma_{n=1}^{\infty} r_n^i < \infty$ for $i = 1, 2, 3, \ldots, k$. It follows that $\Sigma_{n=1}^{\infty} \delta_n < \infty$. By Lemma 2.6.1, we get $\lim_{n\to\infty} \|x_n - p\|$ exists. □

Theorem 3.2.2. *Let C be a nonempty closed convex subset of a real Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of asymtotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i), i = 1, 2, \ldots, k$. Suppose that $F = \bigcap_{i=1}^{k} F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (3.1.2). Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1\leq i\leq k}\{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F} \|x - p\|$.*

Proof. The necessity is obvious and then we prove only the sufficiency. Let $p \in F$. By Lemma 3.2.1(6), $\lim_{n\to\infty} \|x_n - p\|$ exists and hence $\{\|x_n - p\|\}$ is bounded. We let $M = \sup_{n\geq 1}\{\|x_n - p\|\}$. From Lemma 3.2.1(5), we get

$$\|x_{n+1} - p\| \leq \|x_n - p\| + M\delta_n, \quad n \geq 1,$$

where $\delta_n = \binom{k}{1}r_n + \binom{k}{2}r_n^2 + \ldots + \binom{k}{k}r_n^k$. Thus, for positive integers m and n, we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq \|x_{n+m-1} - p\| + M\delta_{n+m-1} \\ &\leq \|x_{n+m-2} - p\| + M(\delta_{n+m-1} + \delta_{n+m-2}) \\ &\vdots \\ &\leq \|x_n - p\| + M \sum_{i=n}^{n+m-1} \delta_i \end{aligned} \tag{3.2.2}$$

By Lemma 3.2.1(5), we obtain

$$d(x_{n+1}, F) \leq (1 + \delta_n)\, d(x_n, F).$$

From the given condition $\liminf_{n\to\infty} d(x_n, F) = 0$ and Lemma 2.6.1, we get

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{3.2.3}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. By (3.2.3) and $\sum_{n=1}^{\infty} \delta_n < \infty$, we get that for any $\epsilon > 0$, there exists a positive integer n_0 such that, for all $n \geq n_0$,

$$d(x_n, F) < \frac{\epsilon}{8} \quad \text{and} \quad \sum_{n=n_0}^{\infty} \delta_n < \frac{\epsilon}{2M}. \tag{3.2.4}$$

From the first inequality of (3.2.4), there exists $p_0 \in F$ such that

$$\|x_{n_0} - p_0\| < \frac{\epsilon}{4}. \tag{3.2.5}$$

For any positive integer m, by (3.2.2), (3.2.4) and (3.2.5), we obtain

$$\begin{aligned} \|x_{n_0+m} - x_{n_0}\| &\leq \|x_{n_0+m} - p_0\| + \|x_{n_0} - p_0\| \\ &\leq 2\|x_{n_0} - p_0\| + M \sum_{i=n_0}^{n_0+m-1} \delta_i \\ &< 2\left(\frac{\epsilon}{4}\right) + M\left(\frac{\epsilon}{2M}\right) = \epsilon. \end{aligned} \tag{3.2.6}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\} \to q \in X$. Actually, $q \in C$ because $\{x_n\} \subset C$ and C is a closed subset of X. Next we show that $q \in F$. Since $F(T_i)$ is a closed subset in C for all $i = 1, 2, \ldots, k$, so is $F = \bigcap_{i=1}^{k} F(T_i)$. From the continuity of $d(x, F)$ with $d(x_n, F) \to 0$ and $x_n \to q$ as $n \to \infty$, we get $d(q, F) = 0$ and then $q \in F$. Therefore, the proof is complete. □ □

Since any quasi-nonexpansive mapping is asymptotically quasi-nonexpansive, the next corollary is obtained immediately from Theorem 3.2.2.

Corollary 3.2.3. *Let C be a nonempty closed convex subset of a real Banach space X, and $\{T_i : i = 1, 2, \dots, k\}$ be a family of quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - p_i\| \le \|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i), i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (3.1.2). Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F} \|x - p\|$.*

Remark 3.2.4. *Since any quasi-nonexpansive mapping is asymptotically quasi-nonexpansive, Theorem 3.2.2 can be applied for all quasi-nonexpansive mappings.*

3.3 Weak and Strong Convergence Theorems for a Finite Family of Asymptotically Quasi-nonexpansive Mappings in Uniformly Convex Banach Spaces

In this section, we prove some strong and weak convergence results for the iterative process (3.1.2) on uniformly convex Banach spaces without using the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ appearing in Section 3.2. Instead, we consider $(L - \gamma)$ uniform Lipschitz mappings, condition (A''), semi-compact mappings, Opial property and demiclosed mappings at 0.

Theorem 3.3.1. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X. Let $\{T_i : i = 1, 2, \dots, k\}$ be a family of uniformly $(L - \gamma_i)$-Lipschitzian and asymptotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - T_i^n y\| \le L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \le (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i), i = 1, 2, \dots, k$. Suppose that $\{T_i : i = 1, 2, \dots, k\}$ satisfies condition (A'') and $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let $x_1 \in C$ and the iterative sequence $\{x_n\}$ be defined by (3.1.2). Assume that $\sum_{n=1}^\infty r_n < \infty$, where $r_n = \max_{1\le i\le k}\{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.*

Proof. Let $p \in F$. By Lemma 3.2.1(6), we get that $\lim_{n\to\infty} \|x_n - p\|$ exists. Then there is a real number $c \ge 0$ such that

$$\lim_{n\to\infty} \|x_n - p\| = c. \tag{3.3.1}$$

By Lemma 3.2.1(4), we have

$$\|y_{in} - p\| \le (1 + r_n)^i \|x_n - p\|, \text{ for } i = 1, 2, \dots, k - 1.$$

By taking lim sup on both sides of the above inequality, we get

$$\limsup_{n\to\infty} \|y_{in} - p\| \le c, \quad \text{for } i = 1, 2, \dots, k - 1. \tag{3.3.2}$$

Therefore, by Lemma 3.2.1(3) and (3.3.1), we obtain

$$\limsup_{n\to\infty} \|T_i^n y_{(i-1)n} - p\| \le c, \quad \text{for } i = 1, 2, \dots, k. \tag{3.3.3}$$

Since $\lim_{n\to\infty} \|x_{n+1} - p\| = c$, we have

$$\lim_{n\to\infty} \|(1 - \alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = c.$$

Using (3.3.2), (3.3.3) and Lemma 2.6.2, we conclude that

$$\lim_{n\to\infty} \|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0.$$

We assume that

$$\lim_{n\to\infty} \|y_{(j-1)n} - T_j^n y_{(j-1)n}\| = 0, \text{ for some } 2 \le j \le k. \tag{3.3.4}$$

By (3.1.2) and Lemma 3.2.1 (3), we have

$$\|x_{n+1} - p\| \le (1 + r_n)^{k-i}\|y_{in} - p\|, \text{ for all } i = 1, 2, \ldots, k-1.$$

This together with (3.3.4) and $r_n \to 0$ as $n \to \infty$, we obtain

$$c \le \liminf_{n\to\infty} \|y_{(j-1)n} - p\|. \tag{3.3.5}$$

By Lemma 3.2.1 (4), (3.1.2), and (3.3.5), we get

$$\lim_{n\to\infty} \|(1 - \alpha_{(j-1)n})(y_{(j-2)n} - p) + \alpha_{(j-1)n}(T_{j-1}^n y_{(j-2)n} - p)\| = \lim_{n\to\infty} \|y_{(j-1)n} - p\| = c.$$

Using (3.3.1), (3.3.3), Lemma 3.2.1(3) and Lemma 2.6.2, we conclude that

$$\lim_{n\to\infty} \|y_{(j-2)n} - T_{j-1}^n y_{(j-2)n}\| = 0.$$

Therefore, by mathematical induction, we obtain

$$\lim_{n\to\infty} \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \ldots, k. \tag{3.3.6}$$

From (3.1.2), we have

$$\|y_{in} - y_{(i-1)n}\| = \alpha_{in}\|T_i^n y_{(i-1)n} - y_{(i-1)n}\|, \quad \text{for } i = 1, 2, \ldots, k-1.$$

By (3.3.6), we obtain that

$$\|y_{in} - y_{(i-1)n}\| \to 0 \ \text{ as } n \to \infty, \quad \text{for } i = 1, 2, \ldots, k-1. \tag{3.3.7}$$

From

$$\|x_n - y_{in}\| \le \|x_n - y_{1n}\| + \|y_{1n} - y_{2n}\| + \ldots + \|y_{(i-1)n} - y_{in}\|,$$

for $i = 1, 2, \ldots, k-1$. It follows by (3.3.7) that

$$\|x_n - y_{in}\| \to 0 \ \text{ as } n \to \infty, \quad \text{for } i = 1, 2, \ldots, k-1. \tag{3.3.8}$$

From (3.3.6), when $i = 1$ we get $\lim_{n\to\infty} \|x_n - T_1^n x_n\| = 0$. For $2 \le i \le k$, we have

$$\begin{aligned} \|x_n - T_i^n x_n\| &\le \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - T_i^n x_n\| \\ &\le \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\| + L\|y_{(i-1)n} - x_n\|^{\gamma_i}. \end{aligned}$$

From (3.3.6) and (3.3.8), we conclude that

$$\lim_{n\to\infty} \gamma_{in} = \lim_{n\to\infty} \|x_n - T_i^n x_n\| = 0, \quad \text{for } i = 1, 2, \ldots, k. \tag{3.3.9}$$

where $\gamma_{in} = \|x_n - T_i^n x_n\|$. From (3.1.2), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq (1-\alpha_{kn})\|y_{(k-1)n} - x_n\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - x_n\| \\ &\leq (1-\alpha_{kn})\|y_{(k-1)n} - x_n\| \\ &\quad + \alpha_{kn}(\|T_k^n y_{(k-1)n} - y_{(k-1)n}\| + \|y_{(k-1)n} - x_n\|) \\ &= \|y_{(k-1)n} - x_n\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - y_{(k-1)n}\|. \end{aligned}$$

From (3.3.6) and (3.3.8),

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.3.10}$$

For $i = 1, 2, \ldots, k$, we have

$$\begin{aligned} \|x_{n+1} - T_i x_{n+1}\| &\leq \|x_{n+1} - T_i^{n+1} x_{n+1}\| + \|T_i x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\leq \gamma_{i(n+1)} + L\|x_{n+1} - T_i^n x_{n+1}\|^{\gamma_i} \\ &\leq \gamma_{i(n+1)} + L(\|x_{n+1} - x_n\| + \|x_n - T_i^n x_n\| \\ &\qquad + \|T_i^n x_n - T_i^n x_{n+1}\|)^{\gamma_i} \\ &\leq \gamma_{i(n+1)} + L(\|x_{n+1} - x_n\| + \gamma_{in} + L\|x_n - x_{n+1}\|^{\gamma_i})^{\gamma_i}. \end{aligned}$$

Using (3.3.9) and (3.3.10), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - T_i x_{n+1}\| = 0, \quad \text{for } i = 1, 2, \ldots, k.$$

Therefore, by using condition (A″), there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - T_j x_n\| = 0,$$

for some $1 \leq j \leq k$. That is

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

By Theorem 3.2.2, we conclude that $\{x_n\}$ converges strongly to a point $p \in F$. □

Lemma 3.3.2. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of $(L-\gamma_i)$ uniform Lipschitz and asymtotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - T_i^n y\| \leq L\|x-y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i), i = 1, 2, \ldots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (3.1.2) with $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1\leq i\leq k}\{r_{in}\}$. Then,*

(1) $\lim_{n\to\infty} \|x_n - T_i^n y_{(i-1)n}\| = 0$, *for all* $i = 1, 2, , \ldots, k$;

(2) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$, *for all* $i = 1, 2, , \ldots, k$.

Proof. (i) Let $p \in F$. By Lemma 3.2.1(6), we obtain that $\lim_{n\to\infty} \|x_n - p\|$ exits and we then suppose that

$$\lim_{n\to\infty} \|x_n - p\| = c. \tag{3.3.11}$$

By (3.3.11) and Lemma 3.2.1(4), we have

$$\limsup_{n\to\infty} \|y_{in} - p\| \le c, \quad \text{for } i = 1, 2, \ldots, k-1 \tag{3.3.12}$$

By (3.1.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &\le (1-\alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}\|T_k^n y_{(k-1)n} - p\| \\
&\le (1-\alpha_{kn})\|y_{(k-1)n} - p\| + \alpha_{kn}(1+r_n)\|y_{(k-1)n} - p\| \\
&\le (1+r_n)\|y_{(k-1)n} - p\| \\
&= (1+r_n)\|(1-\alpha_{(k-1)n})(y_{(k-2)n} - p) + \alpha_{(k-1)n}(T_{k-1}^n y_{(k-2)n} - p)\| \\
&\le (1+r_n)\Big((1-\alpha_{(k-1)n})\|y_{(k-2)n} - p\| \\
&\quad + \alpha_{(k-1)n}(1+r_n)\|y_{(k-2)n} - p\|\Big) \\
&\le (1+r_n)^2\|y_{(k-2)n} - p\| \\
&\ \ \vdots \\
&\le (1+r_n)^{k-i}\|y_{in} - p\|,
\end{aligned}$$

for some $i = 1, 2, \ldots, k-1$. It follows that

$$c \le \liminf_{n\to\infty} \|y_{in} - p\|, \quad \text{for } i = 1, 2, \ldots, k-1. \tag{3.3.13}$$

From (3.3.12) and (3.3.13), we obtain

$$\lim_{n\to\infty} \|y_{in} - p\| = c, \tag{3.3.14}$$

and then

$$\lim_{n\to\infty} \|(1-\alpha_{in})(y_{(i-1)n} - p) + \alpha_{in}(T_i^n y_{(i-1)n} - p)\| = c, \tag{3.3.15}$$

for $i = 1, 2, \ldots, k-1$.

By Lemma 3.2.1(3) and (3.3.14), we get

$$\limsup_{n\to\infty} \|T_i^n y_{(i-1)n} - p\| \le c, \quad \text{for } i = 1, 2, \ldots, k-1. \tag{3.3.16}$$

From (3.3.11), (3.3.12), (3.3.15), (3.3.16) and Lemma 2.6.2, we obtain

$$\lim_{n\to\infty} \|T_i^n y_{(i-1)n} - y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \ldots, k-1. \tag{3.3.17}$$

Now we want to show that (3.3.17) is also true for $i = k$. By Lemma 3.2.1 (4), we have

$$\begin{aligned}
\|T_k^n y_{(k-1)n} - p\| &\le (1+r_n)\|y_{(k-1)n} - p\| \\
&\le (1+r_n)(1+r_n)^{k-1}\|x_n - p\| \\
&= (1+r_n)^k\|x_n - p\|.
\end{aligned}$$

This implies by (3.3.11) that

$$\limsup_{n\to\infty} \|T_k^n y_{(k-1)n} - p\| \le c. \tag{3.3.18}$$

We also have

$$\lim_{n\to\infty} \|(1-\alpha_{kn})(y_{(k-1)n} - p) + \alpha_{kn}(T_k^n y_{(k-1)n} - p)\| = \lim_{n\to\infty} \|x_{n+1} - p\| = c.$$

Hence, by (3.3.12), (3.3.18) and Lemma 2.6.2, we obtain

$$\lim_{n\to\infty} \|y_{(k-1)n} - T_k^n y_{(k-1)n}\| = 0. \tag{3.3.19}$$

Then, (3.3.17) and (3.3.19) give us

$$\lim_{n\to\infty} \|T_i^n y_{(i-1)n} - y_{(i-1)n}\| = 0, \quad \text{for } i = 1, 2, \dots, k. \tag{3.3.20}$$

From

$$\|x_n - T_i^n y_{(i-1)n}\| \le \|x_n - y_{(i-1)n}\| + \|y_{(i-1)n} - T_i^n y_{(i-1)n}\|,$$

It implies by (3.3.8) and (3.3.20) that

$$\lim_{n\to\infty} \|x_n - T_i^n y_{(i-1)n}\| = 0, \tag{3.3.21}$$

for some $i = 1, 2, 3, \dots, k$.

(ii) From part (i), for $i = 1$, we have

$$\lim_{n\to\infty} \|T_1^n x_n - x_n\| = 0. \tag{3.3.22}$$

For $i = 2, 3, 4, \dots, k$, we get

$$\begin{aligned} \|T_i^n x_n - x_n\| &\le \|T_i^n x_n - T_i^n y_{(i-1)n}\| + \|T_i^n y_{(i-1)n} - x_n\| \\ &\le L\|x_n - y_{(i-1)n}\|^{\gamma_i} + \|T_i^n y_{(i-1)n} - x_n\| \end{aligned}$$

By part (1) and (3.3.8), we conclude that

$$\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k. \tag{3.3.23}$$

For $1 \le i \le k$, we obtain

$$\begin{aligned} \|x_n - T_i x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\quad + \|T_i^{n+1} x_{n+1} - T_i^{n+1} x_n\| + \|T_i^{n+1} x_n - T_i x_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_i^{n+1} x_{n+1}\| \\ &\quad + L\|x_{n+1} - x_n\|^{\gamma_i} + L\|T_i^n x_n - x_n\|^{\gamma_i}. \end{aligned}$$

From (3.3.23), we then have

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \quad \text{for } i = 1, 2, \dots, k.$$

□

Theorem 3.3.3. *Under the hypotheses of Lemma 3.3.2, assume that T_j^m is semi-compact for some positive integers m and $1 \leq j \leq k$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, k\}$.*

Proof. Suppose that T_j^m is semi-compact for some positive integers $m \geq 1$ and $1 \leq j \leq k$. We have

$$\begin{aligned} \|T_j^m x_n - x_n\| &\leq \|T_j^m x_n - T_j^{m-1} x_n\| + \|T_j^{m-1} x_n - T_j^{m-2} x_n\| + \dots \\ &\quad + \|T_j^2 x_n - T_j x_n\| + \|T_j x_n - x_n\| \\ &\leq (m-1)L\|T_j x_n - x_n\|^{\gamma_j} + \|T_j x_n - x_n\|. \end{aligned}$$

Then, by Lemma 3.3.2(2), we get $\|T_j^m x_n - x_n\| \to 0$ as $n \to \infty$. Since $\{x_n\}$ is bounded and T_j^m is semi-compact, there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \to q \in C$ as $l \to \infty$.

By continuity of T_i and Lemma 3.3.2(2), we obtain

$$\|q - T_j q\| = \lim_{l \to \infty} \|x_{n_l} - T_j\, x_{n_l}\| = 0, \quad \text{for all } i = 1, 2, \dots, k.$$

Therefore, $q \in F$ and then Theorem 3.2.2 implies that $\{x_n\}$ converges strongly to a common fixed point q of the family $\{T_i : i = 1, 2, \dots, k\}$. □

We note that in practical Theorem 3.3.3 is very useful in the case that one of $T_i, i = 1, 2, 3, \dots, k$, is semi-compact.

Theorem 3.3.4. *Let C be a nonempty closed convex subset of an uniformly convex real Banach space X satisfying the Opial property, and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and asymtotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i), i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (3.1.2) with $\alpha_{in} \in [\delta, 1 - \delta]$. Assume that $\sum_{n=1}^\infty r_n < \infty$, where $r_n = \max_{1 \leq i \leq k}\{r_{in}\}$. If $I - T_i, i = 1, 2, \cdots, k$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the family of mappings.*

Proof. Let $p \in F$. As proved in Theorem 3.2.2, we get $\lim_{n \to \infty} \|x_n - p\|$ exits and then $\{x_n\}$ is bounded. Since an uniformly convex Banach space is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to a point $z_1 \in C$. By Lemma 3.3.2, we have $\lim_{n \to \infty} \|x_n - T_i x_n\| = 0$. From $I - T_i$ is demiclosed at 0 for $i = 1, 2, \dots, k$, we obtain $T_i z_1 = z_1$. Therefore, $z_1 \in F$.

Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ converging weakly to a point $z_2 \in C$. We want to show that $z_1 = z_2$ in order to conclude that $\{x_n\}$ converges weakly to

z_1. Assume $z_1 \neq z_2$. By Opial property,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - z_1\| &= \lim_{j\to\infty} \|x_{n_j} - z_1\| \\ &< \lim_{j\to\infty} \|x_{n_j} - z_2\| \\ &= \lim_{n\to\infty} \|x_n - z_2\| \\ &= \lim_{k\to\infty} \|x_{n_k} - z_2\| \\ &< \lim_{k\to\infty} \|x_{n_k} - z_1\| \\ &= \lim_{n\to\infty} \|x_n - z_1\|. \end{aligned}$$

We get a contradiction. Hence, $\{x_n\}$ converges weakly to a common fixed point $p \in F$. □

The following remarks are obtained directly from the results in Section 3.2 and Section 3.3.

Remark 3.3.5. *It is not hard to show that Theorems 3.3.1, 3.3.3 and 3.3.4 can be extended to a finite family of generalized asymptotically quasi-nonexpansive mappings as we can see in [63].*

Remark 3.3.6. *It is clear that Theorems 3.3.1, 3.3.3 and 3.3.4 can be used for any quasi-nonexpansive mapping.*