Chapter 4
Equilibrium Problems and Fixed Point Problems of Quasi-nonexpansive Mappings

4.1 A Strong Convergence Theorem of Hybrid Methods for Generalized Mixed Equilibrium Problems and Fixed Point Problems of an Infinite Family of Lipschitzian Quasi-nonexpansive Mappings in Hilbert Spaces

In this section, motivated by the result in Section 2.5 , we prove a strong convergent theorem of a hybrid projection iterative method defined by (4.1.1) for finding a common element of the set of fixed points of an infinite family of Lipschitzian quasi-nonexpansive mappings, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem in the framework of real Hilbert spaces. Our main result can be deduced for nonexpansive mappings applied for strict pseudo-contraction mappings. It is clear that our result generalizes the work by [14].

Theorem 4.1.1. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4) *and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of Lipschitzian quasi-nonexpansive mappings of C into itself such that $\lim_{n\to\infty} \|S_n x - S_n y\| \leq L_n \|x - y\|$ for all $x, y \in C$, $\sup_n L_n = L$, $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n), \forall y \in C, \\ z_n = P_C(u_n - \lambda_n B u_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.1.1}$$

where $\{\alpha_n\} \subset [0,1), \{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (4.1.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

Proof. We divide our proof into 5 steps.

Step 1: We show that $F \subset C_n$ and C_n is closed and convex for all $n \geq 1$.

From the assumption, $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \geq 1$. Next, we show that C_{m+1} is closed and convex. For any $w \in C_m$, we see that

$$\|y_m - w\| \leq \|x_m - w\|$$

is equivalent to

$$\|x_m\|^2 - \|y_m\|^2 - 2\langle w, x_m - y_m\rangle \geq 0.$$

Therefore, C_{m+1} is closed and convex.

Since A is α-inverse-strongly monotone and B is β-inverse-strongly monotone, by Lemma 2.3.14, we get that $I - r_n A$ and $I - \lambda_n B$ are nonexpansive.

By nonexpansiveness of T_{r_n} and $I - r_n A$, we have

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(p - r_n Ap)\|^2 \\
&\leq \|(x_n - r_n A x_n) - (p - r_n Ap)\|^2 \\
&= \|(x_n - p) - r_n(Ax_n - Ap)\|^2 \\
&= \|x_n - p\|^2 - 2r_n\langle x_n - p, Ax_n - Ap\rangle + r_n^2\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2 - 2r_n\alpha\|Ax_n - Ap\|^2 + r_n^2\|Ax_n - Ap\|^2 \\
&= \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 \\
&\leq \|x_n - p\|^2. \qquad (4.1.2)
\end{aligned}$$

We are now ready to show that $F \subset C_n$ for each $n \geq 1$. From the assumption, we have that $F \subset C = C_1$. Suppose $F \subset C_m$ for some $m \geq 1$. For any $w \in F \subset C_m$, by nonexpansiveness of $I - \lambda_m B$, we have

$$\begin{aligned}
\|y_m - w\| &= \|\alpha_m x_m + (1 - \alpha_m) S_m z_m - w\| \\
&\leq \alpha_m\|x_m - w\| + (1 - \alpha_m)\|z_m - w\| \\
&= \alpha_m\|x_m - w\| + (1 - \alpha_m)\|P_C(I - \lambda_m B)u_m - P_C(I - \lambda_m B)w\| \\
&\leq \alpha_m\|x_m - w\| + (1 - \alpha_m)\|u_m - w\| \\
&\leq \alpha_m\|x_m - w\| + (1 - \alpha_m)\|x_m - w\| \\
&= \|x_m - w\|.
\end{aligned}$$

That is $w \in C_{m+1}$. By mathematical induction, we conclude that $F \subset C_n$ for each $n \geq 1$.

Step 2: We show that $\{x_n\}$ is bounded.

Since $x_n = P_{C_n}x_1$ and $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1} \subset C_n$, we get

$$\begin{aligned} 0 &\leq \langle x_1 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_1 - x_n, x_n - x_1 + x_1 - x_{n+1}\rangle \\ &\leq -\|x_1 - x_n\|^2 + \|x_1 - x_n\|\|x_1 - x_{n+1}\|. \end{aligned} \tag{4.1.3}$$

Thus

$$\|x_1 - x_n\| \leq \|x_1 - x_{n+1}\|. \tag{4.1.4}$$

Since $x_n = P_{C_n}x_1$, for any $w \in F \subset C_n$, we have

$$\|x_1 - x_n\| \leq \|x_1 - w\|. \tag{4.1.5}$$

In particular, we obtain

$$\|x_1 - x_n\| \leq \|x_1 - P_F x_1\|. \tag{4.1.6}$$

By (4.1.4) and (4.1.6), we get that $\lim_{n\to\infty}\|x_n - x_1\|$ exists. It implies that $\{x_n\}$ is bounded.

Step 3: We show that $\lim_{n\to\infty}\|x_n - Sx_n\| = 0$ for all $S \in \mathcal{S}$.

By using (4.1.3), we obtain that

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|x_n - x_1 + x_1 - x_{n+1}\| \\ &= \|x_n - x_1\|^2 + 2\langle x_n - x_1, x_1 - x_{n+1}\rangle + \|x_1 - x_{n+1}\|^2 \\ &= \|x_n - x_1\|^2 - 2\|x_n - x_1\|^2 + 2\langle x_n - x_1, x_n - x_{n+1}\rangle \\ &\quad + \|x_1 - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_n - x_1\|^2 \end{aligned}$$

and so

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{4.1.7}$$

Since $x_{n+1} = P_{C_{n+1}}x_1 \in C_{n+1}$, we have

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$$

and then

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_n - x_{n+1}\| \leq 2\|x_n - x_{n+1}\|.$$

By (4.1.7), we get

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{4.1.8}$$

On the other hand, we have

$$\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)S_n z_n\| = (1-\alpha_n)\|x_n - S_n z_n\|.$$

It follows from (4.1.8) and the assumption $0 \leq \alpha_n \leq a < 1$ that

$$\lim_{n\to\infty} \|x_n - S_n z_n\| = 0. \quad (4.1.9)$$

For any $w \in F$, we have

$$\begin{aligned} \|y_n - w\|^2 &= \|\alpha_n x_n + (1-\alpha_n)S_n z_n - w\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|S_n z_n - w\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|z_n - w\|^2. \end{aligned} \quad (4.1.10)$$

From (4.1.2), we obtain

$$\begin{aligned} \|y_n - w\|^2 &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|P_C(I-\lambda_n B)u_n - w\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|(I-\lambda_n B)u_n - (I-\lambda_n B)w\|^2 \\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|u_n - w\|^2 + \lambda_n^2\|Bu_n - Bw\|^2 \\ &\quad -2\lambda_n\langle u_n - w, Bu_n - Bw\rangle) \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|u_n - w\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2) \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2) \\ &\leq \|x_n - w\|^2 + (1-\alpha_n)\lambda_n(\lambda_n - 2\beta)\|Bu_n - Bw\|^2. \end{aligned}$$

We then have

$$\begin{aligned} (1-a)b(2\beta - c)\|Bu_n - Bw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &= (\|x_n - w\| - \|y_n - w\|)(\|x_n - w\| + \|y_n - w\|) \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|). \end{aligned}$$

By (4.1.8), we obtain that

$$\lim_{n\to\infty} \|Bu_n - Bw\| = 0. \quad (4.1.11)$$

On the other hand, since P_C is firmly nonexpansive and $I - \lambda_n B$ is nonexpansive, we have

$$\begin{aligned} \|z_n - w\|^2 &= \|P_C(I-\lambda_n B)u_n - P_C(I-\lambda_n B)w\|^2 \\ &\leq \langle (I-\lambda_n B)u_n - (I-\lambda_n B)w, z_n - w\rangle \\ &= \frac{1}{2}\{\|(I-\lambda_n B)u_n - (I-\lambda_n B)w\|^2 + \|z_n - w\|^2 \\ &\quad -\|(I-\lambda_n B)u_n - (I-\lambda_n B)w - (z_n - w)\|^2\} \\ &\leq \frac{1}{2}\{\|u_n - w\|^2 + \|z_n - w\|^2 \\ &\quad -\|u_n - z_n - \lambda_n(Bu_n - Bw)\|^2\} \quad (4.1.12) \\ &= \frac{1}{2}\{\|u_n - w\|^2 + \|z_n - w\|^2 - \|u_n - z_n\|^2 \\ &\quad +2\lambda_n\langle u_n - z_n, Bu_n - Bw\rangle - \lambda_n^2\|Bu_n - Bw\|^2\}. \end{aligned}$$

From (4.1.2), it implies that

$$\begin{aligned}\|z_n - w\|^2 &\leq \|u_n - w\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, Bu_n - Bw\rangle \\ &\quad -\lambda_n^2\|Bu_n - Bw\|^2 \\ &\leq \|x_n - w\|^2 - \|u_n - z_n\|^2 \\ &\quad +2\lambda_n\|u_n - z_n\|\|Bu_n - Bw\|. \end{aligned} \tag{4.1.13}$$

By (4.1.10) and (4.1.13), we get

$$\begin{aligned}(1-\alpha_n)\|u_n - z_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\quad +2(1-\alpha_n)\lambda_n\|u_n - z_n\|\|Bu_n - Bw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad +2(1-\alpha_n)\lambda_n\|u_n - z_n\|\|Bu_n - Bw\|.\end{aligned}$$

By (4.1.8), (4.1.11) and the assumption $0 \leq \alpha_n \leq a < 1$, we get

$$\lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{4.1.14}$$

Also, by (4.1.10) and (4.1.12), we obtain that

$$\begin{aligned}\|y_n - w\|^2 &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|u_n - w\|^2 \qquad (4.1.15)\\ &= \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(w - r_nAw)\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)\|(x_n - r_nAx_n) - (w - r_nAw)\|^2 \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 - 2r_n\langle x_n - w, Ax_n - Aw\rangle \\ &\quad +r_n^2\|Ax_n - Aw)\|^2) \\ &\leq \alpha_n\|x_n - w\|^2 + (1-\alpha_n)(\|x_n - w\|^2 \\ &\quad +r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2) \\ &\leq \|x_n - w\|^2 + (1-\alpha_n)r_n(r_n - 2\alpha)\|Ax_n - Aw\|^2.\end{aligned}$$

From the assumptions $0 \leq \alpha_n \leq a < 1$ and $0 < d \leq r_n \leq e < 2\alpha$, we have

$$\begin{aligned}(1-a)d(2\alpha - e)\|Ax_n - Aw\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|).\end{aligned}$$

By (4.1.8), we obtain

$$\lim_{n\to\infty}\|Ax_n - Aw\| = 0. \tag{4.1.16}$$

On the other hand, by using Lemma 2.6.3, we have T_{r_n} is firmly nonexpansive.

Then we get

$$\begin{aligned}\|u_n - w\|^2 &= \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)w\|^2 \\ &\leq \langle (I - r_nA)x_n - (I - r_nA)w, u_n - w\rangle \\ &= \frac{1}{2}(\|(I - r_nA)x_n - (I - r_nA)w\|^2 + \|u_n - w\|^2 \\ &\quad -\|(I - r_nA)x_n - (I - r_nA)w - (u_n - w)\|^2) \\ &\leq \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|(x_n - u_n) - r_n(Ax_n - Aw)\|^2) \\ &= \frac{1}{2}(\|x_n - w\|^2 + \|u_n - w\|^2 - \|x_n - u_n\|^2 \\ &\quad +2r_n\langle x_n - u_n, Ax_n - Aw\rangle - r_n^2\|Ax_n - Aw\|^2)\end{aligned}$$

and so

$$\begin{aligned}\|u_n - w\|^2 &\leq \|x_n - w\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Aw\rangle \\ &\quad -r_n^2\|Ax_n - Aw\|^2 \\ &\leq \|x_n - w\|^2 - \|x_n - u_n\|^2 \\ &\quad +2r_n\|x_n - u_n\|\|Ax_n - Aw\|. \end{aligned} \tag{4.1.17}$$

By (4.1.15) and (4.1.17), we get

$$\begin{aligned}\|y_n - w\|^2 &\leq \|x_n - w\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2 \\ &\quad +2(1 - \alpha_n)r_n\|x_n - u_n\|\|Ax_n - Aw\|,\end{aligned}$$

which implies that

$$\begin{aligned}(1 - \alpha_n)\|x_n - u_n\|^2 &\leq \|x_n - w\|^2 - \|y_n - w\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Aw\| \\ &\leq \|x_n - y_n\|(\|x_n - w\| + \|y_n - w\|) \\ &\quad +2r_n\|x_n - u_n\|\|Ax_n - Aw\|.\end{aligned}$$

From the assumptions $0 \leq \alpha_n \leq a < 1$, $0 < d \leq r_n \leq e < 2\alpha$, (4.1.8) and (4.1.16), we obtain

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{4.1.18}$$

On the other hand, we have

$$\begin{aligned}\|x_n - S_nx_n\| &\leq \|S_nx_n - S_nz_n\| + \|S_nz_n - x_n\| \\ &\leq L\|x_n - z_n\| + \|S_nz_n - x_n\| \\ &\leq L\|x_n - u_n\| + L\|u_n - z_n\| + \|S_nz_n - x_n\|.\end{aligned}$$

Using (4.1.9), (4.1.14) and (4.1.18), we conclude that

$$\lim_{n\to\infty} \|x_n - S_nx_n\| = 0. \tag{4.1.19}$$

From the assumption $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0, \quad \forall S \in \mathcal{S}. \tag{4.1.20}$$

Since $\{x_n\}$ is bounded, we assume that a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to ξ.

Step 4: We show that $\xi \in F = F(\mathcal{S}) \cap VI(C,B) \cap GMEP(f,\varphi,A)$.

First, we show that $\xi \in F(\mathcal{S})$. Suppose that $\xi \neq S\xi$ for some $S \in \mathcal{S}$. From Opial's condition and (4.1.20), we obtain

$$\begin{aligned}\liminf_{i\to\infty} \|x_{n_i} - \xi\| &< \liminf_{i\to\infty} \|x_{n_i} - S\xi\| \\ &= \liminf_{i\to\infty} \|x_{n_i} - Sx_{n_i} + Sx_{n_i} - S\xi\| \\ &\leq \liminf_{i\to\infty} \|x_{n_i} - \xi\|,\end{aligned}$$

which give us a contradiction. Hence, $\xi \in F(\mathcal{S})$.

Now, we prove that $\xi \in VI(C,B)$. Let T be the maximal monotone mapping defined by (2.3.11):

$$Tx = \begin{cases} Bx + N_c x & \text{if } x \in C \\ \emptyset & \text{if } x \notin C. \end{cases}$$

For any given $(x,y) \in G(T)$, we get $y - Bx \in N_C x$. By $z_n \in C$ and the definition of N_C, we have

$$\langle x - z_n, y - Bx\rangle \geq 0. \tag{4.1.21}$$

On the other hand, since $z_n = P_C(I - \lambda_n B)u_n$, we obtain

$$\langle x - z_n, z_n - (I - \lambda_n B)u_n\rangle \geq 0$$

and then

$$\langle x - z_n, \frac{z_n - u_n}{\lambda_n} + Bu_n\rangle \geq 0. \tag{4.1.22}$$

Since $u_n - z_n \to 0$ as $n \to \infty$, we have that

$$Bu_n - Bz_n \to 0 \text{ as } n \to \infty. \tag{4.1.23}$$

From (4.1.21), (4.1.22) and the β-inverse monotonicity of B, we obtain

$$\begin{aligned}\langle x - z_{n_i}, y\rangle &\geq \langle x - z_{n_i}, Bx\rangle \\ &\geq \langle x - z_{n_i}, Bx\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i}\rangle \\ &= \langle x - z_{n_i}, Bx - Bz_{n_i}\rangle + \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle \\ &\quad -\langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle x - z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &= \langle x, Bz_{n_i} - Bu_{n_i}\rangle - \langle z_{n_i}, Bz_{n_i} - Bu_{n_i}\rangle - \langle x - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle x, Bz_{n_i} - Bu_{n_i}\rangle - \|z_{n_i}\|\|Bz_{n_i} - Bu_{n_i}\| \\ &\quad -\frac{1}{\lambda_{n_i}}\|x - z_{n_i}\|\|z_{n_i} - u_{n_i}\|.\end{aligned} \tag{4.1.24}$$

By (4.1.14) and (4.1.18), we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Since $x_{n_i} \rightharpoonup \xi$, we obtain $z_{n_i} \rightharpoonup \xi$. From (4.1.14), (4.1.23) and (4.1.24), we obtain

$$\langle x - \xi, y\rangle = \lim_{n_i\to\infty} \langle x - z_{n_i}, y\rangle \geq 0.$$

Since T is maximal monotone, we obtain that $0 \in T\xi$. It follows that $\xi \in VI(C, B)$.

Next, we show that $\xi \in GMEP(f, \varphi, A)$. For any $y \in C$,

$$f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n).$$

From the condition (A2), we get that

$$\varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq f(y, u_n) + \varphi(u_n).$$

Replacing n by n_i, we obtain

$$\varphi(y) + \langle Ax_{n_i}, y - u_{n_i}\rangle + \langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq f(y, u_{n_i}) + \varphi(u_{n_i}). \quad (4.1.25)$$

For any t with $0 < t \leq 1$ and $y \in C$, put $\rho_t = ty + (1-t)\xi$. Since $y \in C$ and $\xi \in C$, we obtain $\rho_t \in C$. It follows from (4.1.25) and the monotonicity of A that

$$\begin{aligned}
\langle \rho_t - u_{n_i}, A\rho_t\rangle \;\geq\; & \langle \rho_t - u_{n_i}, A\rho_t\rangle - \langle Ax_{n_i}, \rho_t - u_{n_i}\rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \\
& + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\
= \;& \langle \rho_t - u_{n_i}, A\rho_t - Au_{n_i}\rangle + \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\
& - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t) \\
\geq \;& \langle \rho_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle - \langle \rho_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \\
& + f(\rho_t, u_{n_i}) + \varphi(u_{n_i}) - \varphi(\rho_t). \qquad (4.1.26)
\end{aligned}$$

Since A is Lipschitzian, by (4.1.18), we get $Au_{n_i} - Ax_{n_i} \to 0$ as $i \to \infty$. From (A3) and (4.1.26), we arrive at

$$\langle \rho_t - \xi, A\rho_t\rangle \geq f(\rho_t, \xi) + \varphi(\xi) - \varphi(\rho_t). \quad (4.1.27)$$

From (A1), (A3), (4.1.27) and convexity of φ, we have that

$$\begin{aligned}
0 \;=\; & f(\rho_t, \rho_t) \leq tf(\rho_t, y) + (1-t)f(\rho_t, \xi) \\
\leq \;& tf(\rho_t, y) + (1-t)\big(\langle \rho_t - \xi, A\rho_t\rangle + \varphi(\rho_t) - \varphi(\xi)\big) \\
\leq \;& tf(\rho_t, y) + (1-t)t\big(\langle y - \xi, A\rho_t\rangle + \varphi(y) - \varphi(\xi)\big),
\end{aligned}$$

which implies that

$$f(\rho_t, y) + (1-t)\big(\langle y - \xi, A\rho_t\rangle + \varphi(y) - \varphi(\xi)\big) \geq 0.$$

Letting $t \to 0$, by (A4), we arrive at

$$f(\xi, y) + \langle y - \xi, A\xi\rangle + \varphi(y) - \varphi(\xi) \geq 0.$$

This shows that $\xi \in GMEP(f, \varphi, A)$.

Step 5: We show that $x_n \to P_F x_1$.

Let $\overline{x} = P_F x_1$ Since $\overline{x} = P_F x_1 \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}} x_1$, we get

$$\|x_1 - x_{n+1}\| \leq \|x_1 - \overline{x}\|.$$

On the other hand, we have

$$\begin{aligned} \|x_1 - \overline{x}\| &\leq \|x_1 - \xi\| \\ &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \|x_1 - \overline{x}\|. \end{aligned}$$

Therefore, we get

$$\|x_1 - \xi\| = \lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - \overline{x}\|.$$

This implies that $\overline{x} = \xi$. Since H has the Kadec-Klee property and $x_1 - x_{n_i} \rightharpoonup x_1 - \overline{x}$, it follows that $x_{n_i} \to \overline{x}$. Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $x_n \to \overline{x}$ as $n \to \infty$. The proof is now complete. □

4.2 Deduced Strong Convergence Theorems and Applications of Hybrid Methods in Hilbert Spaces

Theorem 4.1.1 can be reduced to many different results. By putting $S_n = S$ for all $n \geq 1$ in Theorem 4.1.1, we obtain the following theorem:

Theorem 4.2.1. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4) *and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a L-Lipschitzian quasi-nonexpansive mapping such that $F = F(S) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \varphi(u_n), \forall y \in C, \\ z_n = P_C(u_n - \lambda_n B u_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{4.2.1}$$

where $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \textit{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (4.2.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

When $\{S_n\}$ and $\mathcal{S}$ are families of nonexpansive mappings, we get the following theorem:

Theorem 4.2.2. *Let* C *be a closed convex subset of a real Hilbert space* H, $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4) *and* $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and* B *be a* β*-inverse-strongly monotone mapping of* C *into* H, *respectively. Let* $\{S_n\}$ *and* $\mathcal{S}$ *be families of nonexpansive mappings of* C *into itself such that* $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ *and* $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. *Suppose that* $\{S_n\}$ *satisfies the NST-condition with* $\mathcal{S}$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$ *be a sequence generated by the algorithm (4.1.1), where* $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (4.1.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

Now we show how to apply Theorem 4.2.2 for families of strict pseudo-contraction mappings.

Theorem 4.2.3. *Let* C *be a closed convex subset of a real Hilbert space* H, $f : C \times C \to \mathbb{R}$ *be a bifunction satisfying* (A1)-(A4) *and* $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *and* B *be a* β*-inverse-strongly monotone mapping of* C *into* H, *respectively. Let* $\{R_n\}$ *and* $\mathcal{R}$ *be families of* $k-$*strict pseudo-contraction mappings of* C *into itself such that* $\bigcap_{n=1}^{\infty} F(R_n) = F(\mathcal{R})$ *and* $F = F(\mathcal{R}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. *Define a mapping* $S_n : C \to C$ *by* $S_n x = kx + (1 - k)R_n x$ *for all* $x \in C$. *Suppose that* $\{R_n\}$ *satisfies the NST-condition with* $\mathcal{R}$. *Assume that either (B1) or (B2) holds. Let* $\{x_n\}$ *be a sequence generated by the algorithm (4.1.1), where* $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. *Then the sequence* $\{x_n\}$ *defined by the algorithm (4.1.1) converges strongly to a point* $\overline{x} = P_F x_1$, *where* P_F *is the metric projection of* H *onto* F.

Proof. By Lemma 2.6.4, we obtain that S_n is nonexpansive for all positive integer n. We also get that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S} = \{kI + (1 - k)T : T \in \mathcal{R}\}$. The proof is now complete because of the direct result of Theorem 4.2.2. □

By putting $S_n = S$ for all $n \geq 1$ in Theorem 4.2.3, we obtain the following corallary.

Corollary 4.2.4. *Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4) *and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $R : C \to C$ be a $k-$strict pseudo-contraction such that $F = F(R) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S : C \to C$ by $Sx = kx + (1-k)Rx$ for all $x \in C$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm (4.2.1), where $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,*

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \textit{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (4.2.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

Remark 4.2.5. By letting $\varphi = 0$ in Corollary 4.2.4, we obtain Theorem 2.1 of [14].

Remark 4.2.6. Since Theorems 4.1.1, 4.2.1, 4.2.2, 4.2.3 and Corollary 4.2.4 are for finding a common element of the set of fixed points, the set of solutions of the general system of the variational inequality and the set of solutions of the generalized mixed equilibrium problem, we can reduce each theorem or corollary by letting $B = 0$, $A = 0$, $\varphi = 0$ or $f(x, y) = 0$.