Chapter 5
Conclusion

In this chapter, we conclude all main results obtained in this thesis. The results in Chapters 3 and 4 are shown in Sections 5.1 and 5.2, respectively.

5.1 Common Fixed Points of Asymptotically Quasi-nonexpansive Mappings

5.1.1 Strong Convergence Theorems for a Finite Family of Asymptotically Quasi-nonexpansive Mappings in Banach Spaces

(1) Let C be a nonempty closed convex subset of a real Banach space X, and $\{T_i : i = 1, 2, \ldots, k\}$ be a family of asymtotically quasi-nonexpansive self-mappings of C, i.e., $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x \in C$ and $p_i \in F(T_i), i = 1, 2, \ldots, k$. Suppose that $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_{kn})y_{(k-1)n} + \alpha_{kn}T_k^n y_{(k-1)n}, \\ y_{(k-1)n} &= (1-\alpha_{(k-1)n})y_{(k-2)n} + \alpha_{(k-1)n}T_{k-1}^n y_{(k-2)n}, \\ &\vdots \\ y_{2n} &= (1-\alpha_{2n})y_{1n} + \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} &= (1-\alpha_{1n})x_n + \alpha_{1n}T_1^n y_{0n}, \end{aligned} \tag{5.1.1}$$

where $x_1 \in C$ and $y_{0n} = x_n$ for all n. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1 \leq i \leq k}\{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p \in F} \|x - p\|$.

5.1.2 Weak and Strong Convergence Theorems for a Finite Family of Asymptotically Quasi-nonexpansive Mappings in Uniformly Convex Banach Spaces

(1) Let C be a nonempty closed convex subset of an uniformly convex real Banach space X. Let $\{T_i : i = 1, 2, \ldots, k\}$ be a family of uniformly $(L-\gamma_i)$-Lipschitzian and asymptotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - T_i^n y\| \leq L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \leq (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i), i = 1, 2, \ldots, k$. Suppose that $\{T_i : i = 1, 2, \ldots, k\}$ satisfies condition (A$''$) and $F = \bigcap_{i=1}^k F(T_i) \neq \emptyset$. Let $x_1 \in C$ and the iterative

sequence $\{x_n\}$ be defined by (5.1.1). Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1\le i\le k}\{r_{in}\}$. Then $\{x_n\}$ converges strongly to a common fixed point of the family of mappings.

(2) Let C be a nonempty closed convex subset of an uniformly convex real Banach space X, and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and asymtotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - T_i^n y\| \le L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \le (1+r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i), i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^{k} F(T_i) \ne \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (5.1.1) with $\alpha_{in} \in [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1\le i\le k}\{r_{in}\}$, and T_j^m is semi-compact for some positive integers m and $1 \le j \le k$. Then $\{x_n\}$ converges strongly to a common fixed point of the family $\{T_i : i = 1, 2, \dots, k\}$.

(3) Let C be a nonempty closed convex subset of an uniformly convex real Banach space X satisfying the Opial property, and $\{T_i : i = 1, 2, \dots, k\}$ be a family of $(L - \gamma_i)$ uniform Lipschitz and asymtotically quasi-nonexpansive selfmappings of C, i.e., $\|T_i^n x - T_i^n y\| \le L\|x - y\|^{\gamma_i}$ and $\|T_i^n x - p_i\| \le (1 + r_{in})\|x - p_i\|$, for all $x, y \in C$ and $p_i \in F(T_i), i = 1, 2, \dots, k$. Suppose that $F = \bigcap_{i=1}^{k} F(T_i) \ne \emptyset$, $x_1 \in C$ and the iterative sequence $\{x_n\}$ is defined by (5.1.1) with $\alpha_{in} \in [\delta, 1-\delta]$. Assume that $\sum_{n=1}^{\infty} r_n < \infty$, where $r_n = \max_{1\le i\le k}\{r_{in}\}$. If $I - T_i, i = 1, 2, \cdots, k$, is demiclosed at 0, then $\{x_n\}$ converges weakly to a common fixed point of the family of mappings.

5.2 Equilibrium Problems and Fixed Point Problems of Quasi-nonexpansive Mappings

5.2.1 A Strong Convergence Theorem of Hybrid Methods for Generalized Mixed Equilibrium Problems and Fixed Point Problems of an Infinite Family of Lipschitzian Quasi-nonexpansive Mappings in Hilbert Spaces

(1) Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying *(A1)-(A4)* and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of Lipschitzian quasi-nonexpansive mappings of C into itself such that $\lim_{n\to\infty} \|S_n x - S_n y\| \le L_n\|x - y\|$ for all $x, y \in C$, $\sup_n L_n = L$, $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \ne \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be

a sequence generated by the algorithm:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n), \\ \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n B u_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{5.2.1}$$

where $\{\alpha_n\} \subset [0,1), \{\lambda_n\} \subset (0, 2\beta), \{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (5.2.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

5.2.2 Deduced Strong Convergence Theorems and Applications of Hybrid Methods in Hilbert Spaces

(1) Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying *(A1)-(A4)* and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $S : C \to C$ be a L-Lipschitzian quasi-nonexpansive mapping such that $F = F(S) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_1 \in C, \\ C_1 = C, \\ f(u_n, y) + \varphi(y) + \langle Ax_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq \varphi(u_n), \\ \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n B u_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S z_n, \\ C_{n+1} = \{w \in C_n : \|y_n - w\| \leq \|x_n - w\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \geq 1, \end{cases} \tag{5.2.2}$$

where $\{\alpha_n\} \subset [0,1), \{\lambda_n\} \subset (0, 2\beta), \{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (5.2.2) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

(2) Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying *(A1)-(A4)* and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{S_n\}$ and $\mathcal{S}$ be families of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) = F(\mathcal{S})$ and $F = F(\mathcal{S}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Suppose that $\{S_n\}$ satisfies the NST-condition with $\mathcal{S}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (5.2.1), where $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (5.2.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

(3) Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying *(A1)-(A4)* and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $\{R_n\}$ and $\mathcal{R}$ be families of $k-$strict pseudo-contraction mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(R_n) = F(\mathcal{R})$ and $F = F(\mathcal{R}) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S_n : C \to C$ by $S_n x = kx + (1-k)R_n x$ for all $x \in C$. Suppose that $\{R_n\}$ satisfies the NST-condition with $\mathcal{R}$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the algorithm (5.2.1), where $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (5.2.1) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.

(4) Let C be a closed convex subset of a real Hilbert space H, $f : C \times C \to \mathbb{R}$ be a bifunction satisfying *(A1)-(A4)* and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be an α-inverse-strongly monotone mapping of C into H and B be a β-inverse-strongly monotone mapping of C into H, respectively. Let $R : C \to$ be a $k-$strict pseudo-contraction such that $F = F(R) \cap VI(C, B) \cap GMEP(f, \varphi, A) \neq \emptyset$. Define a mapping $S : C \to C$ by $Sx = kx + (1-k)Rx$ for all $x \in C$. Assume that either (B1) or (B2) holds. Let $\{x_n\}$ be a sequence generated by the following algorithm (5.2.2), where $\{\alpha_n\} \subset [0, 1), \{\lambda_n\} \subset (0, 2\beta)$, $\{r_n\} \subset (0, 2\alpha)$,

$$0 \leq \alpha_n \leq a < 1, \quad 0 < b \leq \lambda_n \leq c < 2\beta, \quad \text{and} \quad 0 < d \leq r_n \leq e < 2\alpha,$$

for some $a, b, c, d, e \in \mathbb{R}$. Then the sequence $\{x_n\}$ defined by the algorithm (5.2.2) converges strongly to a point $\overline{x} = P_F x_1$, where P_F is the metric projection of H onto F.