Chapter 2

Preliminaries

In this chapter, we will briefly review some concepts and some results of Semigroup Theory.

2.1 Elementary Concepts

Definition 2.1.1. Let S be a semigroup, then a non-empty subset T of S is called a *subsemigroup* of S, that is, if

$$xy \in T \quad \text{for all} \quad x, y \in T.$$

Definition 2.1.2. Let S be a semigroup and for each non-empty subset A of S, we set

$$\langle A \rangle = \cap \{T : T \text{ is a subsemigroup of } S \text{ containing } A\}.$$

It is characterized that $\langle A \rangle$ is the smallest subsemigroup of S containing A, and

$$\langle A \rangle = \{a_1.a_2 \ldots a_n : a_i \in A \text{ for all } i = 1, 2, \ldots n \text{ and } n \in \mathbb{N}\}.$$

Definition 2.1.3. Let S be a semigroup. A proper subsemigroup M of S is called a *maximal subsemigroup* of S if, whenever $M \subseteq N \subsetneq S$ and N is a subsemigroup of S, then $M = N$.

From Definition 2.1.3, we can easily prove the following lemma.

Lemma 2.1.4. *Let S be a semigroup. Then the following are equivalent.*

(i) *M is a maximal subsemigroup of S;*

(ii) *$\langle M \cup \{a\} \rangle = S$ for all $a \in S \setminus M$;*

(iii) *for any $a, b \in S \setminus M$, a can be written as a finite product of elements of $M \cup \{b\}$.*

Proof. Suppose that M is a maximal subsemigroup of S and $a \in S \setminus M$. Then $M \subsetneq \langle M \cup \{a\} \rangle$ where $\langle M \cup \{a\} \rangle$ is a subsemigroup of S. Hence $\langle M \cup \{a\} \rangle = S$ by the maximality of M, that is, (i) implies (ii).

Next, suppose that (ii) holds and let $a, b \in S \setminus M$. Then $a \in S = \langle M \cup \{b\} \rangle$ and thus a is a finite product of elements of $M \cup \{b\}$ by Definition 2.1.2. Therefore (ii) implies (iii).

Finally, to show that (iii) implies (i), we suppose that (iii) holds and $M \subseteq N \subsetneq S$ where N is a subsemigroup of S. Then there exists $a \in S \setminus N \subseteq S \setminus M$. If there exists $b \in N \setminus M$, then $b \in S \setminus M$. So, (iii) implies that a is a finite product of elements of $M \cup \{b\}$. Thus, $a \in \langle M \cup \{b\} \rangle \subseteq N$, a contradiction. Hence $M = N$ and therefore M is maximal in S. ■

Definition 2.1.5. Let S be a semigroup. A subsemigroup U of S is called a *left unitary subsemigroup* if U satisfies the property:

$$\text{for } \; u \in U, s \in S \; \text{ if } \; us \in U \; \text{ then } \; s \in U.$$

A *right unitary subsemigroup* of S is defined dually, and U is an *unitary subsemigroup* if it is both left and right unitary.

Definition 2.1.6. Let S be a semigroup.

(i) If there exists an element 1 of S such that

$$x1 = x = 1x \; \text{ for all } x \in S,$$

then 1 is called an *identity* element of S and S is called a *semigroup with identity.*

(ii) If there exists an element 0 of S such that

$$x0 = 0 = 0x \; \text{ for all } x \in S,$$

then 0 is called a *zero* element of S and S is called a *semigroup with zero.*

Definition 2.1.7. A semigroup S is called *left cancellative* (*right cancellative*) if, for all a, b and c in S,

$$ca = cb \; \text{ implies } \; a = b \; (ac = bc \; \text{ implies } \; a = b).$$

Definition 2.1.8. A semigroup S is called *left reductive* (*right reductive*) if, for any a, b in S,

$$xa = xb \text{ for all } x \in S \text{ implies } a = b \ (ax = bx \text{ for all } x \in S \text{ implies } a = b).$$

From Definition 2.1.7 and Definition 2.1.8 we see that, a left cancellative (right cancellative) semigroup is left reductive (right reductive).

Definition 2.1.9. An element e of a semigroup S is called an *idempotent* if $e = e^2$. The set of all idempotents in S is denoted by $E(S)$. We call S an *idempotent-free* semigroup if S has no idempotent element.

Definition 2.1.10. An element a of a semigroup S is called *regular* if there exists x in S such that $a = axa$. A semigroup S is *regular* if all elements in S are regular.

Definition 2.1.11. A semigroup S is called an *inverse semigroup* if every a in S possesses a unique inverse, i.e. if there exists a unique element a^{-1} in S such that

$$a = aa^{-1}a \quad \text{and} \quad a^{-1} = a^{-1}aa^{-1}.$$

Theorem 2.1.12. [3] *Let S be a semigroup. Then S is an inverse semigroup if and only if S is regular and idempotent elements commute.*

Definition 2.1.13. Let S and T be semigroups. A mapping φ from S into T is called a *homomorphism* if

$$(xy)\varphi = (x\varphi)(y\varphi) \text{ for all } x, y \in S.$$

An injective homomorphism is called a *monomorphism*. A surjective homomorphism is called an *epimorphism*, and if a homomorphism is bijective then we call it an *isomorphism*. If there exists an isomorphism from S onto T then we say that S and T are *isomorphic* and write $S \cong T$. If φ is a homomorphism from S into S then we call it an *endomorphism* of S. An isomorphism from S onto S will be called an *automorphism* of S.

2.2 Ideals and Green's Relations

Definition 2.2.1. Let S be a semigroup.

(i) A non-empty subset A of S is called a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$, and a (*two-sided*) *ideal* if it is both a left and a right ideal.

(ii) An ideal I of S is called a *prime ideal* if $I \neq S$ and whenever $ab \in I$ for elements a and b of S, then either $a \in I$ or $b \in I$.

From Definition 2.2.1, it is equivalent to say that, I is a prime ideal of S if and only if $S \setminus I$ is a subsemigroup of S. Also, if S has a zero element, then $\{0\}$ and S are ideals of S. We call an ideal I of S a *proper* ideal if $\{0\} \neq I \neq S$.

For any semigroup S, the notation S^1 means S itself if S contains the identity element, otherwise, we let $S^1 = S \cup \{1\}$ and define the binary operation on S^1 by

$$1 \cdot s = s = s \cdot 1 \text{ for all } s \in S \text{ and } 1 \cdot 1 = 1.$$

Then S^1 becomes a semigroup with the identity element 1.

For any element a in S,

the smallest left ideal of S containing a is $Sa \cup \{a\} = S^1a$,

the smallest right ideal of S containing a is $aS \cup \{a\} = aS^1$, and

the smallest ideal of S containing a is $SaS \cup aS \cup Sa \cup \{a\} = S^1aS^1$,

which we call the *principal left ideal*, *principal right ideal* and *principal ideal generated by a*, respectively.

In 1951, Green defined the equivalence relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ on S by the rules that, for $a, b \in S$,

$a \, \mathcal{L} \, b$ if and only if $S^1a = S^1b$,

$a \, \mathcal{R} \, b$ if and only if $aS^1 = bS^1$, and

$a \, \mathcal{J} \, b$ if and only if $S^1aS^1 = S^1bS^1$.

Then he defined the equivalence relations

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R} \text{ and } \mathcal{D} = \mathcal{L} \circ \mathcal{R},$$

and obtained that the composition of $\mathcal{L}$ and $\mathcal{R}$ is commutative. This follows that $\mathcal{D}$ is the *join* $\mathcal{L} \vee \mathcal{R}$, that is, $\mathcal{D}$ is the smallest equivalence relation containing $\mathcal{L} \cup \mathcal{R}$. Moreover, $\mathcal{H} \subseteq \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{H} \subseteq \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. But, in commutative semigroups, we have $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}$.

Definition 2.2.2. A semigroup S is called *left simple* (*right simple, bi-simple*) if,

$$\mathcal{L} = S \times S \ \ (\mathcal{R} = S \times S,\ \mathcal{D} = S \times S).$$

2.3 Transformation Semigroups

In this section, we give some useful results about transformation semigroups which will be used in this thesis.

2.3.1 The semigroups $P(X), T(X), I(X)$ and $G(X)$

Let X be a non-empty set. As usual, $P(X)$ denotes the set of all *partial transformations of* X, that is, all transformations α whose *domain*, $\operatorname{dom}\alpha$, and *range*, $X\alpha$ (or $\operatorname{ran}\alpha$) are subsets of X. Then $P(X)$ is a semigroup under the *composition of mappings*, that is, if $\alpha, \beta \in P(X)$, then $\alpha\beta \in P(X)$ is defined by

$$x(\alpha\beta) = (x\alpha)\beta \ \text{ for all } x \in \operatorname{dom}\alpha.$$

We also have

$$\operatorname{dom}\alpha\beta = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \ \text{ and } \ \operatorname{ran}\alpha\beta = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta.$$

In the case that $\operatorname{ran}\alpha \cap \operatorname{dom}\beta = \emptyset$, we define $\alpha\beta$ to be the *empty transformation*, which is a partial transformation of X with empty domain and it is denoted by $\emptyset$.

Let $T(X), I(X)$ and $G(X)$ be the following sets:

$$\begin{aligned} T(X) &= \{\alpha \in P(X) : \operatorname{dom}\alpha = X\}, \\ I(X) &= \{\alpha \in P(X) : \alpha \text{ is injective}\}, \\ G(X) &= \{\alpha \in P(X) : \alpha \text{ is bijective}\}. \end{aligned}$$

Then $T(X)$ and $I(X)$ are subsemigroups of $P(X)$, which are called the *full transformation semigroup* and the *symmetric inverse semigroup on* X, respectively. Also, we call $G(X)$ the *permutation group on* X, which is a subgroup of $P(X), T(X)$ and $I(X)$.

For a non-empty subset A of X, we let id_A denote the identity mapping on A. Then it is clear that id_X is the identity element of $P(X), T(X), I(X)$ and $G(X)$.

It is well known that every group can be embedded up to isomorphism in a permutation group $G(X)$ for some set X (Cayley's Theorem). Comparing with this result, in semigroup theory we have the following well known theorems.

Theorem 2.3.1. [3] *If S is a semigroup and $X = S^1$, then there exists a monomorphism $\rho : S \to T(X)$.*

Theorem 2.3.2. [3] (*The Vagner-Preston Representation Theorem*)
If S is an inverse semigroup, then there exists a set X and a monomorphism $\phi : S \to I(X)$.

2.3.2 Baer-Levi semigroup

For any $\alpha \in P(X)$, we write

$$G(\alpha) = X \setminus \operatorname{dom} \alpha \quad \text{and} \quad D(\alpha) = X \setminus \operatorname{ran} \alpha.$$

We also let

$$g(\alpha) = |G(\alpha)|, \quad d(\alpha) = |D(\alpha)|, \quad r(\alpha) = |\operatorname{ran} \alpha|,$$

and refer to these cardinals as the *gap*, the *defect* and the *rank* of α, respectively.

In 1932, R. Baer and F. Levi constructed a right cancellative right simple semigroup which is not a group on an infinite set X with cardinal p. The semigroup is defined by

$$BL(q) = \{\alpha \in I(X) : g(\alpha) = 0, d(\alpha) = q\},$$

where $\aleph_0 \leq q \leq p$. This semigroup is called a *Baer-Levi semigroup of type* (p, q) on X. From [1] vol 2, Section 8.1, we have the following well known results on $BL(q)$.

Theorem 2.3.3. [1] *For any two infinite cardinals p, q such that $p \geq q$, there exists a Baer-Levi semigroup of type (p, q).*

Theorem 2.3.4. [1] *Let S be a Baer-Levi semigroup. Then S is a right cancellative, right simple semigroup without idempotents.*

Theorem 2.3.5. [1] *Let S be a right cancellative, right simple semigroup without idempotents. Then S can be embedded in a Baer-Levi semigroup of type (p, p), where $p = |S|$.*

In 1984, Levi and Wood determined a maximal subsemigroup of $BL(q)$ by letting

$$M_A = \{\alpha \in BL(q) : A \not\subseteq X\alpha \text{ or } (A\alpha \subseteq A \text{ or } |X\alpha \backslash A| < q)\}$$

where A is a non-empty subset of X with $|X\backslash A| \geq q$. That is, α in $BL(q)$ belongs to M_A if and only if

(i) $A \not\subseteq X\alpha$, or

(ii) $A \subseteq X\alpha$ and either $A\alpha \subseteq A$, or $|X\alpha \setminus A| < q$.

The authors showed that M_A is a maximal subsemigroup of $BL(q)$ ([9] Theorem 1). Later, Hotzel [2] studied maximal subsemigroups and maximal left unitary subsemigroups of $BL(q)$. He showed that there are many other maximal subsemigroups of $BL(q)$ and they are very complicated to describe.

2.3.3 Partial Baer-Levi semigroup

Let X be an infinite set with cardinal p, and let q be a cardinal such that $p \geq q \geq \aleph_0$. In this thesis, we examine a related semigroup of $BL(q)$: namely, the *partial Baer-Levi semigroup* on X defined by

$$PS(q) = \{\alpha \in I(X) : d(\alpha) = q\}.$$

This semigroup was first defined in [13] p 82. In contrast with $BL(q)$, $PS(q)$ is neither right simple nor right cancellative. Moreover, this semigroup always contains idempotents. In [12], Pinto and Sullivan described some algebraic properties of $PS(q)$ as follows.

Theorem 2.3.6. *If $p \geq q \geq \aleph_0$, then $PS(q)$ is a right and left reductive semigroup with idempotents. Moreover, $PS(q)$ contains a zero precisely when $p = q$.*

Theorem 2.3.7. *If $p \geq q \geq \aleph_0$ and $\alpha \in PS(q)$, then the following statements are equivalent.*

(i) *α is regular,*

(ii) *$g(\alpha) = q$,*

(iii) *$\alpha^{-1} \in PS(q)$.*

They also studied the set of all regular elements in $PS(q)$: namely,

$$R(q) = \{\alpha \in PS(q) : g(\alpha) = q\}.$$

They showed that $R(q)$ is the largest regular subsemigroup of $PS(q)$. Moreover, they obtained the following result.

Theorem 2.3.8. *If $p \geq q \geq \aleph_0$, then $R(q)$ is an inverse semigroup.*

They characterized Green's relations of $PS(q)$ as follows.

Theorem 2.3.9. *If $\alpha, \beta \in PS(q)$, then $\alpha = \beta\mu$ for some $\mu \in PS(q)$ if and only if* $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$*. Hence $\alpha\mathcal{R}\beta$ in $PS(q)$ if and only if* $\operatorname{dom}\alpha = \operatorname{dom}\beta$.

Theorem 2.3.10. *If $\alpha, \beta \in PS(q)$, then $\alpha = \lambda\beta$ for some $\lambda \in PS(q)$ if and only if $X\alpha \subseteq X\beta$ and*

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q).$$

Hence, $\alpha\ \mathcal{L}\ \beta$ in $PS(q)$ if and only if

$$(X\alpha = X\beta \text{ and } g(\alpha) = g(\beta) \geq q) \text{ or } (\alpha = \beta \text{ and } g(\alpha) < q).$$

Theorem 2.3.11. *If $\alpha, \beta \in PS(q)$, then $\alpha\mathcal{H}\beta$ in $PS(q)$ if and only if*

$$(X\alpha = X\beta, \operatorname{dom}\alpha = \operatorname{dom}\beta \text{ and } g(\alpha) \geq q) \text{ or } (\alpha = \beta \text{ and } g(\alpha) < q).$$

Theorem 2.3.12. *If $\alpha, \beta \in PS(q)$, then $\alpha\mathcal{D}\beta$ in $PS(q)$ if and only if*

$$(\operatorname{dom}\alpha = \operatorname{dom}\beta \text{ and } g(\alpha) < q) \text{ or } (r(\alpha) = r(\beta) \text{ and } g(\alpha) = g(\beta) \geq q).$$

Theorem 2.3.13. *If $\alpha, \beta \in PS(q)$, then $\alpha\mathcal{J}\beta$ in $PS(q)$ if and only if*

$$(\max(g(\alpha), g(\beta)) \leq q \text{ and } r(\alpha) = r(\beta)) \text{ or } (g(\alpha) = g(\beta) > q).$$

Let u be a cardinal number. The *successor* of u, denoted by u', is defined as

$$u' = min\{v : v > u\}.$$

Note that u' always exists since the cardinals are well-ordered, and when u is finite we have $u' = u + 1$.

Consequently, Pinto and Sullivan described the ideals of $PS(q)$ as follows.

Theorem 2.3.14. *The proper ideals of $PS(q)$ for $p > q$ are precisely the sets:*

$$T_r = \{\alpha \in PS(q) : g(\alpha) \geq r\}$$

where $q < r \leq p$. Moreover, each T_r is a principal ideal.

Theorem 2.3.15. *If $p = q$, the ideals of $PS(q)$ are precisely the sets:*

$$J_r = \{\alpha \in PS(q) : r(\alpha) < r\}$$

where $1 \leq r \leq p'$. Moreover, J_r is principal precisely when $r = s'$ where $0 \leq s \leq p$.

For $\aleph_0 \leq r \leq p$, Pinto and Sullivan [12] defined a subsemigroup

$$S_r = \{\alpha \in PS(q) : g(\alpha) \leq r\}$$

of $PS(q)$. Then they gave the following result

Corollary 2.3.16. *If $p \geq r > q \geq \aleph_0$, then $G_r = S_r \cap T_r$ is bi-simple and idempotent-free.*

Moreover, they showed that S_q is generated by $BL(q)$ and $R(q)$ in very specific ways.

Theorem 2.3.17. *If $p \geq q \geq \aleph_0$, then $S_q = BL(q).R(q)$. In fact, $S_q = \alpha.R(q)$ for each $\alpha \in BL(q)$.*

Theorem 2.3.18. *If $p > q$, then $S_q = BL(q).\mu.BL(q)$ for each $\mu \in R(q)$.*

2.4 Automorphisms of Transformation Semigroups

Definition 2.4.1. Let X be an infinite set. A semigroup S of partial transformations of X is said to be G_X*-normal* if for every $\alpha \in G(X)$, $\alpha S \alpha^{-1} \subseteq S$.

Example 2.4.2. The semigroup of all partial transformations $P(X)$, the full transformation semigroup $T(X)$, the symmetric inverse semigroup $I(X)$ and all ideals of $P(X), T(X)$ and $I(X)$ are G_X-normal.

Example 2.4.3. The partial Baer-Levi semigroup $PS(q)$ is G_X-normal. To see this, we let $\alpha \in G(X)$ and $\beta \in PS(q)$. Since $X\alpha = X$, we have $X\alpha\beta\alpha^{-1} = X\beta\alpha^{-1}$. Thus,

$$d(\alpha\beta\alpha^{-1}) = |X \setminus X\alpha\beta\alpha^{-1}| = |X \setminus X\beta\alpha^{-1}| = |(X \setminus X\beta)\alpha^{-1}| = q$$

since $d(\beta) = |X \setminus X\beta| = q$. Clearly $\alpha\beta\alpha^{-1}$ is injective, hence $\alpha\beta\alpha^{-1} \in PS(q)$, that is, $\alpha.PS(q).\alpha^{-1} \subseteq PS(q)$.

Definition 2.4.4. Let X be an infinite set and S be a semigroup of total or partial transformations of X. An automorphism φ of S is said to be *inner* if there exists $\gamma \in G(X)$ such that $(\beta)\varphi = \gamma\beta\gamma^{-1}$ for all $\beta \in S$.

In what follows, we let Aut S denote the set of all automorphisms of the subsemigroup S of $P(X)$. The following results are the characterization of Aut $PS(q)$.

Theorem 2.4.5. [13] *If S is the partial Baer-Levi semigroup of type (p, p), then every automorphism of S is inner and* Aut $S \cong G(X)$.

Theorem 2.4.6. [12] *If $p > q$, then* Aut $PS(q)$ *is isomorphic to $G(X)$.*

When necessary, we will use the notation $PS(X,p,q)$ in place of $PS(q)$ to highlight the set X and its cardinal p. The following result is quoted from [12] Theorem 3.

Theorem 2.4.7. [12] *The semigroups $PS(X,p,q)$ and $PS(Y,r,s)$ are isomorphic if and only if $p=r$ and $q=s$. Moreover, for each isomorphism φ, there is a bijection $\gamma : X \to Y$ such that $\alpha\varphi = \gamma^{-1}\alpha\gamma$ for each $\alpha \in PS(X,p,q)$.*

In [12], the authors let $\mathcal{B}(X,q)$ denote the family of all $A \subseteq X$ such that $|X \setminus A| = q$ where $|X| = p \geq q \geq \aleph_0$. If Y is a set with $|Y| = r \geq s \geq \aleph_0$, then we call a mapping $H : \mathcal{B}(X,q) \to \mathcal{B}(Y,s)$ an *order monomorphism* if H is injective and, for $A, B \in \mathcal{B}(X,q)$,

$$A \subseteq B \text{ if and only if } AH \subseteq BH.$$

Moreover, when H is bijective we call H an *order isomorphism.*

In order to prove Theorem 2.4.7, the authors used the following lemma.

Lemma 2.4.8. [12] *Suppose $|X| = p \geq q \geq \aleph_0$ and $|Y| = r \geq s \geq \aleph_0$. Every order isomorphism $H : \mathcal{B}(X,q) \to \mathcal{B}(Y,s)$ is induced by a bijection $h : X \to Y$, that is, for each $A \in \mathcal{B}(X,q)$, we have $AH = Ah$, the image of A under h.*

2.5 Partial Orders on Semigroups

Definition 2.5.1. A binary relation $\leq$ on a set X is called a *partial order* if

(i) $x \leq x$ for all $x \in X$,

(ii) for all $x, y \in X$, if $x \leq y$ and $y \leq x$, then $x = y$, and

(iii) for all $x, y, z \in X$, if $x \leq y$ and $y \leq z$, then $x \leq z$.

We shall refer to $(X, \leq)$, or just to X, as a *partially ordered set* and we sometimes write $(x, y) \in \leq$ instead of $x \leq y$.

Definition 2.5.2. Let $(X, \leq)$ be a partially ordered set.

(i) an element $a \in X$ is called *maximal* if $a \leq x$ and $x \in X$ imply $a = x$; and $b \in X$ is called *maximum* if $x \leq b$ for all $x \in X$.

(ii) an element $a \in X$ is called *minimal* if $x \leq a$ and $x \in X$ imply $x = a$; and $b \in X$ is called *minimum* if $b \leq x$ for all $x \in X$.

Definition 2.5.3. Let $(X, \leq)$ be a partially ordered set and Y a non-empty subset of X.

(i) a *lower bound* of Y is an element $c \in X$ such that $c \leq y$ for all $y \in Y$. A lower bound c_0 of Y is the *greatest lower bound* of Y if $c \leq c_0$ for any lower bound c of Y. When $Y = \{a, b\}$, we let $a \wedge b$ denote the greatest lower bound of Y and call it the *meet* of a and b.

(ii) an *upper bound* of Y is an element $d \in X$ such that $y \leq d$ for all $y \in Y$. An upper bound d_0 of Y is the *least upper bound* of Y if $d_0 \leq d$ for any upper bound d of Y. When $Y = \{a, b\}$, we let $a \vee b$ denote the least upper bound of Y and call it the *join* of a and b.

Definition 2.5.4. Let ρ be a relation on a semigroup S.

(i) an element $c \in S$ is called *left compatible* with ρ if $(ca, cb) \in \rho$ for all $(a, b) \in \rho$.

(ii) an element $c \in S$ is called *right compatible* with ρ if $(ac, bc) \in \rho$ for all $(a, b) \in \rho$.

It is well known that if S is a regular semigroup, then $(S, \leq)$ is a partially ordered set under the relation $\leq$ defined on S by

$$a \leq b \text{ if and only if } a = eb = bf \text{ for some } e, f \in E(S).$$

In [5] the authors investigated properties of this order for the regular semigroup $T(X)$. In particular, they characterised when $\alpha \leq \beta$ for $\alpha, \beta \in T(X)$, and they determined the maximal and minimal elements of $(T(X), \leq)$. Later, in 1986, Mitsch [11] extended the above partial order to any semigroup S by defining $\leq$ on S as follows:

$$a \leq b \text{ if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1,$$

and we call $\leq$ the *natural partial order on S*. In 2003, Marques-Smith and Sullivan [10] studied various properties of the natural partial order $\leq$ and the *containment order* $\subseteq$ on $P(X)$, where $\subseteq$ is defined by, for $\alpha, \beta \in P(X)$,

$$\alpha \subseteq \beta \text{ if and only if } \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \text{ and } x\alpha = x\beta \text{ for all } x \in \operatorname{dom}\alpha.$$

They determined an upper bound Ω' and the join Ω of $\leq$ and $\subseteq$, which defined by

$$(\alpha,\beta) \in \Omega' \text{ if and only if } X\alpha \subseteq X\beta, \quad \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \text{ and } \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1},$$

$$(\alpha,\beta) \in \Omega \text{ if and only if } (\alpha,\beta) \in \Omega' \text{ and } \beta\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}.$$

They gave the useful results for this thesis as follows:

Theorem 2.5.5. *If $\alpha, \beta \in P(X)$ then $\alpha \leq \beta$ if and only if $X\alpha \subseteq X\beta$, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ and $\beta\beta^{-1} \cap (\operatorname{dom}\beta \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}$.*

Theorem 2.5.6. *If $\alpha, \beta \in P(X)$ then the following are equivalent.*

(i) *$\alpha \subseteq \beta$,*

(ii) *$X\alpha \subseteq X\beta$ and $\alpha\beta^{-1} \subseteq \beta\beta^{-1}$,*

(iii) *$X\alpha \subseteq X\beta$ and $\alpha\alpha^{-1} \subseteq \alpha\beta^{-1}$.*

Theorem 2.5.7. *Suppose $g \in P(X)$ is non-zero and $|X| \geq 3$, then the following statements hold.*

(i) *g is left compatible with Ω on $P(X)$ if and only if g is surjective,*

(ii) *g is right compatible with Ω on $P(X)$ if and only if $g \in T(X)$ and either g is injective or g is constant.*