Chapter 3

Partial Orders on the Baer-Levi Semigroups of Partial Transformations

Throughout this thesis, $|X| = p \geq q \geq \aleph_0$. Also, $Y = A \dot{\cup} B$ means that Y is a *disjoint* union of A and B. We modify the convention introduced in [1] vol 2, p 241: namely, if $\alpha \in I(X)$ is non-zero then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $\operatorname{ran}\alpha = \{x_i\}$, $x_i\alpha^{-1} = \{a_i\}$ and $\operatorname{dom}\alpha = \{a_i : i \in I\}$. For simplicity, we often write $X\alpha$ in place of $\operatorname{ran}\alpha$, in which case $X\alpha^{-1} = \operatorname{ran}\alpha^{-1} = \operatorname{dom}\alpha$.

In this chapter, we consider the natural partial order $\leq$, the containment order $\subseteq$ and other partial orders defined on $I(X)$ and $PS(q)$.

3.1 Partial Orders

In order to characterise $\leq$ on $PS(q)$, we first define the relation $\mathbb{L}$ on $PS(q)$ by

$$(\alpha, \beta) \in \mathbb{L} \quad \text{if and only if} \quad PS(q)^1\alpha \subseteq PS(q)^1\beta.$$

It is easy to see that $\mathbb{L}$ is reflexive and transitive. However, in general, it is not anti-symmetric. For example, Let $X = A \dot{\cup} B \dot{\cup} \{c, d, e\}$ where $|A| = p$ and $|B| = q$, and define $\alpha, \beta, \lambda, \mu \in PS(q)$ by

$$\alpha = \operatorname{id}_A \cup \begin{pmatrix} d \\ c \end{pmatrix}, \beta = \operatorname{id}_A \cup \begin{pmatrix} e \\ c \end{pmatrix}, \lambda = \operatorname{id}_A \cup \begin{pmatrix} d \\ e \end{pmatrix}, \mu = \operatorname{id}_A \cup \begin{pmatrix} e \\ d \end{pmatrix}.$$

Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, so $(\alpha, \beta) \in \mathbb{L}$ and $(\beta, \alpha) \in \mathbb{L}$, but $\alpha \neq \beta$.

Nonetheless, if ρ is any partial order on $PS(q)$, then $\rho \cap \mathbb{L}$ is also a partial order on $PS(q)$. This idea leads to a simple description of $\leq$ on $PS(q)$.

Theorem 3.1.1. *When restricted to $PS(q)$, $\leq$ equals $\subseteq \cap \mathbb{L}$. Moreover, $\leq$ is properly contained in $\subseteq$.*

Proof. Suppose that $\alpha, \beta \in PS(q)$ are distinct and $\alpha \leq \beta$ in $PS(q)$. Then $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$ for some $\lambda, \mu \in PS(q)$, and so

$$PS(q)^1\alpha = PS(q)^1\lambda\beta \subseteq PS(q)^1\beta,$$

that is, $(\alpha, \beta) \in \mathbb{L}$. Also, $\alpha = \lambda\beta$ implies $X\alpha \subseteq X\beta$ and $\alpha = \alpha\mu$ implies $X\alpha \subseteq \operatorname{dom}\mu$. Hence

$$\alpha\alpha^{-1} = \alpha\mu(\beta\mu)^{-1} = \alpha(\mu\mu^{-1})\beta^{-1} = \alpha\beta^{-1},$$

and so $\alpha \subseteq \beta$ by Theorem 2.5.6. Therefore, $\leq$ is a subset of $\subseteq \cap \mathbb{L}$.

Conversely, suppose that $(\alpha, \beta) \in \subseteq \cap \mathbb{L}$ and $\alpha \neq \beta$. Then $PS(q)^1\alpha \subseteq PS(q)^1\beta$ and so $\alpha = \lambda\beta$ for some $\lambda \in PS(q)$. Moreover, since $\alpha \subseteq \beta$, we can write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \beta = \begin{pmatrix} a_i & a_j \\ x_i & x_j \end{pmatrix}, \mu = \begin{pmatrix} x_i \\ x_i \end{pmatrix},$$

where $d(\mu) = d(\alpha) = q$. Hence $\mu \in PS(q)$ and clearly $\alpha = \beta\mu$ and $\alpha = \alpha\mu$. Therefore, $\alpha \leq \beta$ in $PS(q)$.

Now we deduce that $\leq$ is a subset of $\subseteq$ on $PS(q)$ and we assert that this containment is always proper on $PS(q)$. To see this, we suppose $X = A \dot\cup B \dot\cup \{c\}$ where $|A| = p$ and $|B| = q$, and let $\alpha : A \cup B \to A$ be a bijection. Then $d(\alpha) = |B \cup \{c\}| = q$ and so $\alpha \in PS(q)$. Likewise, if $\beta \in T(X)$ equals α on $A \cup B$ and satisfies $c\beta = c$, then $\beta \in PS(q)$ and $\alpha \subseteq \beta$. But $g(\beta) = 0 < q$ and $|X\beta \setminus X\alpha| = 1 < q$, hence there is no $\lambda \in PS(q)$ such that $\alpha = \lambda\beta$ by Theorem 2.3.10. This follows that $(\alpha, \beta) \notin \mathbb{L}$ and so $\alpha \not\leq \beta$. ∎

In [11] p 384 and Lemma 1(x), Mitsch observed that, if S is an inverse semigroup, then the natural partial order on S equals the order $\preceq$ defined on S by

$$a \preceq b \quad \text{if and only if} \quad a = eb \ \text{ for some idempotent } e \in S.$$

Moreover, from [3] Proposition V.2.3, we know that $\preceq$ equals $\subseteq$ on $I(X)$, and thus $\leq\, = \,\subseteq$ on $I(X)$.

We recall from Chapter 2, Section 2.5 that Ω' and Ω are partial orders defined on $P(X)$ by

$$(\alpha,\beta) \in \Omega' \quad \text{if and only if} \quad X\alpha \subseteq X\beta, \qquad \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta \text{ and}$$
$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1},$$
$$(\alpha,\beta) \in \Omega \quad \text{if and only if} \quad (\alpha,\beta) \in \Omega' \qquad \text{and } \beta\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}.$$

In [10], the authors showed that Ω' is an upper bound for $\leq$ and $\subseteq$, and that $\Omega = \,\leq \vee \subseteq\, = \,\subseteq \circ \leq$ on $P(X)$. Clearly $\Omega \subseteq \Omega'$ and these are also partial orders on $I(X)$, a semigroup in which $\leq\, = \,\subseteq$. Therefore, we get the following result.

Theorem 3.1.2. *$\Omega = \Omega'$ on $I(X)$ and $PS(q)$.*

Proof. Suppose that $\alpha, \beta \in I(X)$ and $(\alpha,\beta) \in \Omega'$. Then $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and $\beta\beta^{-1} = \mathrm{id}_{\operatorname{dom}\beta}$, so

$$\beta\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) = \mathrm{id}_{\operatorname{dom}\alpha} = \alpha\alpha^{-1}.$$

Hence $(\alpha,\beta) \in \Omega$, and thus $\Omega' \subseteq \Omega$ on $I(X)$ as required. When $\alpha, \beta \in PS(q)$, we can use the same proof for $\alpha, \beta \in I(X)$ to obtain $\Omega' \subseteq \Omega$ on $PS(q)$. ■

Since $\leq\, = \,\subseteq$ and $\Omega = \Omega'$ on $I(X)$, it is natural to ask whether all four orders are equal on $I(X)$. In [10], the authors showed that $\subseteq$ is contained in Ω' in $P(X)$. The next result shows that this also holds in $I(X)$ and $PS(q)$. We first note that $\Omega \,= \,\subseteq$ on $I(X)$ when $|X| = 1$.

Theorem 3.1.3. *If $|X| > 1$, then $\subseteq$ is properly contained in Ω on $I(X)$.*

Proof. Assume that $|X| > 1$. If α, β are injective and $\alpha \subseteq \beta$, then $\alpha\beta^{-1} = \alpha\alpha^{-1}$. So $X\alpha \subseteq X\beta$, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) = \alpha\alpha^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) = \alpha\alpha^{-1}.$$

That is, $(\alpha, \beta) \in \Omega' = \Omega$. Since $|X| > 1$, we can choose distinct $x, y \in X$ and define $\alpha, \beta \in I(X)$ by

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix}, \beta = \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

Then $X\alpha \subseteq X\beta$, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) = \emptyset \subseteq \alpha\alpha^{-1}.$$

Hence $(\alpha, \beta) \in \Omega' = \Omega$ but $\alpha \not\subseteq \beta$, so $\subseteq$ is properly contained in Ω on $I(X)$. ■

Theorem 3.1.4. *$\subseteq$ is properly contained in Ω on $PS(q)$.*

Proof. From the first part of the proof in Theorem 3.1.3, we see that $\subseteq$ is also contained in Ω on $PS(q)$. Suppose that $X = A \,\dot\cup\, B \,\dot\cup\, \{x\} \,\dot\cup\, \{y\}$ where $|A| = p$ and $|B| = q$, and let $\theta : A \cup B \to A$ be a bijection. Define $\alpha, \beta \in PS(q)$ by

$$\alpha = \begin{pmatrix} A \cup B & x \\ A & x \end{pmatrix}, \beta = \begin{pmatrix} A \cup B & x & y \\ A & y & x \end{pmatrix}$$

where $\alpha|(A \cup B) = \theta = \beta|(A \cup B)$, we see that $\alpha \not\subseteq \beta$. Since $y \notin \operatorname{dom}\alpha$, we have

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) = \mathrm{id}_{A\cup B} \subseteq \mathrm{id}_{\operatorname{dom}\alpha} = \alpha\alpha^{-1},$$

that is, $(\alpha, \beta) \in \Omega$. Therefore $\subseteq$ is always properly contained in Ω. ■

From [10], Ω is the join of $\subseteq$ and $\leq$ on $P(X)$ and it equals $\subseteq \circ \leq$. But, $\Omega \neq \subseteq \circ \leq$ on $PS(q)$: otherwise, Ω is contained in $\subseteq \circ \subseteq$ (since $\leq$ is contained in $\subseteq$), so Ω is contained in $\subseteq$, which is a contradiction.

From Theorem 3.1.1 and Theorem 3.1.4, on $PS(q)$ we always have:

$$\leq = \subseteq \cap \mathbb{L} \quad \subsetneq \quad \subseteq \quad \subsetneq \quad \Omega.$$

3.2 Compatible Partial Orders

We first note that $\subseteq$ is left and right compatible on $P(X)$. Therefore, it is also left and right compatible on $PS(q)$ since $PS(q)$ is contained in $P(X)$. In this section, since we know that $\Omega = \Omega'$ on $PS(q)$, we will only characterize the compatibility of $\leq$ and Ω on $PS(q)$.

Theorem 3.2.1. *Suppose that $\gamma \in PS(q)$.*

(i) *γ is left compatible with $\leq$ on $PS(q)$ if and only if $q \leq g(\gamma)$,*

(ii) *$\leq$ is right compatible on $PS(q)$.*

Proof. To prove (i), suppose that γ is left compatible with $\leq$. If $\gamma = \emptyset$ (in case $p = q$), then $g(\gamma) = p = q$. If $\gamma \neq \emptyset$, we choose $x \in X\gamma$ and let $\alpha = \mathrm{id}_{X\gamma \setminus \{x\}}$ and $\beta = \mathrm{id}_{X\gamma}$. Then $\alpha \subseteq \beta$ and $g(\beta) = d(\beta) = d(\gamma) = q$ and $g(\alpha) = d(\alpha) = q + 1 = q$ (since $q \geq \aleph_0$). Hence $\alpha, \beta \in PS(q)$ and

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) = q = \max(g(\alpha), q).$$

Then $\alpha = \lambda\beta$ for some $\lambda \in PS(q)$ by Theorem 2.3.10 and so $PS(q)^1\alpha \subseteq PS(q)^1\beta$, that is, $(\alpha, \beta) \in \mathbb{L}$. It follows that $\alpha \leq \beta$ by Theorem 3.1.1. Since γ is left compatible, we have $\gamma\alpha \leq \gamma\beta$ and so $(\gamma\alpha, \gamma\beta) \in \mathbb{L}$ (since $\leq \, = \, \subseteq \cap \, \mathbb{L}$) where $\gamma\alpha \neq \gamma\beta = \gamma$. Then Theorem 2.3.10 implies that

$$q \leq \max(g(\gamma\beta), |X\gamma\beta \setminus X\gamma\alpha|).$$

But, since $|X\gamma\beta \setminus X\gamma\alpha| = 1 < q$, this implies $q \leq g(\gamma\beta) = g(\gamma)$.

Conversely, suppose that $q \leq g(\gamma)$. If $\alpha, \beta \in PS(q)$ and $\alpha \leq \beta$, then $\alpha \subseteq \beta$ and $(\alpha, \beta) \in \mathbb{L}$ by Theorem 3.1.1. Since $\subseteq$ is left compatible, we have $\gamma\alpha \subseteq \gamma\beta$. Also, $\mathrm{dom}\,\gamma\beta \subseteq \mathrm{dom}\,\gamma$ implies $q \leq g(\gamma) \leq g(\gamma\beta)$. By the definition of $\leq$, we have $\alpha = \beta\mu$ for some $\mu \in PS(q)^1$. If $\mu = 1$, then $\alpha = \beta$ and so $\gamma\alpha \leq \gamma\beta$. If $\mu \in PS(q)$, then $\gamma\alpha = (\gamma\beta)\mu$ and hence $g(\gamma\beta) \leq g(\gamma\alpha)$. Moreover, since $\gamma\alpha \in PS(q)$,

$$|X\gamma\beta \setminus X\gamma\alpha| = |X\gamma\beta \cap (X \setminus X\gamma\alpha)| \leq q$$

and so

$$q \leq g(\gamma\beta) = \max(g(\gamma\beta), |X\gamma\beta \setminus X\gamma\alpha|) \leq g(\gamma\alpha) = \max(g(\gamma\alpha), q),$$

that is, $(\gamma\alpha, \gamma\beta) \in \mathbb{L}$. Since $\leq \, = \, \subseteq \cap \, \mathbb{L}$, we have $\gamma\alpha \leq \gamma\beta$. In both cases we deduce that γ is left compatible with respect to $\leq$ as required. Finally, note that

$\subseteq$ is right compatible, and $PS(q)^1\alpha \subseteq PS(q)^1\beta$ implies $PS(q)^1\alpha\gamma \subseteq PS(q)^1\beta\gamma$ for any $\gamma \in PS(q)$, that is, $\mathbb{L}$ is also right compatible on $PS(q)$. Hence, (ii) follows from Theorem 3.1.1. ■

Here, for simplicity, we write x_y for the $\alpha \in I(X)$ with domain $\{x\}$ and range $\{y\}$.

Theorem 3.2.2. *Suppose that $p = q$ and let $\gamma \in PS(q)$. Then*

(i) *$\emptyset$ is the only element of $PS(q)$ which is left compatible with Ω,*

(ii) *γ is right compatible with Ω if and only if $\gamma = \emptyset$ or $\operatorname{dom}\gamma = X$.*

Proof. Clearly $\emptyset \in PS(q)$ and it is left and right compatible with Ω. Let γ be a non-zero element in $PS(q)$. If we choose $x \in X\gamma$, $y \in X \setminus X\gamma$ and define

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix}, \quad \beta = \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

then $\alpha, \beta \in PS(q)$ (since $p = q$) and it is easy to check that $(\alpha, \beta) \in \Omega$. However, since $X\gamma\alpha = \{x\} \nsubseteq \{y\} = X\gamma\beta$, we have $(\gamma\alpha, \gamma\beta) \notin \Omega$ (by definition) and so γ is not left compatible with Ω. Therefore (i) holds.

Next, suppose that $\gamma \in PS(q)$ is non-empty and right compatible with Ω. If $a \in \operatorname{dom}\gamma$, $x \in X \setminus \operatorname{dom}\gamma$ and $Y = \{a, x\}$ then $x_a, \operatorname{id}_Y \in PS(q)$ and $(x_a, \operatorname{id}_Y) \in \Omega$ (note that $x_a.\operatorname{id}_Y^{-1} \cap \{(x, x)\} = \emptyset$). Hence $(x_a.\gamma, \operatorname{id}_Y.\gamma) \in \Omega$ and so $\operatorname{dom}(x_a.\gamma) = \{x\} \subseteq \operatorname{dom}(\operatorname{id}_Y.\gamma) = \{a\}$, a contradiction. Thus, we have shown that $\operatorname{dom}\gamma = X$. Therefore, to prove (ii), it remains to show that, if $\operatorname{dom}\gamma = X$, then γ is right compatible with Ω. To do this, let $\alpha, \beta \in PS(q)$ and $(\alpha, \beta) \in \Omega$. Then, since $\Omega = \Omega'$, we have $X\alpha \subseteq X\beta$, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}.$$

Clearly $X\alpha\gamma \subseteq X\beta\gamma$ and, since $\operatorname{dom}\gamma = X$, $\operatorname{dom}\alpha\gamma = \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta = \operatorname{dom}\beta\gamma$. Also $\gamma\gamma^{-1} = \operatorname{id}_X$ (but note that $\operatorname{id}_X \notin PS(q)$), and hence

$$\alpha\gamma(\beta\gamma)^{-1} \cap (\operatorname{dom}\alpha\gamma \times \operatorname{dom}\alpha\gamma) = \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1} = \alpha\gamma(\alpha\gamma)^{-1},$$

from which it follows that $(\alpha\gamma, \beta\gamma) \in \Omega$. ■

Theorem 3.2.3. *Suppose that $p > q$ and let $\gamma \in PS(q)$. Then*

(i) *no element of $PS(q)$ is left compatible with Ω,*

(ii) *γ is right compatible with Ω if and only if* $\operatorname{dom}\gamma = X$.

Proof. To prove (i), let $\theta \in PS(q)$, choose $x \in X\theta$, $y \in X \setminus X\theta$ and define

$$\alpha = \mathrm{id}_{X\theta}, \quad \beta = \begin{pmatrix} X\theta \setminus \{x\} & x & y \\ X\theta \setminus \{x\} & y & x \end{pmatrix},$$

where $z\beta = z$ for all $z \in X\theta \setminus \{x\}$. Then $\alpha, \beta \in PS(q)$ and $(\alpha, \beta) \in \Omega$. Since $x \in X\theta\alpha \setminus X\theta\beta$, we have $(\theta\alpha, \theta\beta) \notin \Omega$ (by definition). That is, θ is not left compatible with Ω. The proof of (ii) is the same as that for Theorem 3.2.2(ii), except that now $\emptyset \notin PS(q)$. ■

For completeness, we note the following result for Ω on $I(X)$.

Theorem 3.2.4. *If $\gamma \in I(X)$ is non-zero, then*

(i) *γ is left compatible with Ω on $I(X)$ if and only if $X\gamma = X$,*

(ii) *γ is right compatible with Ω on $I(X)$ if and only if* $\operatorname{dom}\gamma = X$.

Proof. Suppose that $X\gamma \neq X$. Then, as in the proof of Theorem 3.2.3(i), there exists $(\alpha, \beta) \in \Omega$ on $I(X)$ but $(\gamma\alpha, \gamma\beta) \notin \Omega$. Therefore γ is not left compatible with Ω. For the converse of (i), if $\gamma \in PS(q)$ is surjective, then Theorem 2.5.7 (i) implies that it is left compatible with Ω on $P(X)$, and so the same is true for $I(X)$. To see (ii), suppose that $\gamma \in I(X)$ is non-empty and right compatible with Ω. If $a \in \operatorname{dom}\gamma$, $x \in X \setminus \operatorname{dom}\gamma$ and $Y = \{a, x\}$ then $x_a, \mathrm{id}_Y \in I(X)$ and $(x_a, \mathrm{id}_Y) \in \Omega$ (note that $x_a.\mathrm{id}_Y^{-1} \cap \{(x,x)\} = \emptyset$). Hence $(x_a.\gamma, \mathrm{id}_Y.\gamma) \in \Omega$ and so $\operatorname{dom}(x_a.\gamma) = \{x\} \subseteq \operatorname{dom}(\mathrm{id}_Y.\gamma) = \{a\}$, a contradiction. Thus, we have shown that $\operatorname{dom}\gamma = X$. It remains to show that, if $\operatorname{dom}\gamma = X$, then γ is right compatible with Ω. To do this, let $\alpha, \beta \in I(X)$ and $(\alpha, \beta) \in \Omega$. Then, since $\Omega = \Omega'$, we have $X\alpha \subseteq X\beta$, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}.$$

Clearly $X\alpha\gamma \subseteq X\beta\gamma$ and, since $\operatorname{dom}\gamma = X$, $\operatorname{dom}\alpha\gamma = \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta = \operatorname{dom}\beta\gamma$. Also $\gamma\gamma^{-1} = \mathrm{id}_X$, and hence

$$\alpha\gamma(\beta\gamma)^{-1} \cap (\operatorname{dom}\alpha\gamma \times \operatorname{dom}\alpha\gamma) = \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1} = \alpha\gamma(\alpha\gamma)^{-1}.$$

Therefore $(\alpha\gamma, \beta\gamma) \in \Omega$. ■

3.3 Minimal and Maximal Elements

In this section, we consider the existence of minimal (maximal) elements in $PS(q)$ with respect to each of the orders $\leq$, $\subseteq$ and Ω.

First, recall that, if $\preceq$ is any partial order on a set T, and if $x \in S \subseteq T$ is minimal (maximal) in T, then x is minimal (maximal) in S. Similarly, suppose $<_1$ and $<_2$ are partial orders on a set S such that $<_2$ contains $<_1$. Clearly, if $x \in S$ is minimal (maximal) with respect to $<_2$, then x is minimal (maximal) with respect to $<_1$. Consequently, under the same supposition, if x is a minimum (maximum) with respect to $<_1$, then x is a minimum (maximum) with respect to $<_2$.

Theorem 3.3.1. *$PS(q)$ has no maximum element with respect to $\leq$, $\subseteq$ or Ω.*

Proof. Write $X = A \dot{\cup} B \dot{\cup} C$ where $|A| = p$ and $|B| = q = |C|$. Clearly, if $\alpha = \mathrm{id}_{A\cup B}$ and $\beta = \mathrm{id}_{A\cup C}$, then $\alpha, \beta \in PS(q)$. If $\gamma \in PS(q)$ is a maximum with respect to Ω, then $(\alpha, \gamma) \in \Omega$ and $(\beta, \gamma) \in \Omega$. Consequently $X\alpha \subseteq X\gamma$ and $X\beta \subseteq X\gamma$, hence $X\alpha \cup X\beta \subseteq X\gamma$ and so $X\gamma = X$, which contradicts $d(\gamma) = q$. Therefore $PS(q)$ has no maximum element with respect to Ω. Next recall that $\leq$ is properly contained in $\subseteq$ which is properly contained in Ω on $PS(q)$. So, if α is a maximum under $\subseteq$, then it is also a maximum under Ω, a contradiction. Likewise, there is no maximum under $\leq$. ■

Theorem 3.3.2. *The following are equivalent for $\alpha \in PS(q)$.*

(i) *α is maximal with respect to Ω,*

(ii) *α is maximal with respect to $\subseteq$,*

(iii) *$\operatorname{dom}\alpha = X$.*

Proof. (i) implies (ii) since $\subseteq$ is contained in Ω. To show (ii) implies (iii), suppose that (ii) holds and assume $\operatorname{dom}\alpha \subsetneq X$. Choose $x \in X \setminus \operatorname{dom}\alpha$ and $y \in X \setminus X\alpha$ (recall that $d(\alpha) = q$) and let β be the mapping such that $\operatorname{dom}\beta = \operatorname{dom}\alpha \cup \{x\}$, $\beta|\operatorname{dom}\alpha = \alpha$ and $x\beta = y$. Then $\beta \in PS(q)$ and $\alpha \subseteq \beta$ with $\alpha \neq \beta$, contradicting our supposition.

Finally, to show (iii) implies (i), suppose that $\operatorname{dom}\alpha = X$ and let $\beta \in PS(q)$ satisfy $(\alpha, \beta) \in \Omega$. Then by Theorem 3.1.2, we have

$$\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta,\ X\alpha \subseteq X\beta \text{ and } \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1}.$$

So $\operatorname{dom}\beta = X$. For each $x \in X$, if $x\alpha = y\beta$ for some $y \in X$, then $x\alpha\beta^{-1} = y$ and thus

$$(x, y) \in \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1} = \operatorname{id}_X .$$

This follows that $x = y$, that is, $x\alpha = x\beta$ and hence $\alpha = \beta$. This shows that (i) holds. ∎

The corresponding result for $\leq$ is substantially different.

Theorem 3.3.3. *Let $\alpha \in PS(q)$. Then α is maximal with respect to $\leq$ if and only if $g(\alpha) < q$.*

Proof. Suppose that $g(\alpha) \geq q$. Then $X \setminus \operatorname{dom}\alpha \neq \emptyset$. Choose $x \in X \setminus \operatorname{dom}\alpha$ and $y \in X \setminus X\alpha$ (recall that $d(\alpha) = q$) and let β be the mapping such that $\operatorname{dom}\beta = \operatorname{dom}\alpha \cup \{x\}$, $\beta|\operatorname{dom}\alpha = \alpha$ and $x\beta = y$. Then $\beta \in PS(q)$ and $\alpha \subseteq \beta$ with $\alpha \neq \beta$, $X\beta = X\alpha \dot\cup \{y\}$ and $g(\alpha) = g(\beta)$. So $|X\beta \setminus X\alpha| = 1$ and hence the inequation in Theorem 2.3.10 is satisfied, that is, $\alpha = \lambda\beta$ for some $\lambda \in PS(q)$. Then $(\alpha, \beta) \in \mathbb{L}$ and it follows that $\alpha \leq \beta$ by Theorem 3.1.1, but $\alpha \neq \beta$, so α is not maximal. Conversely, suppose that $g(\alpha) < q$ and assume that $\alpha < \beta$ for some $\beta \in PS(q)$. Thus, by Theorem 3.1.1, we have $\alpha \subsetneq \beta$ and, by Theorem 2.3.10

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q) = q.$$

Therefore, $g(\beta) \leq g(\alpha) < q$ and so $|X\beta \setminus X\alpha| = q$. Consequently, since $X\alpha \subseteq X\beta$, then

$$q = |(X\beta \setminus X\alpha)\beta^{-1}| = |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha| \leq g(\alpha) < q,$$

a contradiction. This shows that α is maximal. ∎

As in many algebraic settings, it is interesting to know when $\alpha \in PS(q)$ lies below some maximal element of $PS(q)$.

Theorem 3.3.4. *The following are equivalent for $\alpha \in PS(q)$.*

(i) $g(\alpha) \leq q$,

(ii) $\alpha \leq \beta$ *for some* $\beta \in PS(q)$ *maximal with respect to* $\leq$,

(iii) $\alpha \subseteq \beta$ *for some* $\beta \in PS(q)$ *maximal with respect to* $\subseteq$,

(iv) $(\alpha, \beta) \in \Omega$ *for some* $\beta \in PS(q)$ *maximal with respect to* Ω.

Proof. Suppose that (i) holds. If $g(\alpha) < q$, then $\alpha \leq \alpha$ and α is maximal under $\leq$ by Theorem 3.3.3. Therefore, suppose that $g(\alpha) = q$. Since $d(\alpha) = q$, we can write $X \setminus X\alpha = A \dot{\cup} B$ where $|A| = |B| = q$. Let $\theta : X \setminus \operatorname{dom}\alpha \to A$ be any bijection and define $\beta \in PS(q)$ by letting $\operatorname{dom}\beta = X$, $\beta|\operatorname{dom}\alpha = \alpha$ and $\beta|(X \setminus \operatorname{dom}\alpha) = \theta$. Then $g(\beta) = 0$ and $X\beta = X\alpha \dot{\cup} A$, so

$$q = |A| = \max(g(\beta), |X\beta \setminus X\alpha|) = \max(g(\alpha), q).$$

That is, $(\alpha, \beta) \in \mathbb{L}$ and clearly $\alpha \subsetneq \beta$. Hence $\alpha < \beta$ where β is maximal with respect to $\leq$ since $g(\beta) = 0 < q$.

Now suppose that (ii) holds: namely, suppose $\alpha \leq \beta$ where $g(\beta) = r < q$. Then $\alpha \subseteq \beta$ and $d(\beta) = q$, so we can write $X \setminus X\beta = A \dot{\cup} B$ where $|A| = r$ and $|B| = q$. Let $\theta : X \setminus \operatorname{dom}\beta \to A$ be any bijection and define $\beta^+ \in PS(q)$ by letting $\operatorname{dom}\beta^+ = X$, $\beta^+|\operatorname{dom}\beta = \beta$ and $\beta^+|(X \setminus \operatorname{dom}\beta) = \theta$. Then $\alpha \subseteq \beta \subseteq \beta^+$ where β^+ is maximal with respect to $\subseteq$: that is, (iii) holds by Theorem 3.3.2.

Next, suppose that (iii) holds. Then $\alpha \subseteq \beta$ for some $\beta \in PS(q)$ maximal with respect to $\subseteq$. By Theorem 3.3.2, β is maximal with respect to Ω. Since $\subseteq$ is contained in Ω, we deduce that (iv) also holds.

Finally, suppose that (iv) holds: that is, suppose $(\alpha, \beta) \in \Omega$ where $\operatorname{dom}\beta = X$, and write

$$A = \{x \in \operatorname{dom}\alpha : x\alpha\beta^{-1} = x\},$$
$$B = \{x \in \operatorname{dom}\alpha : x\alpha\beta^{-1} \notin \operatorname{dom}\alpha\}.$$

By the definition of Ω, if $x \in \operatorname{dom}\alpha$ and $x\alpha = y\beta$ (possible since $X\alpha \subseteq X\beta$) then either $y \in \operatorname{dom}\alpha$ (so $y = x$ and $x \in A$) or $y \notin \operatorname{dom}\alpha$ (so $x \in B$). It follows that $\operatorname{dom}\alpha = A \dot{\cup} B$, $A\alpha = A\beta$ and $B\alpha = C\beta$ for some $C \subseteq \operatorname{dom}\beta \setminus \operatorname{dom}\alpha$. Note that $X\alpha = (A \cup C)\beta$ and $(A \cup C) \cap B = \emptyset$. Therefore $(A \cup C)\beta \cap B\beta = \emptyset$ (since β is injective). This follows that $X\alpha \cap B\beta = \emptyset$, that is, $B\beta \subseteq X \setminus X\alpha$. So, since $\operatorname{dom}\beta = X$,

$$|B| = |B\alpha| = |B\beta| \leq |X \setminus X\alpha| = q.$$

Next let $D = X \setminus (A \cup B \cup C)$ and observe that $D\beta \cap X\alpha = D\beta \cap (A \cup C)\beta = \emptyset$. Therefore

$$|D\beta| \leq |X \setminus X\alpha| = q.$$

Now $X\beta = A\beta \dot{\cup} B\beta \dot{\cup} C\beta \dot{\cup} D\beta$ and thus

$$(X \setminus \operatorname{dom}\alpha)\beta = (X \setminus (A \cup B))\beta = X\beta \setminus (A \cup B)\beta = C\beta \cup D\beta.$$

Consequently

$$g(\alpha) = |X \setminus \operatorname{dom}\alpha| = |(X \setminus \operatorname{dom}\alpha)\beta| = |C\beta| + |D\beta| \leq |B\alpha| + q = q,$$

and so (i) holds. ■

Observe that, if $p = q$, then $g(\alpha) \leq q$ for all $\alpha \in PS(q)$. Hence, in this case, every $\alpha \in PS(q)$ is contained in some maximal element.

Theorem 3.3.5. *If $p > q$, then $PS(q)$ has no minimal element with respect to $\leq, \subseteq$ or Ω, and hence also no minimum element.*

Proof. Suppose that $p > q$ and let $\alpha \in PS(q)$. Since $|X \setminus X\alpha| = q < p$, we have $p = |X\alpha| = |\operatorname{dom}\alpha|$ and we can write $\operatorname{dom}\alpha = A \dot{\cup} B$ where $|A| = p$ and $|B| = q$. If $\gamma = \alpha|A$, then $d(\gamma) = |B\alpha| + d(\alpha) = q$, thus $\gamma \in PS(q)$ and clearly $\gamma \subsetneq \alpha$. Also, let $C = X \setminus \operatorname{dom}\alpha$ and $\lambda = \operatorname{id}_{A \cup C}$, then $d(\lambda) = |B| = q$, so $\lambda \in PS(q)$ and $\gamma = \lambda\alpha$. Consequently, $(\gamma, \alpha) \in \mathbb{L}$ and so $\gamma < \alpha$ by Theorem 3.1.1. Therefore, there is no minimal element under $\leq$, and hence none for $\subseteq$ and Ω (due to them containing $\leq$). Hence, there is also no minimum element under each of these orders. ■

When $p = q$, it is easy to see that $\emptyset$ is the minimum under $\leq, \subseteq$ and Ω. In this case, we say $\alpha \in PS(q)$ is *non-zero minimal* with respect to an order $\preceq$ on $PS(q)$ if α is minimal among the non-zero elements of $PS(q)$ under $\preceq$.

Theorem 3.3.6. *If $p = q$, then the following are equivalent for $\alpha \in PS(q)$.*

(i) *α is non-zero minimal with respect to Ω,*

(ii) *α is non-zero minimal with respect to $\subseteq$,*

(iii) *α is non-zero minimal with respect to $\leq$,*

(iv) $|\operatorname{dom}\alpha| = 1$.

Proof. Since Ω contains $\subseteq$, and $\subseteq$ contains $\leq$, we have (i) implies (ii), and (ii) implies (iii). To show that (iii) implies (iv), suppose that (iii) holds and assume that $|\operatorname{dom}\alpha| > 1$. Now, as in the proof of Theorem 3.3.5, if $|\operatorname{dom}\alpha| = p$, then we can write $\operatorname{dom}\alpha = A \dot\cup B$ where $|A| = p$ and $|B| = q$. If $\gamma = \alpha|A$, then $d(\gamma) = |B\alpha| + d(\alpha) = q$, thus $\gamma \in PS(q)$ and clearly $\gamma \subsetneq \alpha$. Also, if $X = A \dot\cup B \dot\cup C$ and $\lambda = \mathrm{id}_{A\cup C}$, then $d(\lambda) = |B| = q$, so $\lambda \in PS(q)$ and $\gamma = \lambda\alpha$ (since $C = X \setminus \operatorname{dom}\alpha$). Consequently, $(\gamma, \alpha) \in \mathbb{L}$ and so $\emptyset < \gamma < \alpha$ by Theorem 3.1.1, contradicting (iii). On the other hand, if $|\operatorname{dom}\alpha| < p$ then $g(\alpha) = p$. In this case, choose $a \in \operatorname{dom}\alpha$ and write $C = \operatorname{dom}\alpha \setminus \{a\}$ (which is non-empty by assumption). If $\beta = \alpha|C$ and $\lambda = \mathrm{id}_C$ then $\beta, \lambda \in PS(q)$ and $\beta = \lambda\alpha$. Therefore, $(\beta, \alpha) \in \mathbb{L}$ and clearly $\beta \subsetneq \alpha$. That is, $\emptyset < \beta < \alpha$, contradicting (iii) again.

Finally, to show (iv) implies (i), suppose that $|\operatorname{dom}\alpha| = 1$, say $\operatorname{dom}\alpha = \{x\}$. Since $\Omega = \Omega'$ and by the definition of Ω', if there exists $\beta \neq \emptyset$ such that $(\beta, \alpha) \in \Omega$, then $\operatorname{dom}\beta = \{x\}$ and $X\beta = \{x\alpha\}$. Hence $\alpha = \beta$ and so α is non-zero minimal under Ω. ■

Recall that $\leq$ is properly contained in $\subseteq$, and $\subseteq$ is properly contained in Ω on $PS(q)$. Thus, it is interesting to consider the following problems: Are there other partial orders lie between $\leq$ and $\subseteq$, and between $\subseteq$ and Ω on $PS(q)$? The following two theorems answer this question.

Theorem 3.3.7. *Let $\alpha, \beta \in PS(q)$. For any (α, β) in $\subseteq \setminus \leq$,*

$$\rho_{\alpha,\beta} = \leq \cup \{(\alpha, \beta)\}$$

is a minimal partial order on $PS(q)$ containing $\leq$.

Proof. It is clear that a pair (α, β) in $\subseteq \setminus \leq$ always exists since $\leq$ is properly contained in $\subseteq$ on $PS(q)$. Since $\leq$ is a partial order contained in $\rho_{\alpha,\beta}$, we have $\rho_{\alpha,\beta}$ is reflexive. To see that $\rho_{\alpha,\beta}$ is anti-symmetric, we let $(\gamma, \lambda), (\lambda, \gamma) \in \rho_{\alpha,\beta}$. We consider only the case that $\gamma \leq \lambda, (\lambda, \gamma) = (\alpha, \beta)$, otherwise, it is easy to see that $\gamma = \lambda$. In this case we have $\lambda = \alpha, \gamma = \beta$, and so $\beta \leq \alpha$. It follows that $\beta \subseteq \alpha$ (since $\subseteq$ contains $\leq$), and hence $\alpha = \beta$ (since $\alpha \subseteq \beta$). Therefore $\lambda = \gamma$. To see that $\rho_{\alpha,\beta}$ is transitive, let $(\gamma, \lambda), (\lambda, \mu) \in \rho_{\alpha,\beta}$. It is clear that $(\gamma, \mu) \in \rho_{\alpha,\beta}$ when $\gamma \leq \lambda, \lambda \leq \mu$, and if $(\gamma, \lambda), (\lambda, \mu) \in \{(\alpha, \beta)\}$, then $\gamma = \mu$ and thus $(\gamma, \mu) \in \rho_{\alpha,\beta}$ since $\rho_{\alpha,\beta}$ is reflexive. For the rest, we have either

$$\gamma \leq \lambda,\ \lambda = \alpha,\ \mu = \beta \quad \text{or} \quad \lambda \leq \mu,\ \gamma = \alpha,\ \lambda = \beta.$$

In the first case, we have $\gamma \leq \alpha$. If $g(\gamma) < q$, then γ is maximal under $\leq$, and hence $\alpha = \gamma$. This implies that $(\gamma, \mu) = (\alpha, \beta) \in \rho_{\alpha,\beta}$. Otherwise, $q \leq g(\gamma)$ where $\gamma \subseteq \alpha \subseteq \beta$ (since $\subseteq$ contains $\leq$). Consequently, since

$$X \setminus \operatorname{dom} \gamma = (X \setminus \operatorname{dom} \beta) \,\dot{\cup}\, (\operatorname{dom} \beta \setminus \operatorname{dom} \gamma),$$

we have

$$q \leq |X \setminus \operatorname{dom} \beta| = g(\beta) \quad \text{or} \quad q \leq |\operatorname{dom} \beta \setminus \operatorname{dom} \gamma| = |X\beta \setminus X\gamma|,$$

that is, $q \leq \max(g(\beta), |X\beta \setminus X\gamma|)$. Also, since $g(\beta) \leq g(\gamma)$ and $|X\beta \setminus X\gamma| \leq d(\gamma) = q$, we have

$$q \leq \max(g(\beta), |X\beta \setminus X\gamma|) \leq \max(g(\gamma), q),$$

that is $(\gamma, \beta) \in \mathbb{L}$, and so $\gamma \leq \beta = \mu$. Therefore $(\gamma, \mu) \in \rho_{\alpha,\beta}$. For the latter, we have $\beta \leq \mu$ and hence $\alpha \subseteq \beta \subseteq \mu$. Like in the first case, if $g(\beta) < q$, then $\beta = \mu$ and so $(\gamma, \mu) = (\alpha, \beta) \in \rho_{\alpha,\beta}$. Otherwise, we have $q \leq g(\beta) \leq g(\alpha)$. Consequently, since

$$X \setminus \operatorname{dom} \alpha = (X \setminus \operatorname{dom} \mu) \,\dot{\cup}\, (\operatorname{dom} \mu \setminus \operatorname{dom} \alpha),$$

we have

$$q \leq |X \setminus \operatorname{dom}\mu| = g(\mu) \quad \text{or} \quad q \leq |\operatorname{dom}\mu \setminus \operatorname{dom}\alpha| = |X\mu \setminus X\alpha|,$$

that is, $q \leq \max(g(\mu), |X\mu \setminus X\alpha|)$. Also, since $g(\mu) \leq g(\alpha)$ and $|X\mu \setminus X\alpha| \leq d(\alpha) = q$, we have

$$q \leq \max(g(\mu), |X\mu \setminus X\alpha|) \leq \max(g(\alpha), q),$$

that is $(\alpha, \mu) \in \mathbb{L}$, and so $\alpha \leq \mu$. Therefore $(\gamma, \mu) = (\alpha, \mu) \in \rho_{\alpha,\beta}$ as required. By the definition of $\rho_{\alpha,\beta}$, it is clear that

$$\leq \quad \subsetneq \quad \rho_{\alpha,\beta} \quad \subsetneq \quad \subseteq,$$

and obviously, there is no other partial orders lie between $\leq$ and $\rho_{\alpha,\beta}$. Therefore $\rho_{\alpha,\beta}$ is a minimal partial order containing $\leq$. ■

For the following result, we sometimes write $\alpha \sim_\Omega \beta$ instead of $(\alpha, \beta) \in \Omega$ for convenience.

Theorem 3.3.8. *For distinct $x, y \in X$, there is a partial order $\delta_{x,y}$ on $PS(q)$ lies strictly between $\subseteq$ and Ω.*

Proof. For distinct x, y in X, we write $X = A \,\dot\cup\, B \,\dot\cup\, \{x\} \,\dot\cup\, \{y\}$ where $|A| = p$ and $|B| = q$. Let $\theta : A \cup B \to A$ be a bijection and define $\alpha, \beta \in PS(q)$ by

$$\alpha = \begin{pmatrix} A \cup B & x \\ A & x \end{pmatrix}, \quad \beta = \begin{pmatrix} A \cup B & x & y \\ A & y & x \end{pmatrix}$$

where $\alpha|(A \cup B) = \theta = \beta|(A \cup B)$. Since $y \notin \operatorname{dom}\alpha$, we have

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) = \operatorname{id}_{A \cup B} \subseteq \operatorname{id}_{\operatorname{dom}\alpha} = \alpha\alpha^{-1}.$$

That is, $(\alpha, \beta) \in \Omega$, but $\alpha \nsubseteq \beta$ since $x\alpha \neq x\beta$. Let

$$T_{x,y} = \{(\gamma, \beta) : \gamma \in PS(q) \text{ and } \gamma \subseteq \alpha\} \quad \text{and} \quad \delta_{x,y} = \subseteq \cup\, T_{x,y}.$$

Since $(\alpha, \beta) \in T_{x,y} \setminus \subseteq$, we have $T_{x,y} \neq \emptyset$ and $\subseteq$ is properly contained in $\delta_{x,y}$. Moreover, for each pair (γ, β) in $T_{x,y}$, $\gamma \subseteq \alpha$. It follows that $\gamma \sim_\Omega \alpha \sim_\Omega \beta$

(since Ω contains $\subseteq$), that is, Ω contains $T_{x,y}$. For $y' \in X, x \neq y' \neq y$, we write $X = A' \dot\cup B' \dot\cup \{x\} \dot\cup \{y'\}$ where $|A'| = p$ and $|B'| = q$. Then define $\alpha', \beta' \in PS(q)$ in the same way as α and β. We have $(\alpha', \beta') \in \Omega \setminus \subseteq$ and also, $(\alpha', \beta') \notin T_{x,y}$ since $\beta \neq \beta'$. It follows that $\subseteq \cup\, T_{x,y} \subsetneq \Omega$, that is $\delta_{x,y}$ is properly contained in Ω. To see that $\delta_{x,y}$ is a partial order, we observe that $\delta_{x,y}$ contains a partial order $\subseteq$, then it is reflexive. To show that $\delta_{x,y}$ is anti-symmetric, let $(\lambda, \mu), (\mu, \lambda) \in \delta_{x,y}$. If both of these pairs belong to $T_{x,y}$, then $\beta \subseteq \alpha$, a contradiction since $\operatorname{dom} \beta \nsubseteq \operatorname{dom} \alpha$. Also, if $\lambda \subseteq \mu$ and $(\mu, \lambda) \in T_{x,y}$, then $\beta \subseteq \mu \subseteq \alpha$ and we get a contradiction again and, similarly, this also happen when $\mu \subseteq \lambda$ and $(\lambda, \mu) \in T_{x,y}$. It follows that both $(\lambda, \mu), (\mu, \lambda)$ belong to $\subseteq$, and hence $\lambda = \mu$. To show $\delta_{x,y}$ is transitive, let $(\lambda, \mu), (\mu, \theta) \in \delta_{x,y}$. Like before, if $(\lambda, \mu), (\mu, \theta) \in T_{x,y}$, then $\beta \subseteq \alpha$, that is, we get a contradiction again. So we consider only the following three cases. If both $(\lambda, \mu), (\mu, \theta)$ belong to $\subseteq$, then $\lambda \subseteq \theta$ and so $(\lambda, \theta) \in \delta_{x,y}$. If $\lambda \subseteq \mu$ and $(\mu, \theta) \in T_{x,y}$, then $\lambda \subseteq \mu \subseteq \alpha$ and $\beta = \theta$. Thus $(\lambda, \theta) \in T_{x,y} \subseteq \delta_{x,y}$. Finally, if $\mu \subseteq \theta$ and $(\lambda, \mu) \in T_{x,y}$, then $\lambda \subseteq \alpha$ and $\beta = \mu \subseteq \theta$. Since $\operatorname{dom} \beta = X$, we have β is maximal under $\subseteq$, so $\beta = \theta$ and this implies that $(\lambda, \theta) \in T_{x,y} \subseteq \delta_{x,y}$ as required. ∎

3.4 Meets and Joins

In this section, we study the existence of a meet $\alpha \wedge \beta$ and a join $\alpha \vee \beta$ for α, β in the semigroups $I(X)$, $PS(q)$ and $R(q)$ for each of the orders $\leq$ and $\subseteq$. To do this, we first define the *equaliser* of $\alpha, \beta \in I(X)$ (compare [14] p 416 for linear transformations) as follows:

$$E(\alpha, \beta) = \{x \in \operatorname{dom} \alpha \cap \operatorname{dom} \beta : x\alpha = x\beta\}.$$

The next result may be well-known, but we do not know a reference in the literature (recall that $\subseteq$ equals $\leq$ on $I(X)$).

Theorem 3.4.1. *Let $\alpha, \beta \in I(X)$ and $E = E(\alpha, \beta)$. Then, under $\subseteq$, $\alpha \wedge \beta = \alpha|E = \beta|E$.*

Proof. Clearly $\alpha|E = \beta|E \subseteq \alpha, \beta$. If $\gamma \subseteq \alpha, \beta$, then for each $x \in \operatorname{dom}\gamma$, $x\alpha = x\gamma = x\beta$ and this follows that $\operatorname{dom}\gamma \subseteq E$. Also, $\gamma \subseteq \alpha|E$, then we have $\alpha \wedge \beta = \alpha|E$. ■

Theorem 3.4.2. *Let $\alpha, \beta \in PS(q)$ and $E = E(\alpha, \beta)$. Then $\gamma \subseteq \alpha, \beta$ for some non-empty $\gamma \in PS(q)$ if and only if*

(i) $E \neq \emptyset$, *and*

(ii) $\max(|X\alpha \setminus E\alpha|, |X\beta \setminus E\beta|) \leq q$.

Moreover, when this occurs, $\alpha|E$ (equals $\beta|E$) is the non-empty meet of α, β under $\subseteq$.

Proof. Suppose that $\emptyset \neq \gamma \subseteq \alpha, \beta$ in $PS(q)$. Then $\emptyset \neq \operatorname{dom}\gamma \subseteq \operatorname{dom}\alpha \cap \operatorname{dom}\beta$ and $x\alpha = x\gamma = x\beta$ for all $x \in \operatorname{dom}\gamma$. That is, $\emptyset \neq \operatorname{dom}\gamma \subseteq E$ and this implies $X\gamma = E\gamma$. Now $E\gamma = (E \cap \operatorname{dom}\gamma)\gamma \subseteq E\alpha \subseteq X\alpha$ and so

$$|X\alpha \setminus E\alpha| \leq |X\alpha \setminus E\gamma| = |X\alpha \setminus X\gamma| \leq |X \setminus X\gamma| = q.$$

Similarly, $|X\beta \setminus E\beta| \leq q$ and hence the conditions hold. Conversely, if the conditions hold then $\gamma = \alpha|E = \beta|E$ is a non-empty element of $I(X)$ with domain $E = E(\alpha, \beta)$ and $\gamma \subseteq \alpha, \beta$. Moreover, since $X\gamma = E\gamma = E\alpha \subseteq X\alpha$, we have

$$X \setminus X\gamma = (X \setminus X\alpha) \mathbin{\dot{\cup}} (X\alpha \setminus E\alpha)$$

and it follows that $d(\gamma) = q$ since $d(\alpha) = q$ and $|X\alpha \setminus E\alpha| \leq q$. That is, $\gamma \in PS(q)$. Finally, as shown in the proof of Theorem 3.4.1, we have $\alpha \wedge \beta = \alpha|E$. ■

Remark 3.4.3. Suppose S is any inverse subsemigroup of $I(X)$. If $\alpha \leq \beta$ in S, then $\alpha = \gamma\beta$ for some $\gamma \in E(S)$. That is, $\alpha = \operatorname{id}_A \circ \beta$ for some $A \subseteq X$ and we deduce that $\alpha \subseteq \beta$. On the other hand, if $\alpha \subseteq \beta$ in the inverse semigroup $R(q) = \{\alpha \in PS(q) : g(\alpha) = q\}$, then $\alpha = \operatorname{id}_{\operatorname{dom}\alpha} \circ \beta$, where $\operatorname{id}_{\operatorname{dom}\alpha}$ is an idempotent in $R(q)$, and so $\alpha \leq \beta$ in $R(q)$. That is, $\leq \,=\, \subseteq$ on $R(q)$.

Of course, when we turn to $R(q)$, we expect a further condition to be needed in order to characterise meets in $R(q)$ under $\subseteq$.

Theorem 3.4.4. *Let $\alpha, \beta \in R(q)$ and $E = E(\alpha, \beta)$. Then $\gamma \subseteq \alpha, \beta$ for some non-empty $\gamma \in R(q)$ if and only if*

(i) $E \neq \emptyset$,

(ii) $\max(|X\alpha \setminus E\alpha|, |X\beta \setminus E\beta|) \leq q$, *and*

(iii) $\max(|\operatorname{dom}\alpha \setminus E|, |\operatorname{dom}\beta \setminus E|) \leq q$.

Moreover, when this occurs, $\alpha|E$ (equals $\beta|E$) is the non-empty meet of α, β under $\subseteq$.

Proof. Suppose that $\emptyset \neq \gamma \subseteq \alpha, \beta \in R(q)$. Since $R(q) \subseteq PS(q)$, Theorem 3.4.2 implies that (i) and (ii) hold. Since $\operatorname{dom}\gamma \subseteq E \subseteq \operatorname{dom}\alpha$, we have

$$|\operatorname{dom}\alpha \setminus E| \leq |\operatorname{dom}\alpha \setminus \operatorname{dom}\gamma| \leq |X \setminus \operatorname{dom}\gamma| = q.$$

Similarly, $|\operatorname{dom}\beta \setminus E| \leq q$ and hence (iii) holds. Conversely, suppose the conditions hold. By Theorem 3.4.2 again, (i) and (ii) imply that $\gamma = \alpha|E = \beta|E$ is a non-empty element of $PS(q)$ and it is also the meet of α, β in $PS(q)$ under $\subseteq$. Also, since $\operatorname{dom}\gamma = E \subseteq \operatorname{dom}\alpha$ and $g(\alpha) = q$, we have

$$X \setminus \operatorname{dom}\gamma = (X \setminus \operatorname{dom}\alpha) \ \dot{\cup} \ (\operatorname{dom}\alpha \setminus E).$$

Then (iii) implies that $g(\gamma) = q$, hence $\gamma \in R(q)$. ■

From Theorem 3.1.1, we have that $\leq$ equals $\subseteq \cap \mathbb{L}$ on $PS(q)$, where $\mathbb{L}$ is the relation defined on $PS(q)$ by

$$(\alpha, \beta) \in \mathbb{L} \quad \text{if and only if} \quad PS(q)^1\alpha \subseteq PS(q)^1\beta.$$

It is equivalent to say that

$$(\alpha, \beta) \in \mathbb{L} \quad \text{if and only if} \quad \alpha = \lambda\beta \ \text{ for some } \ \lambda \in PS(q)^1.$$

Hence, by using Theorem 2.3.10, we can simplify the relation $\mathbb{L}$ by $(\alpha, \beta) \in \mathbb{L}$ if and only if

$$\alpha = \beta \quad \text{or} \quad X\alpha \subseteq X\beta \ \text{ and } \ q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q).$$

Note that if $\alpha \wedge \beta = \emptyset$ in $PS(q)$ under $\leq$, then $p = q$. In this case, if $x \in E = E(\alpha, \beta)$ and $x\alpha = x\beta = y$ then $x_y \in PS(q)$ and $x_y \subseteq \alpha, \beta$. Also, since

$$|X\alpha \setminus \{y\}| = |\operatorname{dom}\alpha \setminus \{x\}| \quad \text{and} \quad g(\alpha) = |X \setminus \operatorname{dom}\alpha|,$$

we have

$$q = p = \max(g(\alpha), |X\alpha \setminus \{y\}|) \leq \max(g(x_y), q) = p = q.$$

Similarly,

$$q = \max(g(\beta), |X\beta \setminus \{y\}|) \leq \max(g(x_y), q) = q.$$

That is, $x_y \leq \alpha, \beta$, so $x_y \leq \alpha \wedge \beta = \emptyset$, a contradiction. In other words, if $\alpha \wedge \beta = \emptyset$ then $E = \emptyset$ and so $\alpha|E = \beta|E = \emptyset$.

As usual, if $\preceq$ is a partial order on a set S, we say $a, b \in S$ are *non-comparable* if $a \not\preceq b$ and $b \not\preceq a$.

Theorem 3.4.5. *Suppose that $\alpha, \beta \in PS(q)$ are non-comparable under $\leq$ and let $E = E(\alpha, \beta)$. Then $\gamma \leq \alpha, \beta$ for some non-empty $\gamma \in PS(q)$ if and only if there exists a non-empty $Y \subseteq E$ such that*

(i) $\max(|X\alpha \setminus Y\alpha|, |X\beta \setminus Y\beta|) \leq q$, *and*

(ii) $q \leq \max(g(\alpha), |X\alpha \setminus Y\alpha|)$ *and* $q \leq \max(g(\beta), |X\beta \setminus Y\beta|)$.

In this event, $\gamma = \alpha|Y = \beta|Y$.

Proof. Suppose that $\emptyset \neq \gamma \leq \alpha, \beta$ and let $Y = \operatorname{dom}\gamma$. Then $\gamma \subseteq \alpha, \beta$ and so $x\alpha = x\gamma = x\beta$ for all $x \in Y$. That is, $\emptyset \neq Y \subseteq E$ and $X\gamma = Y\gamma = Y\alpha = Y\beta$. Since $d(\gamma) = q$, we see that

$$|X\alpha \setminus Y\alpha| \leq |X \setminus Y\alpha| \leq |X \setminus X\gamma| = q$$

and likewise $|X\beta \setminus Y\beta| \leq q$, so (i) holds. Also, since $\leq$ equals $\subseteq \cap\, \mathbb{L}$, we have $(\gamma, \alpha) \in \mathbb{L}$ and $(\gamma, \beta) \in \mathbb{L}$ and these imply

$$q \leq \max(g(\alpha), |X\alpha \setminus Y\alpha|) \quad \text{and} \quad q \leq \max(g(\beta), |X\beta \setminus Y\beta|),$$

that is (ii) holds.

Conversely, suppose the conditions hold. We write $Y = \{y_i\}$ and $E = Y \dot{\cup} \{e_j\}$ (possibly $J = \emptyset$). We also write

$$\alpha = \begin{pmatrix} y_i & e_j & u_m \\ a_i & a_j & a_m \end{pmatrix}, \quad \beta = \begin{pmatrix} y_i & e_j & v_n \\ a_i & a_j & b_n \end{pmatrix}, \quad \gamma = \begin{pmatrix} y_i \\ a_i \end{pmatrix} \qquad (1)$$

Clearly, $\gamma \subseteq \alpha, \beta$ and thus $g(\alpha) \leq g(\gamma)$ and $g(\beta) \leq g(\gamma)$. By condition (i),

$$|X\alpha \setminus X\gamma| = |J| + |M| = |X\alpha \setminus Y\alpha| \leq q,$$

thus $d(\gamma) = |J| + |M| + d(\alpha) = q$ and so $\gamma \in PS(q)$. These also imply

$$\max(g(\alpha), |X\alpha \setminus X\gamma|) \leq \max(g(\gamma), q).$$

Hence, the above and condition (ii) imply that $(\gamma, \alpha) \in \mathbb{L}$ and similarly $(\gamma, \beta) \in \mathbb{L}$. Thus, we have shown that $\gamma \leq \alpha, \beta$ and, we also see that $\gamma = \alpha|Y = \beta|Y$. ■

Corollary 3.4.6. *Suppose that $\alpha, \beta \in PS(q)$ are non-comparable under $\leq$ and let $E = E(\alpha, \beta)$. Then $\alpha \wedge \beta$ exists in $PS(q)$ under $\leq$ and it is non-empty if and only if E is non-empty and α, β satisfy conditions*

(i) $\max(|X\alpha \setminus E\alpha|, |X\beta \setminus E\beta|) \leq q$, *and*

(ii) $q \leq \max(g(\alpha), |X\alpha \setminus E\alpha|)$ *and* $q \leq \max(g(\beta), |X\beta \setminus E\beta|)$.

In this case $\alpha \wedge \beta = \alpha|E = \beta|E$.

Proof. Suppose that $\alpha \wedge \beta = \gamma \in PS(q)$ and it is non-empty. Then $\gamma \leq \alpha, \beta$. Thus, Theorem 3.4.5 implies that there exists a non-empty $Y = \operatorname{dom}\gamma \subseteq E$ and α and β satisfy (i) and (ii) in Theorem 3.4.5. So we can write α, β as in (1) in Theorem 3.4.5. If $g(\gamma) < q$, then Theorem 3.3.3 implies that γ is maximal under $\leq$ and so $\gamma = \alpha = \beta$, contradicting the supposition. Hence $g(\gamma) \geq q$. If there exists $e_0 \in E \setminus Y$ for some $0 \in J$, we can define $\gamma' \in PS(q)$ by

$$\gamma' = \begin{pmatrix} y_i & e_0 \\ a_i & a_0 \end{pmatrix}.$$

Then $\gamma \subseteq \gamma' \subseteq \alpha, \beta$ and $|X\gamma' \setminus X\gamma| = 1$, and we see that

$$\begin{aligned} g(\gamma) &= |J| + |M| + g(\alpha) \text{ (and this implies } |J| + |M| \geq q \text{ or } g(\alpha) \geq q), \\ g(\gamma') &= |J \setminus \{0\}| + |M| + g(\alpha). \end{aligned}$$

Thus, if $|J|+|M| \geq \aleph_0$ then $g(\gamma) = g(\gamma') \geq g(\alpha)$; and if $|J|+|M| < \aleph_0$ then $\gamma \leq \alpha$ implies $q \leq \max(g(\alpha), |J| + |M|)$, so $g(\alpha) \geq q$ and hence $g(\gamma) = g(\gamma') = g(\alpha)$. Since $q \leq g(\gamma)$, we have $g(\gamma') = g(\gamma) \geq q$ in both cases. Therefore,

$$q \leq g(\gamma') = \max(g(\gamma'), 1) \leq \max(g(\gamma), q),$$

that is, $(\gamma, \gamma') \in \mathbb{L}$. Next, since $\gamma \leq \alpha$, we have $(\gamma, \alpha) \in \mathbb{L}$ and so

$$q \leq \max(g(\alpha), |J| + |M|).$$

This implies

$$q \leq \max(g(\alpha), |J \setminus \{0\}| + |M|).$$

We also recall that $|X\alpha \setminus X\gamma'| \leq |X \setminus X\gamma'| = q$ and $g(\alpha) \leq g(\gamma')$ (since $\gamma' \subseteq \alpha$). Then we have

$$q \leq \max(g(\alpha), |J \setminus \{0\}| + |M|) = \max(g(\alpha), |X\alpha \setminus X\gamma'|) \leq \max(g(\gamma'), q),$$

that is, $(\gamma', \alpha) \in \mathbb{L}$ and likewise we can show $(\gamma', \beta) \in \mathbb{L}$. In other words, we have shown that $\gamma < \gamma' \leq \alpha, \beta$, and this contradicts to $\gamma = \alpha \wedge \beta$. Hence, it follows that $Y = E$, that is, α and β satisfy (i) and (ii).

Conversely, suppose E is non-empty and α and β satisfy (i) and (ii). Then, by Theorem 3.4.5, $\gamma \leq \alpha, \beta$ where $\gamma = \alpha|E = \beta|E \in PS(q)$. Moreover, if $\gamma \leq \gamma' \leq \alpha, \beta$ for some $\gamma' \in PS(q)$ then, $\gamma \subseteq \gamma' \subseteq \alpha, \beta$ and thus $x\gamma' = x\alpha = x\beta$ for all $x \in \operatorname{dom}\gamma'$, so $E = \operatorname{dom}\gamma \subseteq \operatorname{dom}\gamma' \subseteq E$, and it follows that $\gamma = \gamma'$. That is, $\gamma = \alpha \wedge \beta$. ■

In effect, by Theorem 3.3.3, the next result determines when two elements of $PS(q)$, which are maximal under $\leq$, possess a meet under $\leq$.

Corollary 3.4.7. *Suppose that $\alpha, \beta \in PS(q)$ are non-comparable under $\leq$ and let $E = E(\alpha, \beta)$. If $g(\alpha) < q$ and $g(\beta) < q$, then $\alpha \wedge \beta$ exists in $PS(q)$ under $\leq$ if and only if $|X\alpha \setminus E\alpha| = q = |X\beta \setminus E\beta|$.*

Proof. Suppose that $g(\alpha), g(\beta) < q$. If $\alpha \wedge \beta$ exists under $\leq$, then Theorem 3.4.5 (ii) implies that $q \leq |X\alpha \setminus E\alpha|$ which is at most q by Theorem 3.4.5 (i). Thus $|X\alpha \setminus E\alpha| = q$ and likewise $g(\beta) < q$ implies $|X\beta \setminus E\beta| = q$.

Conversely, if $|X\alpha \setminus E\alpha| = q = |X\beta \setminus E\beta|$ then both (i) and (ii) in Theorem 3.4.5 hold for $E = E(\alpha, \beta)$, so $\alpha \wedge \beta$ exists. ■

Example 3.4.8. Suppose that $X = M \dot{\cup} N \dot{\cup} \{b, c\}$, where $|M| = p, |N| = q$ and

$$\alpha = \begin{pmatrix} M \cup N & b \\ M & b \end{pmatrix}, \quad \beta = \begin{pmatrix} M \cup N & c \\ M & c \end{pmatrix}$$

where $E = E(\alpha, \beta) = M \cup N$. Then $d(\alpha) = q = d(\beta)$, so $\alpha, \beta \in PS(q)$ and $\alpha|E = \beta|E \in PS(q)$. But, $|X\alpha \setminus E\alpha| = 1 = |X\beta \setminus E\beta|$ and $g(\alpha) = 1 = g(\beta)$, so E satisfies condition (i) in Theorem 3.4.5 but not condition (ii), and hence $\alpha \wedge \beta$ does not exist in $(PS(q), \leq)$. That is, although $\alpha|E$ may be the greatest lower bound under $\subseteq$, that may not be true for $\leq$ since $\leq \neq \subseteq$ on $PS(q)$.

Theorem 3.4.9. *Let $\alpha, \beta \in I(X)$ under $\subseteq$. Then $\alpha, \beta \subseteq \gamma$ for some $\gamma \in I(X)$ if and only if*

(i) $\operatorname{dom}\alpha \cap \operatorname{dom}\beta \subseteq E(\alpha, \beta)$ *and*

(ii) $(\operatorname{dom}\alpha \setminus \operatorname{dom}\beta)\alpha \cap (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta = \emptyset$.

Moreover, in this case, $\alpha \vee \beta$ exists and equals $\alpha \cup \beta$.

Proof. Suppose that $\alpha, \beta \subseteq \gamma \in I(X)$. If $x \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$ then $x\alpha = x\gamma = x\beta$, and so $x \in E(\alpha, \beta)$. On the other hand, if there exist $y \in \operatorname{dom}\alpha \setminus \operatorname{dom}\beta$ and $z \in \operatorname{dom}\beta \setminus \operatorname{dom}\alpha$ such that $y\alpha = z\beta$, then $y\gamma = z\gamma$. Since γ is injective, this implies that $y = z$, a contradiction.

Conversely, suppose that the conditions hold and let $\gamma = \alpha \cup \beta$ (as sets). Then (i) says that γ is a mapping and (ii) says it is injective, so $\gamma \in I(X)$ and clearly it is an upper bound of $\{\alpha, \beta\}$. Moreover, if (i) and (ii) hold, then $\gamma = \alpha \vee \beta$, since $\alpha, \beta \subseteq \lambda \in I(X)$ implies $\alpha, \beta \subseteq \alpha \cup \beta \subseteq \lambda$ (as sets) where $\alpha \cup \beta \in I(X)$. ■

Like before, the result for joins in $PS(q)$ under $\subseteq$ involves an extra condition.

Theorem 3.4.10. *Let $\alpha, \beta \in PS(q)$ under $\subseteq$. Then $\alpha, \beta \subseteq \gamma$ for some $\gamma \in PS(q)$ if and only if the following conditions hold.*

(i) $\operatorname{dom}\alpha \cap \operatorname{dom}\beta \subseteq E(\alpha, \beta)$,

(ii) $(\operatorname{dom}\alpha \setminus \operatorname{dom}\beta)\alpha \cap (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta = \emptyset$, *and*

(iii) $|X \setminus (X\alpha \cup X\beta)| = q$.

Moreover, in this case, $\alpha \vee \beta$ exists and equals $\alpha \cup \beta$.

Proof. Suppose that $\alpha, \beta \subseteq \gamma$ in $PS(q)$. Then, conditions (i) and (ii) hold since $PS(q) \subseteq I(X)$. Since $X\alpha \cup X\beta \subseteq X\gamma$, we also have

$$q = |X \setminus X\gamma| \leq |X \setminus (X\alpha \cup X\beta)| \leq |X \setminus X\alpha| = q.$$

Hence (iii) holds. Conversely, suppose (i), (ii) and (iii) hold and let $\gamma = \alpha \cup \beta$. Then (i) and (ii) imply that $\gamma \in I(X)$, and (iii) implies that

$$d(\gamma) = |X \setminus X\gamma| = |X \setminus (X\alpha \cup X\beta)| = q,$$

that is, $\gamma \in PS(q)$. Since $\gamma = \alpha \cup \beta$, it follows that $\alpha, \beta \subseteq \gamma$. Finally, as in Theorem 3.4.9, we can show that $\alpha \vee \beta = \gamma$. ∎

Theorem 3.4.11. *Let $\alpha, \beta \in R(q)$. Then $\alpha, \beta \subseteq \gamma$ for some $\gamma \in R(q)$ if and only if the following conditions hold.*

(i) $\operatorname{dom}\alpha \cap \operatorname{dom}\beta \subseteq E(\alpha, \beta)$,

(ii) $(\operatorname{dom}\alpha \setminus \operatorname{dom}\beta)\alpha \cap (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta = \emptyset$,

(iii) $|X \setminus (X\alpha \cup X\beta)| = q$, *and*

(iv) $|X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)| = q$.

Moreover, when this occurs, $\alpha \cup \beta$ is the join of α, β under $\subseteq$.

Proof. Suppose that $\alpha, \beta \subseteq \gamma$ in $R(q)$. Since $R(q) \subseteq PS(q)$, Theorem 3.4.10 implies that (i), (ii) and (iii) hold. Since $\operatorname{dom}\alpha \cup \operatorname{dom}\beta \subseteq \operatorname{dom}\gamma$, we have

$$q = |X \setminus \operatorname{dom}\gamma| \leq |X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)| \leq |X \setminus \operatorname{dom}\alpha| = q.$$

Hence (iv) holds. Conversely, suppose that the conditions hold. By Theorem 3.4.10 again, (i), (ii) and (iii) imply that $\gamma = \alpha \cup \beta$ is an element of $PS(q)$ and it is also a join of α, β under $\subseteq$. Also, (iv) implies that

$$g(\gamma) = |X \setminus \operatorname{dom}\gamma| = |X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)| = q,$$

so $\gamma \in R(q)$. ■

To characterize joins in $PS(q)$ under $\leq$, we need two lemmas. In effect, the first provides a description of $\leq$ in terms of $\subseteq$ which differs from that in Theorem 3.1.1.

Lemma 3.4.12. *Suppose that $\alpha, \beta \in PS(q)$ and $\alpha \neq \beta$. Then $\alpha < \beta$ if and only if $\alpha \subsetneq \beta$ and $g(\alpha) \geq q$.*

Proof. If $\alpha < \beta$, then $\alpha \subsetneq \beta$ and $(\alpha, \beta) \in \mathbb{L}$. Therefore, $\operatorname{dom}\alpha \subsetneq \operatorname{dom}\beta$ and $X\alpha \subseteq X\beta$, and hence

$$X \setminus \operatorname{dom}\alpha = (X \setminus \operatorname{dom}\beta) \,\dot{\cup}\, (\operatorname{dom}\beta \setminus \operatorname{dom}\alpha), \text{ and} \tag{2}$$

$$X\beta = [(\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta] \,\dot{\cup}\, [(\operatorname{dom}\alpha)\beta].$$

Now, $(\operatorname{dom}\alpha)\beta = (\operatorname{dom}\alpha)\alpha = X\alpha$ (since $\alpha \subsetneq \beta$) and so

$$|X\beta \setminus X\alpha| = |(\operatorname{dom}\beta \setminus \operatorname{dom}\alpha)\beta| = |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha|. \tag{3}$$

By Theorem 2.3.10, we also know that

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q).$$

Hence, if $\max(g(\beta), |X\beta \setminus X\alpha|) = g(\beta)$, then $q \leq g(\beta) \leq g(\alpha)$ by (2); and if $\max(g(\beta), |X\beta \setminus X\alpha|) = |X\beta \setminus X\alpha|$, then

$$q \leq |X\beta \setminus X\alpha| = |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha| \leq |X \setminus \operatorname{dom}\alpha| = g(\alpha)$$

by (3). That is, the conditions hold.

Conversely, suppose that the conditions hold. Then (2) and (3) hold (since $\alpha \subsetneq \beta$), $\max(g(\alpha), q) = g(\alpha) \geq g(\beta)$ and $|X\beta \setminus X\alpha| \leq |X \setminus X\alpha| = d(\alpha) = q$. Since $g(\alpha) \geq q$, (2) implies that $|X \setminus \operatorname{dom}\beta| \geq q$ or $|\operatorname{dom}\beta \setminus \operatorname{dom}\alpha| \geq q$. By this result together with (3), we deduce that

$$g(\beta) = |X \setminus \operatorname{dom}\beta| \geq q \;\text{ or }\; |X\beta \setminus X\alpha| = |\operatorname{dom}\beta \setminus \operatorname{dom}\alpha| \geq q.$$

Consequently,

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q),$$

and so $(\alpha, \beta) \in \mathbb{L}$. By Theorem 3.1.1, it follows that $\alpha < \beta$. ■

Lemma 3.4.13. *Suppose that* $\alpha, \beta \in PS(q)$ *are non-comparable under* $\leq$. *Then* $\alpha, \beta \leq \gamma$ *for some* $\gamma \in PS(q)$ *if and only if*

(i) $\alpha, \beta \subseteq \theta$ *for some* $\theta \in PS(q)$, *and*

(ii) $g(\alpha) \geq q$ *and* $g(\beta) \geq q$.

Proof. If $\alpha, \beta \leq \gamma \in PS(q)$, then $\alpha, \beta \subseteq \gamma$, so (i) holds. In addition, if $g(\alpha) < q$, then α is maximal under $\leq$ (by Theorem 3.3.3). Hence $\alpha \leq \gamma$ implies $\alpha = \gamma$ and so $\beta \leq \alpha$, contradicting the supposition. Therefore, $g(\alpha) \geq q$ and likewise $g(\beta) \geq q$. That is, (ii) holds.

Conversely, suppose that (i) and (ii) hold. Then (i) and Theorem 3.4.10 imply that $\pi = \alpha \cup \beta \in PS(q)$ is the join of $\{\alpha, \beta\}$ under $\subseteq$. So, if $\alpha = \pi$, then $\beta \subsetneq \alpha$ (since they are non-comparable). Thus, (ii) and Lemma 3.4.12 imply $\beta < \alpha$, which contradicts the supposition. Therefore, $\alpha \subsetneq \pi$ and $g(\alpha) \geq q$, so $\alpha < \pi$ by Lemma 3.4.12 again. Similarly, $\beta < \pi$ and so α, β have an upper bound in $PS(q)$ under $\leq$. ■

Example 3.4.14. Surprisingly, (i) and (ii) in Lemma 3.4.13 do not ensure that $\alpha \cup \beta$ equals $\alpha \vee \beta$ in $PS(q)$ under $\leq$. For example, write $X = A \dot\cup B \dot\cup C \dot\cup D \dot\cup \{a\}$ where $|A| = p = |X|$ and $|B| = |C| = |D| = q$. Let

$$\alpha = \begin{pmatrix} A \cup B \\ A \end{pmatrix} \cup \mathrm{id}_C, \quad \beta = \begin{pmatrix} A \cup B \\ A \end{pmatrix} \cup \mathrm{id}_D$$

where $x\alpha = x\beta$ for all $x \in A \cup B$. Then $\alpha, \beta \in PS(q)$ and they are non-comparable under $\leq$ (since $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$). If $\theta = \alpha \cup \beta$, then $\alpha, \beta \subseteq \theta \in PS(q)$ (since $d(\theta) = |B| = q$), hence α and β satisfy (i). Also, $g(\alpha) = |D| = q = |C| = g(\beta)$, and hence α and β satisfy (ii). By Lemma 3.4.12, $\alpha, \beta < \theta' = \theta \cup \mathrm{id}_{\{a\}} \in PS(q)$, but $\theta \not\leq \theta'$ since $g(\theta) = 1 \not\geq q$, and thus $\alpha \cup \beta$ does not equal $\alpha \vee \beta$.

Theorem 3.4.15. *Suppose that* $\alpha, \beta \in PS(q)$ *are non-comparable under* $\leq$. *Then* $\alpha \vee \beta$ *exists if and only if*

(i) $\alpha, \beta < \theta$ *for some* $\theta \in PS(q)$, *and*

(ii) *either* $X = \operatorname{dom}\alpha \cup \operatorname{dom}\beta$ *or* $|X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)| \geq q$.

Moreover, when this occurs, $\alpha \vee \beta$ *equals* $\alpha \cup \beta$.

Proof. Suppose that $\alpha \vee \beta$ exists under $\leq$ and write $\gamma = \alpha \vee \beta$. Then $\alpha, \beta < \gamma$ (since α and β are non-comparable), so (i) holds. Consequently, $\alpha, \beta \subsetneq \gamma$ and so Theorem 3.4.10 implies that $\pi = \alpha \cup \beta \in PS(q)$ is the join of $\{\alpha, \beta\}$ under $\subseteq$ and this follows that $\pi \subseteq \gamma$. Now, to prove (ii), suppose $\operatorname{dom}\alpha \cup \operatorname{dom}\beta \subsetneq X$. Choose $a \in X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta) = X \setminus \operatorname{dom}\pi$ and, for any $x \in X \setminus X\pi$ (non-empty since $d(\pi) = q$), we let

$$\mu_x = \begin{pmatrix} \operatorname{dom}\pi & a \\ X\pi & x \end{pmatrix}$$

where $\mu_x|\operatorname{dom}\pi = \pi$. Then $\mu_x \in PS(q)$ since $d(\mu_x) = |X \setminus X\pi| = d(\pi) = q$. Clearly, $\alpha \subseteq \mu_x$ and $\alpha \neq \mu_x$ (since $a \in \operatorname{dom}\mu_x \setminus \operatorname{dom}\alpha$). Therefore, using the fact that $\alpha < \gamma$, Lemma 3.4.12 implies that $g(\alpha) \geq q$ and thus $\alpha < \mu_x$ by Lemma 3.4.12 again. Similarly, $\beta < \mu_x$. It follows that $\gamma \leq \mu_x$ for all $x \in X \setminus X\pi$ since $\gamma = \alpha \vee \beta$ under $\leq$. If $\gamma = \mu_x$ for all $x \in X \setminus X\pi$, then $\mu_x = \mu_y$ for all $x \neq y$ in $X \setminus X\pi$, a contradiction. Hence, $\gamma < \mu_z$ for some $z \in X \setminus X\pi$, and so γ is not maximal. Therefore, by Theorem 3.3.3,

$$q \leq g(\gamma) \leq g(\pi) = |X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)|$$

since $\pi \subseteq \gamma$, and so we have proved (ii).

Conversely, suppose the conditions hold. Then (i) implies that $\alpha, \beta \subseteq \theta$ since $\subseteq$ contains $\leq$. Therefore, Lemma 3.4.13 (i) and Theorem 3.4.10 imply (say) $\pi = \alpha \cup \beta \in PS(q)$ is the join of $\{\alpha, \beta\}$ under $\subseteq$ and we claim that it is also the join under $\leq$. In addition, (i) and Lemma 3.4.12 imply that $g(\alpha), g(\beta) \geq q$. Now, if $\pi = \alpha$, then $\beta \subsetneq \alpha$ (since they are non-comparable) and so $\beta < \alpha$ by Lemma 3.4.12, which contradicts the supposition. Thus, $\alpha \subsetneq \pi$ and this follows that $\alpha < \pi$ by Lemma 3.4.12 again. Likewise, we have $\beta < \pi$. Finally, if $\alpha, \beta \leq \mu$ for some $\mu \in PS(q)$, then $\alpha, \beta \subseteq \mu$ and so $\pi \subseteq \mu$. Since (ii) holds, if $X = \operatorname{dom}\alpha \cup \operatorname{dom}\beta$, then $X = \operatorname{dom}\pi$ and so $\pi = \mu$. Otherwise, if $|X \setminus (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)| \geq q$, then

$g(\pi) \geq q$ and so $\pi < \mu$ by Lemma 3.4.12. In other words, π is the join of α and β in $PS(q)$ under $\leq$. ■