Chapter 4

Automorphisms and Isomorphisms of $R(q)$

In [13] Theorem 3, Sullivan showed that Aut $PS(q)$ and $G(X)$ are isomorphic when $p = q$. Later, in [12] Theorem 2, Pinto and Sullivan showed that this is also true when $p > q$. Here, we first consider the problem of describing all automorphisms of

$$R(q) = \{\alpha \in PS(q) : g(\alpha) = q\},$$

the largest regular (and also inverse) subsemigroup of $PS(q)$.

4.1 Automorphisms

From Example 2.4.3, $PS(q)$ is $G(X)$-normal and consequently the same is true for $R(q)$.

Lemma 4.1.1. *$R(q)$ is $G(X)$-normal.*

Proof. Let $h \in G(X)$, $\alpha \in R(q)$. Since $R(q) \subseteq PS(q)$ and $PS(q)$ is $G(X)$-normal, we have $h\alpha h^{-1} \in PS(q)$. To show that $h\alpha h^{-1} \in R(q)$, consider $\operatorname{dom} h\alpha h^{-1} = \operatorname{dom} h\alpha = (Xh \cap \operatorname{dom}\alpha)h^{-1} = (\operatorname{dom}\alpha)h^{-1}$. So

$$g(h\alpha h^{-1}) = |X \setminus (\operatorname{dom}\alpha)h^{-1}| = |(X \setminus \operatorname{dom}\alpha)h^{-1}| = q$$

since $\alpha \in R(q)$. Therefore $h\alpha h^{-1} \in R(q)$, that is, $R(q)$ is $G(X)$-normal. ■

Levi showed in [6] that every automorphism of a $G(X)$-normal subsemigroup of $P(X)$ is inner. Then by lemma 4.1.1, we have φ is inner for all $\varphi \in$ Aut $R(q)$, that is there exists $g \in G(X)$ such that $\alpha\varphi = g\alpha g^{-1}$ for all $\alpha \in R(q)$. The next result gives more details on Aut $R(q)$.

Lemma 4.1.2. *For each* $\varphi \in \text{Aut } R(q)$*, there exists a unique* $\gamma \in G(X)$ *such that* $\alpha\varphi = \gamma^{-1}\alpha\gamma$ *for all* $\alpha \in R(q)$ *and, in this event, we write* $\gamma = \gamma_\varphi$*.*

Proof. Let $\varphi \in \text{Aut } R(q)$. Then φ is inner, so there exists $\gamma \in G(X)$ such that $\alpha\varphi = \gamma^{-1}\alpha\gamma$ for all $\alpha \in R(q)$. Suppose there exists $\mu \in G(X)$ such that $\gamma^{-1}\alpha\gamma = \alpha\varphi = \mu^{-1}\alpha\mu$ for all $\alpha \in R(q)$. Let $x \in X$ and write $X = A \,\dot{\cup}\, B \,\dot{\cup}\, \{x\}$ where $|A| = p$ and $|B| = q$. If $\alpha = \text{id}_A$ and $\beta = \text{id}_{A \dot{\cup} \{x\}}$, then $\alpha, \beta \in R(q)$. This implies that

$$A\gamma = X\gamma^{-1}\alpha\gamma = X\mu^{-1}\alpha\mu = A\mu$$

and

$$(A \,\dot{\cup}\, \{x\})\gamma = X\gamma^{-1}\beta\gamma = X\mu^{-1}\beta\mu = (A \,\dot{\cup}\, \{x\})\mu.$$

Since γ and μ are injective, we have

$$A\gamma \,\dot{\cup}\, \{x\gamma\} = A\mu \,\dot{\cup}\, \{x\mu\}$$

where $A\gamma = A\mu$. Thus $x\gamma = x\mu$ for all $x \in X$, that is, $\gamma = \mu$. ■

The proof of the next result is similar to that for [12] Theorem 2.

Theorem 4.1.3. Aut $R(q)$ *and* $G(X)$ *are isomorphic.*

Proof. Define $\theta : \text{Aut } R(q) \to G(X)$ by $\varphi \mapsto \gamma_\varphi$ the unique permutation on X such that $\alpha\varphi = \gamma_\varphi^{-1}\alpha\gamma_\varphi$ for all $\alpha \in R(q)$ (possible by Lemma 4.1.2). To show θ is a homomorphism, let $\varphi, \psi \in \text{Aut } R(q)$. Then for all $\alpha \in R(q)$, we have

$$\alpha(\varphi\psi) = \alpha(\varphi)\psi = (\gamma_\varphi^{-1}\alpha\gamma_\varphi)\psi = \gamma_\psi^{-1}(\gamma_\varphi^{-1}\alpha\gamma_\varphi)\gamma_\psi = (\gamma_\varphi\gamma_\psi)^{-1}\alpha(\gamma_\varphi\gamma_\psi).$$

Thus $\gamma_{\varphi\psi} = \gamma_\varphi\gamma_\psi$ by the uniqueness of $\gamma_{\varphi\psi}$ (Lemma 4.1.2). So θ is a homomorphism. To show θ is surjective, let $\lambda \in G(X)$ and define

$$\varphi : R(q) \to R(q) \;\text{ by }\; \alpha \mapsto \lambda^{-1}\alpha\lambda.$$

Since $R(q)$ is $G(X)$-normal, we have φ is a well-defined automorphism of $R(q)$. Thus $\gamma_\varphi = \lambda$, so $(\varphi)\theta = \gamma_\varphi = \lambda$, that is, θ is onto. Finally, if $\gamma_\varphi = \gamma_\psi$, then $\alpha\varphi = \gamma_\varphi^{-1}\alpha\gamma_\varphi = \gamma_\psi^{-1}\alpha\gamma_\psi = \alpha\psi$ for all $\alpha \in R(q)$, that is, $\varphi = \psi$ and therefore θ is one-to-one. ■

4.2 Isomorphisms

In what follows, we sometimes write $R(X,p,q)$ in place of $R(q)$ to highlight the underlying set X and its cardinal p. Since $R(X,p,q)$ played an important role in both [4] and [12], it is natural to ask whether any of the semigroups $R(X,p,q)$ are isomorphic for different cardinals p and q. To answer this question, we first need a result for $R(q)$ which corresponds to [12] Lemma 1 for $PS(q)$.

Lemma 4.2.1. *If $\alpha, \beta \in R(q)$ then the following are equivalent.*

(i) $X\alpha \subseteq X\beta$,

(ii) *for each $\gamma \in R(q)$, $\beta\gamma = \beta$ implies $\alpha\gamma = \alpha$.*

Proof. If $X\alpha \subseteq X\beta$ and $\beta\gamma = \beta$ for some $\gamma \in R(q)$, then $X\alpha \subseteq X\beta \subseteq \operatorname{dom}\gamma$ and $\gamma|X\beta = \mathrm{id}_{X\beta}$. Hence $(x\alpha)\gamma = x\alpha$ for each $x\alpha \in X\alpha \subseteq X\beta$, so $\alpha\gamma = \alpha$.

Conversely, suppose there exists $y = x\alpha \notin X\beta = B$ say. Then $\mathrm{id}_B \in R(q)$ and $\beta \circ \mathrm{id}_B = \beta$ but $y\,\mathrm{id}_B \neq y$; that is, $\alpha \circ \mathrm{id}_B \neq \alpha$ and hence the condition does not hold. ∎

Corollary 4.2.2. *Suppose that $|X| = p \geq q \geq \aleph_0$ and $|Y| = r \geq s \geq \aleph_0$. If $\varphi : R(X,p,q) \to R(Y,r,s)$ is an isomorphism then, for each $\alpha, \beta \in R(X,p,q)$, $X\alpha \subseteq X\beta$ if and only if $Y(\alpha\varphi) \subseteq Y(\beta\varphi)$.*

Proof. Suppose that $\alpha, \beta \in R(X,p,q)$. Then, since φ is an isomorphism, Lemma 4.2.1 provides the following equivalences.

$$\begin{aligned} X\alpha \subseteq X\beta &\iff \text{for each } \gamma \in R(X,p,q),\ \beta\gamma = \beta \text{ implies } \alpha\gamma = \alpha, \\ &\iff \text{for each } \gamma \in R(X,p,q),\ \beta\varphi.\gamma\varphi = \beta\varphi \text{ implies } \alpha\varphi.\gamma\varphi = \alpha\varphi, \\ &\iff \text{for each } \gamma' \in R(Y,r,s),\ \beta\varphi.\gamma' = \beta\varphi \text{ implies } \alpha\varphi.\gamma' = \alpha\varphi, \\ &\iff Y(\alpha\varphi) \subseteq Y(\beta\varphi). \end{aligned}$$

Therefore we have proved the corollary. ∎

Theorem 4.2.3. *The semigroups* $R(X,p,q)$ *and* $R(Y,r,s)$ *are isomorphic if and only if* $p=r$ *and* $q=s$. *Moreover, for each isomorphism* $\varphi : R(X,p,q) \to R(Y,r,s)$, *there is a bijection* $\gamma : X \to Y$ *such that* $\alpha\varphi = \gamma^{-1}\alpha\gamma$ *for each* $\alpha \in R(X,p,q)$.

Proof. We assume that there is an isomorphism $\varphi : R(X,p,q) \to R(Y,r,s)$ and write

$$U = \{X\alpha : \alpha \in R(X,p,q)\}, \quad V = \{Y\beta : \beta \in R(Y,r,s)\}.$$

Let $\Gamma : U \to V$ be defined by $(X\alpha)\Gamma = Y(\alpha\varphi)$. Then, by Corollary 4.2.2, Γ is an order-monomorphism: that is, Γ is injective and $A \subseteq B$ if and only if $A\Gamma \subseteq B\Gamma$ for all $A, B \in U$. Next, if $C = Y\beta$ for some $\beta \in R(Y,r,s)$, then $\beta = \alpha\varphi$ for some $\alpha \in R(X,p,q)$ (since φ is onto). Thus $(X\alpha)\Gamma = Y(\alpha\varphi) = Y\beta = C$, so Γ is onto. In fact, if

$$\mathcal{B}(X,q) = \{A \subseteq X : |X \setminus A| = q\}, \quad \mathcal{B}(Y,s) = \{B \subseteq Y : |Y \setminus B| = s\}$$

then $U = \mathcal{B}(X,q)$ and $V = \mathcal{B}(Y,s)$, since $\mathrm{id}_A \in R(X,p,q)$ and $\mathrm{id}_B \in R(Y,r,s)$ for all $A \in \mathcal{B}(X,q)$ and $B \in \mathcal{B}(Y,s)$. That is, Γ is an order-isomorphism from $\mathcal{B}(X,q)$ onto $\mathcal{B}(Y,s)$. Thus by Lemma 2.4.8, there exists a bijection $\gamma : X \to Y$ such that $A\Gamma = A\gamma$ for all $A \in \mathcal{B}(X,q)$, so $p = r$. Now we aim to show $\alpha\varphi = \gamma^{-1}\alpha\gamma$ for each $\alpha \in R(X,p,q)$. Clearly this holds if $\alpha = \emptyset$ (in this case, $p = q$). So suppose $\alpha \neq \emptyset$ and note that $\mathrm{dom}\,\alpha\gamma = \mathrm{dom}\,\alpha$ since $\mathrm{dom}\,\gamma = X$. Let $x \in \mathrm{dom}\,\alpha$ and write $X = C \,\dot\cup\, D \,\dot\cup\, \{x\}$ where $|C| = p$ and $|D| = q$. Then $\beta = \mathrm{id}_C, \lambda = \mathrm{id}_{C\cup\{x\}} \in R(X,p,q)$. Let $A = X\beta$ and $B = X\lambda$, we have $A, B \in \mathcal{B}(X,q)$, $B \setminus A = X\lambda \setminus X\beta = \{x\}$ and

$$\begin{aligned}
Y((\lambda\alpha)\varphi) \setminus Y((\beta\alpha)\varphi) &= Y((\lambda\varphi)(\alpha\varphi)) \setminus Y((\beta\varphi)(\alpha\varphi)) \\
&= (Y(\lambda\varphi) \setminus Y(\beta\varphi))(\alpha\varphi) \\
&= ((X\lambda)\Gamma \setminus (X\beta)\Gamma)(\alpha\varphi) \\
&= (B\Gamma \setminus A\Gamma)(\alpha\varphi) \\
&= (B\gamma \setminus A\gamma)(\alpha\varphi) \\
&= (B \setminus A)\gamma(\alpha\varphi) \\
&= \{x\}\gamma(\alpha\varphi)
\end{aligned}$$

On the other hand, we see that

$$\begin{aligned} Y((\lambda\alpha)\varphi) \setminus Y((\beta\alpha)\varphi) &= (X(\lambda\alpha))\Gamma \setminus (X(\beta\alpha))\Gamma \\ &= (X(\lambda\alpha))\gamma \setminus (X(\beta\alpha))\gamma \\ &= (X\lambda \setminus X\beta)\alpha\gamma \\ &= \{x\}\alpha\gamma. \end{aligned}$$

Thus $\{x\}\gamma(\alpha\varphi) = \{x\}\alpha\gamma$ for all $x \in \operatorname{dom}\alpha$. We now aim to show that $\operatorname{dom}(\gamma(\alpha\varphi)) = \operatorname{dom}(\alpha\gamma)$. To do this, we first note that, $\alpha \in R(X, p, q)$ implies $\alpha^{-1} \in R(X, p, q)$ since $d(\alpha^{-1}) = g(\alpha) = q$ and $g(\alpha^{-1}) = d(\alpha) = q$. It follows that $\operatorname{dom}\alpha = X\alpha^{-1} \in U = \mathcal{B}(X, q)$. Also, since φ is an isomorphism and $R(X, p, q)$ and $R(Y, r, s)$ are inverse semigroups, we have $(\alpha\varphi)^{-1} = \alpha^{-1}\varphi$. Thus, $(\operatorname{dom}\alpha)\gamma = (X\alpha^{-1})\gamma = (X\alpha^{-1})\Gamma = Y(\alpha^{-1}\varphi) = \operatorname{dom}(\alpha\varphi)$, that is, $\operatorname{dom}\alpha = (\operatorname{dom}(\alpha\varphi))\gamma^{-1}$. Consequently, together with $\operatorname{dom}\gamma = X$ and $X\gamma = Y$, we have

$$\operatorname{dom}(\alpha\gamma) = \operatorname{dom}\alpha = (\operatorname{dom}(\alpha\varphi))\gamma^{-1} = (X\gamma \cap \operatorname{dom}(\alpha\varphi))\gamma^{-1} = \operatorname{dom}(\gamma(\alpha\varphi)).$$

Therefore $\gamma(\alpha\varphi) = \alpha\gamma$ and so $\alpha\varphi = \gamma^{-1}\alpha\gamma$. Finally, since $\alpha\varphi \in R(Y, r, s)$, we have $s = |Y \setminus Y(\alpha\varphi)| = |Y \setminus Y\gamma^{-1}\alpha\gamma| = |X\gamma \setminus X\alpha\gamma| = |(X \setminus X\alpha)\gamma| = q$.

Conversely, if $p = r$ and $q = s$, then Theorem 2.4.7 implies that, there is an isomorphism $\varphi : PS(X, p, q) \to PS(Y, r, s)$. Recall that every elements in $R(X, p, q)$ and $R(Y, r, s)$ are all regular, thus, when restrict φ to $R(X, p, q)$ we have $(R(X, p, q))\varphi = R(Y, r, s)$ since φ and φ^{-1} preserve the regularity. Therefore $R(X, p, q)$ and $R(Y, r, s)$ are isomorphic. ■