Chapter 5

Maximal Subsemigroups of the Baer-Levi Semigroups of Partial Transformations

In this chapter, we study maximal subsemigroups of $PS(q)$. In particular, in Section 5.1 we give necessary and sufficient conditions for the existence of maximal subsemigroups of $PS(q)$ when $p > q$. We also determine some maximal subsemigroups of a subsemigroup S_r of $PS(q)$ defined by

$$S_r = \{\alpha \in PS(q) : g(\alpha) \leq r\},$$

where $q \leq r \leq p$. Moreover, we extend M_A, a maximal subsemigroup of $BL(q)$ (see Section 2.3.2) to determine maximal subsemigroups of $PS(q)$. In Section 5.2, we determine some maximal subsemigroups of $PS(q)$ when $p = q$.

5.1 Maximal Subsemigroups of $PS(q)$ when $p > q$

The characterisation of maximal subsemigroups of a given semigroup is a natural topic to consider when studying its structure. Sometimes, it is difficult to describe all of them (see [2] and [9], for examples), but for a semigroup with some special properties, we can easily describe some of its maximal subsemigroups.

In what follows, if S is a semigroup and M is a subsemigroup of S, we sometimes use Lemma 2.1.4 to show the maximality of M in S. Also, recall that an ideal I of S is a prime ideal if its complement in S is a subsemigroup of S.

Lemma 5.1.1. *Suppose that S is a semigroup and I is a prime ideal of S. Let $T = S \setminus I$. Then,*

(i) *for a maximal subsemigroup M of T, $M \cup I$ is a maximal subsemigroup of S;*

(ii) *for a maximal subsemigroup N of S such that $T \setminus N \neq \emptyset$ and $T \cap N \neq \emptyset$, the set $T \cap N$ is a maximal subsemigroup of T.*

Proof. To see that (i) holds, let M be a maximal subsemigroup of T. Since I is an ideal, we have $M \cup I$ is a subsemigroup of S. Clearly, $M \cup I \subsetneq T \cup I = S$. If $a \in S \setminus (M \cup I)$, then $a \in T \setminus M$ and thus $T = \langle M \cup \{a\} \rangle \subseteq \langle M \cup I \cup \{a\} \rangle$. Since $\langle M \cup I \cup \{a\} \rangle$ contains I, we have $S = T \cup I = \langle M \cup I \cup \{a\} \rangle$ and so $M \cup I$ is maximal in S as required.

To prove (ii), let N be a maximal subsemigroup of S such that $T \setminus N \neq \emptyset$ and $T \cap N \neq \emptyset$, and let $a \in T \setminus N$. Since N is maximal in S, we have $\langle N \cup \{a\} \rangle = S$. Thus, for each $b \in T \setminus N$, $b = c_1 c_2 \ldots c_n$ for some natural n and some $c_i \in N \cup \{a\}$ for all $i = 1, 2, \ldots, n$. Since $b \notin N$, we have $c_i = a$ for some i. Moreover, since $b \notin I$, we have $c_j \in T \cap N$ for all $j \neq i$. It follows that $T \setminus N \subseteq \langle (T \cap N) \cup \{a\} \rangle$, therefore

$$T = (T \setminus N) \cup (T \cap N) \subseteq \langle (T \cap N) \cup \{a\} \rangle,$$

that is, $T = \langle (T \cap N) \cup \{a\} \rangle$ and thus $T \cap N$ is maximal in T. ■

From [12] p 95, for $\aleph_0 \leq k \leq p$, the authors showed that

$$S_k = \{\alpha \in PS(q) : g(\alpha) \leq k\}$$

is a subsemigroup of $PS(q)$. Also, when $p > q$, the proper ideals of $PS(q)$ are precisely the sets:

$$T_s = \{\alpha \in PS(q) : g(\alpha) \geq s\}$$

where $q < s \leq p$ (see Theorem 2.3.14). Thus, for any $q \leq r < p$, it is clear that

$$PS(q) = S_r \,\dot{\cup}\, T_{r'},$$

that is, $PS(q)$ can be written as a disjoint union of the semigroup S_r and the ideal $T_{r'}$. In other words, $T_{r'}$ is a prime ideal of $PS(q)$. Hence, the next result follows directly from Lemma 5.1.1(i).

Corollary 5.1.2. *Suppose that $p > r > q \geq \aleph_0$. If M is a maximal subsemigroup of S_r, then $M \cup T_{r'}$ is a maximal subsemigroup of $PS(q)$.*

Proof. Let M be a maximal subsemigroup of S_r. Since $PS(q) = S_r \dot{\cup} T_{r'}$ and $T_{r'}$ is a proper prime ideal of $PS(q)$, we have $M \cup T_{r'}$ is a maximal subsemigroup of $PS(q)$ by Lemma 5.1.1(i). ■

Lemma 5.1.3. *Let $p > q \geq \aleph_0$ and suppose that M is a maximal subsemigroup of $PS(q)$. Then,*

(i) *$S_r \cap M \neq \emptyset$ for all $q \leq r < p$;*

(ii) *if there exists $\alpha \notin M$ with $g(\alpha) < p$, then $S_k \setminus M \neq \emptyset$ for some $q \leq k < p$.*

Proof. To show that (i) holds, we first note that S_q is contained in S_r for all $q \leq r < p$. If $S_q \cap M = \emptyset$, then $M \subseteq T_{q'} \subsetneq PS(q)$ and thus $M = T_{q'}$ by the maximality of M. But $T_{q'} \subsetneq T_{q'} \cup BL(q) \subsetneq PS(q)$ where $T_{q'} \cup BL(q)$ is a subsemigroup of $PS(q)$ (since $T_{q'}$ is an ideal), this contradicts the maximality of $T_{q'}$. Therefore, $\emptyset \neq S_q \cap M \subseteq S_r \cap M$ for all $q \leq r < p$.

To show that (ii) holds, suppose there is $\alpha \notin M$ with $g(\alpha) = k < p$. If $k < q$, then $\alpha \in S_r \setminus M$ for all $q \leq r \leq p$. Otherwise, if $q \leq k$, then $\alpha \in S_k \setminus M$. Hence (ii) holds. ■

In what follows, for any cardinal $r \leq p$, we let

$$G_r = \{\alpha \in PS(q) : g(\alpha) = r\}.$$

Then $G_0 = BL(q)$ and $G_q = R(q)$. Moreover, if $p > q$ and $r > q$, then $G_r = S_r \cap T_r$, and so G_r is a subsemigroup of S_r (since it is the intersection of two semigroups). Also, G_r is bi-simple and idempotent-free, when $p > q$ and $r > q$ (see Corollary 2.3.16).

From Theorem 2.3.17, if $p \geq q$, then $S_q = \alpha.R(q)$ for each $\alpha \in BL(q)$, and by Theorem 2.3.18, $S_q = BL(q).\mu.BL(q)$ for each $\mu \in R(q)$ when $p \neq q$. This motivates the following result.

Lemma 5.1.4. *Suppose that $p \geq r > q \geq \aleph_0$. Then $G_r = BL(q).\alpha.BL(q)$ for each $\alpha \in G_r$.*

Proof. Let $\alpha \in G_r$ and $\beta, \gamma \in BL(q)$. Since

$$X \setminus \operatorname{dom}\alpha = [X\beta \cap (X \setminus \operatorname{dom}\alpha)] \mathbin{\dot\cup} [(X \setminus X\beta) \cap (X \setminus \operatorname{dom}\alpha)]$$

where $g(\alpha) = |X \setminus \operatorname{dom}\alpha| = r > q$ and the second intersection on the right has cardinal at most q (since $|X \setminus X\beta| = q$), we have $|X\beta \cap (X \setminus \operatorname{dom}\alpha)| = r$. This means that

$$\begin{aligned} r &= |[X\beta \cap (X \setminus \operatorname{dom}\alpha)]\beta^{-1}| \\ &= |(X\beta \setminus \operatorname{dom}\alpha)\beta^{-1}| \\ &= |\operatorname{dom}\beta \setminus \operatorname{dom}(\beta\alpha)| \\ &= |X \setminus \operatorname{dom}(\beta\alpha)|, \end{aligned}$$

that is $g(\beta\alpha) = r$. Since $\operatorname{dom}\gamma = X$, we have $\operatorname{dom}(\beta\alpha\gamma) = \operatorname{dom}(\beta\alpha)$, and so $g(\beta\alpha\gamma) = g(\beta\alpha) = r$. Hence $\beta\alpha\gamma \in G_r$ and therefore $BL(q).\alpha.BL(q) \subseteq G_r$.

For the converse, if $\alpha, \beta \in G_r$, then $|X \setminus \operatorname{dom}\alpha| = r = |X \setminus \operatorname{dom}\beta|$. Since $p > q$, every element in $PS(q)$ has rank p, so we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \ \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix} \ \text{ where } \ |I| = p.$$

Now write $X \setminus \{y_i\} = A \mathbin{\dot\cup} B$ and $X \setminus \{a_i\} = C \mathbin{\dot\cup} D$ where $|A| = |B| = |C| = q$ and $|D| = r$ (note that this is possible since $d(\beta) = q \geq \aleph_0$ and $g(\alpha) = r > q \geq \aleph_0$). Define

$$\delta = \begin{pmatrix} b_i & X \setminus \{b_i\} \\ a_i & D \end{pmatrix}, \epsilon = \begin{pmatrix} x_i & X \setminus \{x_i\} \\ y_i & A \end{pmatrix}$$

where $\delta|(X \setminus \{b_i\})$ and $\epsilon|(X \setminus \{x_i\})$ are bijections. Then $\delta, \epsilon \in BL(q)$ and $\beta = \delta\alpha\epsilon$, that is, $G_r \subseteq BL(q).\alpha.BL(q)$ and equality follows. ∎

Now we can describe all maximal subsemigroups of $PS(q)$ when $p > q$.

Theorem 5.1.5. *Suppose that $p > q \geq \aleph_0$. Then M is a maximal subsemigroup of $PS(q)$ if and only if M equals one of the following sets:*

(i) $PS(q) \setminus G_p = \{\alpha \in PS(q) : g(\alpha) < p\}$;

(ii) $N \cup T_{r'}$, *where $q \leq r < p$ and N is a maximal subsemigroup of S_r.*

Proof. Let $\alpha, \beta \in PS(q)$ be such that $g(\alpha) < p$ and $g(\beta) < p$. Since

$$|X\alpha \setminus \operatorname{dom}\beta| \leq |X \setminus \operatorname{dom}\beta| = g(\beta) < p,$$

we have

$$\begin{aligned} |\operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha\beta)| &= |[X\alpha \setminus (X\alpha \cap \operatorname{dom}\beta)]\alpha^{-1}| \\ &= |(X\alpha \setminus \operatorname{dom}\beta)\alpha^{-1}| \\ &= |X\alpha \setminus \operatorname{dom}\beta| < p. \end{aligned}$$

Hence,

$$|X \setminus \operatorname{dom}(\alpha\beta)| = |X \setminus \operatorname{dom}\alpha| + |\operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha\beta)| < p,$$

and this shows that $PS(q) \setminus G_p$ is a subsemigroup of $PS(q)$. To show that $PS(q) \setminus G_p$ is maximal in $PS(q)$, we let $\alpha, \beta \in PS(q) \setminus (PS(q) \setminus G_p) = G_p$. By Lemma 5.1.4, $\alpha = \lambda\beta\mu$ for some $\lambda, \mu \in BL(q) \subseteq PS(q) \setminus G_p$. Thus, α can be written as a finite product of elements in $(PS(q) \setminus G_p) \cup \{\beta\}$, and hence $PS(q) \setminus G_p$ is maximal in $PS(q)$ by Lemma 2.1.4. Also, if $q \leq r < p$ and N is a maximal subsemigroup of S_r, then $N \cup T_{r'}$ is maximal in $PS(q)$ by Corollary 5.1.2.

Finally, suppose that M is a maximal subsemigroup of $PS(q)$ such that $M \neq PS(q) \setminus G_p$. Then there exists $\alpha \notin M$ with $g(\alpha) < p$. Thus, Lemma 5.1.3 implies that $S_k \setminus M \neq \emptyset$ and $S_k \cap M \neq \emptyset$ for some k, where $q \leq k < p$. Since $T_{k'}$ is a proper prime ideal of $PS(q)$ and $S_k = PS(q) \setminus T_{k'}$, Lemma 5.1.1(ii) implies that $S_k \cap M$ is maximal in S_k. Since $PS(q) = S_k \cup T_{k'}$, we see that

$$M = (S_k \cap M) \cup (T_{k'} \cap M) \subseteq (S_k \cap M) \cup T_{k'},$$

where $(S_k \cap M) \cup T_{k'}$ is maximal in $PS(q)$ by Corollary 5.1.2. This means that $M = (S_k \cap M) \cup T_{k'}$ by the maximality of M. ■

By the previous theorem, when $p > q$, most of the maximal subsemigroups of $PS(q)$ are induced by maximal subsemigroups of S_r where $q \leq r < p$. Hence we now determine some maximal subsemigroups of S_r.

As mentioned in Section 2.3.2, for every non-empty subset A of X such that $|X \setminus A| \geq q$, M_A is a maximal subsemigroup of $BL(q)$. Here we extend the definition of M_A and consider the set $\overline{M}_A$ defined as

$$\overline{M}_A = \{\alpha \in PS(q) : A \nsubseteq X\alpha \text{ or } (A\alpha \subseteq A \subseteq \operatorname{dom}\alpha \text{ or } |X\alpha \setminus A| < q)\},$$

that is, α in $PS(q)$ belongs to $\overline{M}_A$ if and only if

(i) $A \nsubseteq X\alpha$, or

(ii) $A \subseteq X\alpha$ and either $A\alpha \subseteq A \subseteq \operatorname{dom}\alpha$, or $|X\alpha \setminus A| < q$.

The next result gives more details on $\overline{M}_A$.

Lemma 5.1.6. *Suppose that $p \geq q \geq \aleph_0$ and let A be a non-empty subset of X such that $|X \setminus A| \geq q$. Then,*

(i) *for any cardinal k such that $0 \leq k \leq p$, there exist $\alpha, \beta \in PS(q)$ such that $g(\alpha) = k = g(\beta)$ and $\alpha \in \overline{M}_A, \beta \notin \overline{M}_A$;*

(ii) *for each $\gamma \notin \overline{M}_A$, $|\operatorname{dom}\gamma \setminus A\gamma^{-1}| = |X \setminus A| = |X\gamma \setminus A|$ and $|A\gamma^{-1}| = |A|$.*

Proof. To show that (i) holds, let $|X \setminus A| = r \geq q$ and let k be a cardinal such that $0 \leq k \leq p$. We write $X \setminus A = R \dot\cup Q$ where $|R| = r$ and $|Q| = q$. If $r = p$, then $|A \cup R| \geq r = p$; if $r < p$, then $|X \setminus A| < p$, and this implies $|A| = p$, and so $|A \cup R| = p$. Fix $a \in A$ and let $B = (A \setminus \{a\}) \cup R$. Then, $|B| = p$ and $|X \setminus B| = |Q \cup \{a\}| = q$. We write $X = K \dot\cup L$ where $|K| = k$ and $|L| = p$. Then there exists a bijection $\alpha : L \to B$ and so $g(\alpha) = k$, $d(\alpha) = q$. Also, since $A \nsubseteq B = X\alpha$, we have $\alpha \in \overline{M}_A$.

To find $\beta \in PS(q) \setminus \overline{M}_A$ with $g(\beta) = k$, we consider two cases. First, if $r = p$, we write $X \setminus A = P \dot\cup Q \dot\cup K$ where $|P| = p, |Q| = q, |K| = k$. Fix $a \in A$ and define

$$\beta = \begin{pmatrix} P \cup Q \cup \{a\} & A \setminus \{a\} \\ P \cup K \cup \{a\} & A \setminus \{a\} \end{pmatrix}$$

where $\beta|(P\cup Q\cup\{a\})$ and $\beta|(A\setminus\{a\})$ are bijections and $a\beta\neq a$. On the other hand, if $r<p$, then $|A|=p$. In this case we write $A=A'\dot{\cup}K'$ and $X\setminus A=R\dot{\cup}Q$ where $|A'|=p, |K'|=k, |R|=r$ and $|Q|=q$. Fix $a\in A'$ and re-define

$$\beta=\begin{pmatrix}(X\setminus A)\cup\{a\} & A'\setminus\{a\}\\ R\cup\{a\} & A\setminus\{a\}\end{pmatrix}$$

where $\beta|((X\setminus A)\cup\{a\})$ and $\beta|(A'\setminus\{a\})$ are bijections and $a\beta\neq a$. In both cases, we have $d(\beta)=q$, $g(\beta)=k$, $A\subseteq X\beta$, $A\beta\nsubseteq A$ and $|X\beta\setminus A|\geq q$, that is $\beta\in PS(q)\setminus\overline{M}_A$.

To see that (ii) holds, suppose that there is $\gamma\notin\overline{M}_A$, then $A\subseteq X\gamma$ and $|X\gamma\setminus A|\geq q$. So $|A\gamma^{-1}|=|A|$ since γ is injective. Also,

$$X\setminus A=(X\setminus X\gamma)\dot{\cup}(X\gamma\setminus A)$$

where $|X\setminus X\gamma|=q$. Since $|X\setminus A|\geq q$ and $|X\gamma\setminus A|\geq q$, we have $|X\setminus A|=|X\gamma\setminus A|=|(X\gamma\setminus A)\gamma^{-1}|=|\operatorname{dom}\gamma\setminus A\gamma^{-1}|$ as required. ■

In [9] Theorem 1, the authors proved that M_A is a maximal subsemigroup of $BL(q)$ for every non-empty subset A of X such that $|X\setminus A|\geq q$. Using a similar argument, we can show that $\overline{M}_A$ is a subsemigroup of $PS(q)$.

Lemma 5.1.7. *Suppose that $p\geq q\geq\aleph_0$ and let A be a non-empty subset of X such that $|X\setminus A|\geq q$. Then $\overline{M}_A$ is a proper subsemigroup of $PS(q)$.*

Proof. Let $\alpha,\beta\in\overline{M}_A$. If $A\nsubseteq X\alpha\beta$, then $\alpha\beta\in\overline{M}_A$. Now we suppose that $A\subseteq X\alpha\beta$ and this implies $A\subseteq X\beta$. Since $\beta\in\overline{M}_A$, we either have

$$A\beta\subseteq A\subseteq\operatorname{dom}\beta\ \text{ or }\ |X\beta\setminus A|<q.$$

If $|X\beta\setminus A|<q$, then $|X\alpha\beta\setminus A|\leq|X\beta\setminus A|<q$ and so $\alpha\beta\in\overline{M}_A$. If $A\beta\subseteq A\subseteq\operatorname{dom}\beta$, then we have $A\beta\subseteq A\subseteq X\alpha\beta$ and hence $A\subseteq X\alpha$ since β is injective. Since $\alpha\in\overline{M}_A$, we either have $A\alpha\subseteq A\subseteq\operatorname{dom}\alpha$, or $|X\alpha\setminus A|<q$. If the latter occurs, then

$$|X\alpha\beta\setminus A|\leq|X\alpha\beta\setminus A\beta|=|(X\alpha\setminus A)\beta|\leq|X\alpha\setminus A|<q,$$

therefore $\alpha\beta \in \overline{M}_A$. On the other hand, if $A\alpha \subseteq A \subseteq \operatorname{dom}\alpha$, we have $A\alpha\beta \subseteq A\beta \subseteq A$. Moreover, $A\alpha \subseteq X\alpha \cap \operatorname{dom}\beta$, that is, $A \subseteq (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1} = \operatorname{dom}\alpha\beta$. It follows that $A\alpha\beta \subseteq A \subseteq \operatorname{dom}\alpha\beta$. Therefore $\alpha\beta \in \overline{M}_A$, and hence $\overline{M}_A$ is a subsemigroup of $PS(q)$. Finally, this subsemigroup is properly contained in $PS(q)$ by Lemma 5.1.6(i). ■

Remark 5.1.8. By Lemma 5.1.6(i), for any cardinal r such that $q \leq r \leq p$, $S_r \cap \overline{M}_A$ is always non-empty and properly contained in S_r. Therefore, it is a proper subsemigroup of S_r, but it is not maximal when $q < r$. To see this, suppose that $S_r \cap \overline{M}_A$ is maximal and choose $\alpha, \beta \notin \overline{M}_A$ such that $g(\alpha) = r$ and $g(\beta) = 0$ (possible by Lemma 5.1.6(i)). Then $\alpha, \beta \in S_r \setminus \overline{M}_A$ where $\operatorname{dom}\beta = X$. Moreover $\langle (S_r \cap \overline{M}_A) \cup \{\alpha\} \rangle = S_r$, and so

$$\beta = \gamma_1\gamma_2 \ldots \gamma_n \alpha \lambda_1 \lambda_2 \ldots \lambda_m$$

for some $n, m \in \mathbb{N}_0$ and $\gamma_i, \lambda_j \in (S_r \cap \overline{M}_A) \cup \{\alpha\}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$. If $n = 0$ or $\gamma_1 = \alpha$, then $\operatorname{dom}\beta \subseteq \operatorname{dom}\alpha$ and so $g(\alpha) = 0$, a contradiction. Thus, $n \neq 0$ and $\gamma_1 \neq \alpha$. Since $X = \operatorname{dom}\beta \subseteq \operatorname{dom}(\gamma_1\gamma_2 \ldots \gamma_n)$, it follows that $\gamma = \gamma_1\gamma_2 \ldots \gamma_n \in BL(q)$. Moreover, $X\gamma \subseteq \operatorname{dom}\alpha$, and this implies,

$$q \leq r = |X \setminus \operatorname{dom}\alpha| \leq |X \setminus X\gamma| = q,$$

and hence $r = q$.

Since M_A is maximal in $BL(q)$, a subsemigroup of $PS(q)$, it is natural to think that $\overline{M}_A$ is maximal in $PS(q)$. But when $p > q$, by taking $r = p$, the above observation shows that this claim is false since $S_p = PS(q)$. Thus, $\overline{M}_A$ is not always a maximal subsemigroup of $PS(q)$ when $p > q$.

The proof of the next result follows some ideas from [9] Theorem 1.

Theorem 5.1.9. *Suppose that* $p \geq r \geq q \geq \aleph_0$ *and let* A *be a non-empty subset of* X *such that* $|X \setminus A| \geq q$. *Then* $S_r \cap \overline{M}_A$ *is a maximal subsemigroup of* S_r *precisely when* $r = q$.

Proof. In Remark 5.1.8, we have shown that $S_r \cap \overline{M}_A$ is not maximal in S_r when $r > q$. It remains to show $S_q \cap \overline{M}_A$ is maximal in S_q. Let $\alpha, \beta \in S_q \setminus \overline{M}_A$. Then $g(\alpha) \leq q, g(\beta) \leq q$ and Lemma 5.1.6(ii) implies that

$$|A\alpha^{-1}| = |A| = |A\beta^{-1}|, \text{ and}$$

$$|\operatorname{dom}\alpha \setminus A\alpha^{-1}| = |\operatorname{dom}\beta \setminus A\beta^{-1}| = |X\beta \setminus A| = |X\alpha \setminus A| = |X \setminus A| = s \text{ (say)} \geq q.$$

We also have $A\beta \nsubseteq A$ or $A \nsubseteq \operatorname{dom}\beta$. In the case that $A\beta \nsubseteq A$, we have $A\beta \cap (X \setminus A) \neq \emptyset$. Thus, there exists $y \in A \cap (X \setminus A)\beta^{-1}$, so $y \notin A\beta^{-1}$. Since $|\operatorname{dom}\beta \setminus (A\beta^{-1} \cup \{y\})| = s$, we can write

$$\operatorname{dom}\beta \setminus (A\beta^{-1} \cup \{y\}) = \{c_j\} \,\dot\cup\, \{d_k\}$$

where $|J| = s$ and $|K| = q$. Also, since $\alpha, \beta \notin \overline{M}_A$, we have $A \subseteq X\alpha$ and $A \subseteq X\beta$. Thus, for convenience, write $A = \{a_i\}$, let $y_i, z_i \in X$ be such that $y_i\alpha = a_i = z_i\beta$ for each i, and let $\operatorname{dom}\alpha \setminus A\alpha^{-1} = \{b_j\}$. Hence, we can write

$$\beta = \begin{pmatrix} z_i & c_j & d_k & y \\ a_i & c_j\beta & d_k\beta & y\beta \end{pmatrix}.$$

Now define $\gamma \in P(X)$ by

$$\gamma = \begin{pmatrix} y_i & b_j \\ z_i & c_j \end{pmatrix}.$$

Then, $d(\gamma) = |X \setminus (\{z_i\} \cup \{c_j\})| = |\{d_k\} \cup \{y\} \cup (X \setminus \operatorname{dom}\beta)| = |\{d_k\} \cup \{y\}| + g(\beta) = q$, that is, $\gamma \in PS(q)$. Also, since $\operatorname{dom}\gamma = \operatorname{dom}\alpha$, we have $g(\gamma) = g(\alpha) \leq q$ and so $\gamma \in S_q$. Moreover, since $y \in A$ and $y \notin X\gamma$, we have $A \nsubseteq X\gamma$, that is, $\gamma \in \overline{M}_A$. Also, since $d(\alpha) = q$, we can write $X \setminus X\alpha = \{m_k\} \,\dot\cup\, \{n_k\} \,\dot\cup\, \{z\}$ and define μ in $P(X)$ by

$$\mu = \begin{pmatrix} a_i & c_j\beta & d_k\beta & y\beta \\ a_i & b_j\alpha & m_k & z \end{pmatrix}.$$

Then $d(\mu) = |X \setminus (\{a_i\} \cup \{b_j\alpha\} \cup \{m_k\} \cup \{z\})| = |X \setminus (X\alpha \cup \{m_k\} \cup \{z\})| = |\{n_k\}| = q = d(\beta) = g(\mu)$, that is, $\mu \in S_q$. Moreover, $\mu \in \overline{M}_A$ since $A\mu = A \subseteq \operatorname{dom}\mu$. Finally, we can see that $\alpha = \gamma\beta\mu$ where $\gamma, \mu \in S_q \cap \overline{M}_A$.

On the other hand, if $A \nsubseteq \operatorname{dom}\beta$, then there exists $w \in A \setminus \operatorname{dom}\beta$. In this case, we rewrite

$$\operatorname{dom}\beta \setminus A\beta^{-1} = \{c_j\} \,\dot\cup\, \{d_k\} \text{ and } X \setminus X\alpha = \{m_k\} \,\dot\cup\, \{n_k\}$$

where $|J| = s, |K| = q$. Like before, we write $A = \{a_i\}$ and $\operatorname{dom}\alpha = \{y_i\} \,\dot\cup\, \{b_j\}$ where $y_i\alpha = a_i = z_i\beta$ and $\{b_j\} = \operatorname{dom}\alpha \setminus A\alpha^{-1}$, then

$$\beta = \begin{pmatrix} z_i & c_j & d_k \\ a_i & c_j\beta & d_k\beta \end{pmatrix}.$$

Define $\gamma, \mu \in P(X)$ by

$$\gamma = \begin{pmatrix} y_i & b_j \\ z_i & c_j \end{pmatrix},\ \mu = \begin{pmatrix} a_i & c_j\beta & d_k\beta \\ a_i & b_j\alpha & m_k \end{pmatrix}.$$

Then, $d(\gamma) = |\{d_k\}| + g(\beta) = q$, $g(\gamma) = g(\alpha) \leq q$, $d(\mu) = |\{n_k\}| = q = d(\beta) = g(\mu)$, and so $\gamma, \mu \in S_q$. Also, $\gamma, \mu \in \overline{M}_A$ since $A \nsubseteq X\gamma$ (note that $w \in A \setminus \operatorname{dom}\beta \subseteq A \setminus X\gamma$) and $A\mu = A \subseteq \operatorname{dom}\mu$. Moreover, $\alpha = \gamma\beta\mu$. In other words, we have shown that for every $\alpha, \beta \in S_q \setminus \overline{M}_A$, α can be written as a finite product of elements in $(S_q \cap \overline{M}_A) \cup \{\beta\}$. Therefore, $S_q \cap \overline{M}_A$ is maximal in S_q. ■

We now determine some other classes of maximal subsemigroups of S_r.

Lemma 5.1.10. *Suppose that $p \geq r \geq q \geq \aleph_0$. Let k be a cardinal such that $k = 0$ or $q \leq k \leq r$. Then*

$$S_r \setminus G_k = \{\alpha \in PS(q) : k \neq g(\alpha) \leq r\}$$

is a proper subsemigroup of S_r.

Proof. Since $k \leq r$, we have $S_r \setminus G_k \subsetneq S_r$. If $k = 0$, then $S_r \setminus G_0 = S_r \setminus BL(q)$ and this is a subsemigroup of S_r since, for $\alpha, \beta \in S_r \setminus BL(q)$, $\operatorname{dom}(\alpha\beta) \subseteq \operatorname{dom}\alpha \subsetneq X$, and this implies $\alpha\beta \in S_r \setminus BL(q)$. Now suppose $q \leq k \leq r$ and let $\alpha, \beta \in S_r$ be such that $g(\alpha\beta) = k$. We claim that $g(\alpha) = k$ or $g(\beta) = k$. To see this, assume that $g(\alpha) \neq k$. Since

$$k = |X \setminus \operatorname{dom}(\alpha\beta)| = |X \setminus \operatorname{dom}\alpha| + |\operatorname{dom}\alpha \setminus \operatorname{dom}(\alpha\beta)|,$$

we have $|X \setminus \operatorname{dom}\alpha| < k$, thus

$$k = |\operatorname{dom}\alpha \backslash \operatorname{dom}(\alpha\beta)| = |[X\alpha \backslash (X\alpha \cap \operatorname{dom}\beta)]\alpha^{-1}| = |(X\alpha \backslash \operatorname{dom}\beta)\alpha^{-1}| = |X\alpha \backslash \operatorname{dom}\beta|.$$

Note that

$$X \setminus \operatorname{dom}\beta = [X\alpha \setminus \operatorname{dom}\beta] \,\dot{\cup}\, [(X \setminus X\alpha) \cap (X \setminus \operatorname{dom}\beta)]$$

where the intersection on the right has cardinal at most q. Hence

$$g(\beta) = |X \setminus \operatorname{dom}\beta| = k$$

since $k \geq q$, and we have shown that $S_r \setminus G_k$ is a subsemigroup of S_r. ■

Remark 5.1.11. Observe that, if $0 < k < q$ then $S_r \setminus G_k$ is not a semigroup for all $q \leq r \leq p$. To see this, let $\alpha \in BL(q)$ and $\beta = \operatorname{id}_{X\alpha \setminus K}$ for some subset K of $X\alpha$ such that $|K| = k$ (possible since $|X\alpha| = p > k$), then $\alpha, \beta \in PS(q)$ since $d(\beta) = d(\alpha) + k = q$. Moreover, since $g(\alpha) = 0$ and $g(\beta) = q \neq k$, we have $\alpha, \beta \in S_r \setminus G_k$. But

$$\operatorname{dom}(\alpha\beta) = (X\alpha \cap \operatorname{dom}\beta)\alpha^{-1} = (X\alpha \setminus K)\alpha^{-1} = X \setminus K\alpha^{-1},$$

thus $g(\alpha\beta) = |K\alpha^{-1}| = k$, that is, $\alpha\beta \in G_k$.

Theorem 5.1.12. *Suppose that $p \geq r \geq q \geq \aleph_0$. Then the following statements hold:*

(i) *$S_r \setminus G_0$ is a maximal subsemigroup of S_r;*

(ii) *if $p > q$, then for each cardinal k such that $q \leq k \leq r$, $S_r \setminus G_k$ is a maximal subsemigroup of S_r.*

Proof. By Lemma 5.1.10, $S_r \setminus G_0$ is a subsemigroup of S_r. To see that it is maximal in S_r, let $\alpha, \beta \in S_r \setminus (S_r \setminus G_0) = G_0 = BL(q) \subseteq S_q$. By Theorem 2.3.17, $S_q = \beta.R(q)$, and this implies that $\alpha = \beta\gamma$ for some $\gamma \in R(q) \subseteq S_r \setminus G_0$. Hence $S_r \setminus G_0$ is maximal in S_r.

Now suppose that $p > q$ and $q \leq k \leq r$. Let $\alpha, \beta \in S_r \setminus (S_r \setminus G_k) = G_k$. If $k = q$, then $G_k = R(q) \subseteq S_q$ and, by Theorem 2.3.18, $S_q = BL(q).\beta.BL(q)$. If $k > q$, then $G_k = BL(q).\beta.BL(q)$ (by Lemma 5.1.4). In both cases, we have $\alpha = \gamma\beta\mu$ for some $\gamma, \mu \in BL(q) \subseteq S_r \setminus G_k$, and so $S_r \setminus G_k$ is maximal in S_r. ■

Corollary 5.1.13. *Suppose that $p > q \geq \aleph_0$ and let A be a non-empty subset of X such that $|X \setminus A| \geq q$. Then the following sets are maximal subsemigroups of $PS(q)$:*

(i) $\overline{M}_A \cup \ T_{q'}$;

(ii) $N_k = \{\alpha \in PS(q) : g(\alpha) \neq k\}$ *where* $k = 0$ *or* $q \leq k \leq p$.

Proof. By Theorem 5.1.9, $S_q \cap \overline{M}_A$ is maximal in S_q. Then Corollary 5.1.2 implies that $(S_q \cap \overline{M}_A) \cup T_{q'}$ is maximal in $PS(q)$. But

$$(S_q \cap \overline{M}_A) \cup T_{q'} = (S_q \cup T_{q'}) \cap (\overline{M}_A \cup \ T_{q'}) = PS(q) \cap (\overline{M}_A \cup \ T_{q'}) = \overline{M}_A \cup \ T_{q'},$$

and so (i) holds. To show that (ii) holds, let $r = p$ in Theorem 5.1.12. Then $S_p = PS(q)$ and thus $N_k = S_p \setminus G_k$ is maximal in $PS(q)$. ■

Corollary 5.1.14. *Suppose that $p > q \geq \aleph_0$ and k equals 0 or q. Let A be a non-empty subset of X such that $|X \setminus A| \geq q$. Then the two classes of maximal subsemigroups $S_q \cap \overline{M}_A$ and $S_q \setminus G_k$ of S_q are always disjoint.*

Proof. By Theorem 5.1.9 and Theorem 5.1.12, $S_q \cap \overline{M}_A$ and $S_q \setminus G_k$ are maximal subsemigroups of S_q. By Lemma 5.1.6(i), there exists $\alpha \in \overline{M}_A$ with $g(\alpha) = k$. Then $\alpha \in S_k \cap \overline{M}_A \subseteq S_q \cap \overline{M}_A$ but $\alpha \notin S_q \setminus G_k$, that is, $S_q \cap \overline{M}_A \nsubseteq S_q \setminus G_k$. Also, $S_q \setminus G_k \nsubseteq S_q \cap \overline{M}_A$ by the maximality of $S_q \cap \overline{M}_A$ and $S_q \setminus G_k$. Therefore $S_q \cap \overline{M}_A$ is not equal to $S_q \setminus G_k$. ■

5.2 Maximal Subsemigroups of $PS(q)$ when $p = q$

We first recall that, when $p = q$, the empty transformation $\emptyset$ belongs to $PS(q)$ since $d(\emptyset) = p = q$. In this case, the ideals of $PS(q)$ are precisely the sets:

$$J_r = \{\alpha \in PS(q) : r(\alpha) < r\}$$

where $1 \leq r \leq p'$ (see Theorem 2.3.15). Clearly, $J_{p'} = PS(q)$ and

$$J_p = \{\alpha \in PS(q) : r(\alpha) < p\}$$

is the largest proper ideal. In this case, J_r is not a prime ideal of $PS(q)$. To see this, write $X = A \dot{\cup} B \dot{\cup} C$ where $|A| = p$ and $|B| = r = |C|$. Then $\mathrm{id}_B, \mathrm{id}_C \in PS(q) \setminus J_r$ whereas $\mathrm{id}_B \,.\, \mathrm{id}_C = \emptyset \in J_r$. Hence, unlike what was done in Section 5.1, we cannot use Lemma 5.1.1 to find maximal subsemigroups of $PS(q)$ when $p = q$. In this section, we determine some maximal subsemigroups of $PS(q)$, for $p = q$, using a different approach. We first describe some properties of each maximal subsemigroup in this case.

Lemma 5.2.1. *Suppose that $p = q \geq \aleph_0$ and M is a maximal subsemigroup of $PS(q)$. Then the following statements hold:*

(i) *M contains all $\alpha \in PS(q)$ with $r(\alpha) < p$,*

(ii) *if $R(q) \subseteq M$, then $M \cap BL(q) = \emptyset$.*

Proof. Let $\alpha \in PS(q)$ with $r(\alpha) = k < p$. Then $g(\alpha) = p$ and we write in the usual way

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}.$$

Also, write $X \setminus \{a_i\} = P \dot{\cup} Q$ and $X \setminus \{x_i\} = R \dot{\cup} S$ where $|P| = |Q| = p = |R| = |S|$, and define β, γ in $P(X)$ by

$$\beta = \begin{pmatrix} a_i & P \\ a_i & P \end{pmatrix}, \gamma = \begin{pmatrix} a_i & Q \\ x_i & R \end{pmatrix}$$

where $\beta|P$ and $\gamma|Q$ are bijections. Then $\beta, \gamma \in PS(q)$ and also,

$$\alpha = \beta \cdot \alpha \cdot \mathrm{id}_{X\alpha} \in PS(q).\alpha.PS(q).$$

If $M = M \cup (PS(q).\alpha.PS(q))$, then $\alpha \in M$ and we have completed the proof. Otherwise, $M \subsetneq M \cup (PS(q).\alpha.PS(q))$ where $M \cup (PS(q).\alpha.PS(q))$ is a subsemigroup of $PS(q)$. This means that $M \cup (PS(q).\alpha.PS(q)) = PS(q)$ by the maximality of M. Since all mappings in $PS(q).\alpha.PS(q)$ have rank at most k, it follows that M

contains all mappings with rank greater than k. Therefore $\beta, \gamma \in M$ and thus $\alpha = \beta\gamma \in M$ as required.

To show that (ii) holds, suppose that $R(q) \subseteq M$. If there exists $\alpha \in M \cap BL(q)$, then Theorem 2.3.17 implies that $PS(q) = \alpha.R(q) \subseteq M$ (note that $S_q = PS(q)$ when $p = q$), so $M = PS(q)$, contrary to the maximality of M. Thus $M \cap BL(q) = \emptyset$. ■

Remark 5.2.2. The maximality properties in Lemma 5.2.1 hold for $PS(q)$ precisely when $p = q$. If $p > q$, then every $\alpha \in PS(q)$ has rank p. This contrast with Lemma 5.2.1(i). Also, by Corollary 5.1.13, if $p > q$ and $q < k \leq p$, N_k is a maximal subsemigroup of $PS(q)$ containing $R(q) \cup BL(q)$, this contrast with Lemma 5.2.1(ii).

As in Section 5.1, for any cardinal k, we let

$$N_k = \{\alpha \in PS(q) : g(\alpha) \neq k\}.$$

By Lemma 5.1.10 and Remark 5.1.11, if $p = q$, then N_k is a subsemigroup of $PS(q)$ precisely when $k = 0$ or $k = p$. From Corollary 5.1.13(ii), when $p > q$, N_p is a maximal subsemigroup of $PS(q)$, but when $p = q$, Lemma 5.2.1(i) implies that N_p is not maximal since $\emptyset \notin N_p$. Moreover, Lemma 5.2.1(i) implies that every maximal subsemigroup of $PS(q)$ must contain the largest proper ideal

$$J_p = \{\alpha \in PS(q) : r(\alpha) < p\}.$$

Note that J_p itself is a subsemigroup of $PS(q)$ and it is contained in $R(q)$ since, in case $p = q$, $r(\alpha) < p$ implies $g(\alpha) = p = q$. Moreover, this containment is always proper. For example, write $X = A \dot{\cup} B$ where $|A| = p = |B|$ and let $\alpha : A \to B$ be a bijection, thus $\alpha \in R(q) \setminus J_p$. Hence J_p is not maximal in $PS(q)$.

Theorem 5.2.3. *Suppose that $p = q \geq \aleph_0$ and let A be a non-empty subset of X such that $|X \setminus A| \geq q$. The following are maximal subsemigroups of $PS(q)$:*

(i) $\overline{M}_A$;

(ii) N_0;

(iii) $N_p \cup J_p$.

Proof. If $p = q$, then $S_q = PS(q)$, and so (i) holds by Theorem 5.1.9. Also, by taking $r = p$ in Theorem 5.1.12(i), we see that (ii) holds. To show that (iii) holds, take $r = p = k$ in Lemma 5.1.10, we have $N_p = S_p \setminus G_p$ is a subsemigroup of $PS(q)$. Moreover, $N_p \cup J_p$ is also a subsemigroup of $PS(q)$ since J_p is an ideal. To show the maximality of $N_p \cup J_p$, let $\alpha, \beta \in PS(q) \setminus (N_p \cup J_p)$. Then $g(\alpha) = g(\beta) = p = r(\alpha) = r(\beta)$. Write in the usual way

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \ \beta = \begin{pmatrix} b_i \\ y_i \end{pmatrix}$$

where $|I| = p$, and let

$$X \setminus \{a_i\} = A \dot{\cup} B \ \text{ and } \ X \setminus \{y_i\} = C \dot{\cup} D$$

where $|A| = |B| = |C| = |D| = p$. Then define $\gamma, \mu \in P(X)$ by

$$\gamma = \begin{pmatrix} b_i & X \setminus \{b_i\} \\ a_i & A \end{pmatrix}, \ \mu = \begin{pmatrix} x_i & X \setminus \{x_i\} \\ y_i & C \end{pmatrix}$$

where $\gamma|(X \setminus \{b_i\})$ and $\mu|(X \setminus \{x_i\})$ are bijections. Thus $\gamma, \mu \in PS(q)$ since $d(\gamma) = |B| = p = |D| = d(\mu)$. Moreover $\gamma, \mu \in N_p \cup J_p$ since $g(\gamma) = g(\mu) = 0 < p$. It is clear that $\beta = \gamma\alpha\mu$ and therefore $N_p \cup J_p$ is maximal in $PS(q)$. ■

Remark 5.2.4. When $p = q$, if M is a maximal subsemigroup of $PS(q)$ containing $R(q)$, then

$$M \subseteq (PS(q) \setminus BL(q)) = N_0$$

by Lemma 5.2.1(ii). Thus, $M = N_0$ by the maximality of M. So we conclude that N_0 is the only maximal subsemigroup of $PS(q)$ containing $R(q)$ when $p = q$.