APPENDIX

Proof of Propositions

This section contains the technical details of the proofs. Herein we use a generic constant c which exemplifies any finite positive constant value and may be implicitly changed in various places. Nevertheless, the constant c is always independent of l_1.

Proof of Proposition 3.1: First,

$$\begin{aligned}\Big|\frac{\partial W_2}{\partial x_1}\dot{x}_1\Big| &\leq \Big|\int_{x_2^*}^{x_2}\frac{\partial\big(s^{\frac{1}{m_2}}-x_2^{*\frac{1}{m_2}}\big)^{2m_1-\tau-m_2}}{\partial x_1}ds\dot{x}_1\Big| \\ &\leq c\Big|\int_{x_2^*}^{x_2}\big(s^{\frac{1}{m_2}}-x_2^{*\frac{1}{m_2}}\big)^{m_1-\tau-m_2}ds\Big(\frac{\partial x_2^{*\frac{1}{m_2}}}{\partial x_1}\Big)\dot{x}_1\Big| \\ &\leq c\Big|\Big(x_2^{\frac{1}{m_2}}-x_2^{*\frac{1}{m_2}}\Big)^{2m_1-\tau-m_2-1}(x_2-x_2^*)\Big|\Big|\Big(\frac{\partial x_2^{*\frac{1}{m_2}}}{\partial x_1}\Big)\dot{x}_1\Big| \qquad (6.1)\end{aligned}$$

Using (2.6) with the fact that $m_2 \leq 1$ and $\frac{1}{m_2} > 1$,

$$\begin{aligned}\Big|\Big(x_2^{\frac{1}{m_2}}-x_2^{*\frac{1}{m_2}}\Big)^{2m_1-\tau-m_2-1}(x_2-x_2^*)\Big| &\leq |\xi_2|^{2m_1-\tau-m_2-1}\Big|\Big(x_2^{\frac{1}{m_2}}\Big)^{m_2}-\Big(x_2^{*\frac{1}{m_2}}\Big)^{m_2}\Big| \\ &\leq 2^{1-m_2}|\xi_2|^{2m_1-\tau-m_2-1}|\xi_2|^{m_2} \\ &\leq 2^{1-m_2}|\xi_2|^{m_1-\tau} \qquad (6.2)\end{aligned}$$

By the definition of x_2^*,

$$\begin{aligned}\frac{\partial x_2^{*\frac{1}{m_2}}}{\partial x_1} &= \frac{(\bar{\beta}_1\xi_1)}{\partial x_1} \\ &\leq c|x_1|^{\frac{m_1-m_1}{m_1}} = c. \qquad (6.3)\end{aligned}$$

This, together with (3.1), gives

$$\begin{aligned}
\left|\left(\frac{\partial x_2^{*\frac{1}{m_2}}}{\partial x_1}\right)\dot{x}_1\right| &\leq c\Big||x_2|^p + |\xi_1|^{m_1+\tau}\Big| \\
&\leq c\Big||x_2^{\frac{1}{m_2}}|^{pm_2} + |\xi_1|^{m_1+\tau}\Big| \\
&\leq c\Big|\big||\xi_1| + |\xi_2|\big|^{pm_2} + |\xi_1|^{m_1+\tau}\Big| \\
&\leq c\Big||\xi_1|^{pm_2} + |\xi_2|^{pm_2} + |\xi_1|^{m_1+\tau}\Big| \\
&\leq c\Big(|\xi_1|^{m_1+\tau} + |\xi_2|^{m_1+\tau}\Big). \qquad (6.4)
\end{aligned}$$

Clearly, Proposition 3.1 follows from (6.1) and (6.4) with Lemma 2.4.

Proof of Proposition 3.2: We have, with $q_2 = m_1 - \tau - m_2$

$$v(\hat{z})bz_2^{\frac{q_2}{m_2}}\left(z_2^{\frac{m_1}{m_2}} - (\eta_2 + l_1 z_1)\right) \leq c|v(\hat{z})||z_2|^{\frac{q_2}{m_2}}\left|z_2^{\frac{m_1}{m_2}} - \hat{z}_2^{\frac{m_1}{m_2}}\right| \qquad (6.5)$$

By the homogeneity of v, $|v(\hat{z})| \leq c\|\hat{z}\|_\triangle^{\tau+m_2}$, where $\|\hat{z}\|_\triangle = \left(|\hat{z}_1|^{\frac{2}{m_1}} + |\hat{z}_2|^{\frac{2}{m_2}}\right)^{\frac{1}{2}}$, $\triangle = (m_1, m_2)$. From the definition of the homogeneous norm and Lemma 2.3, we have

$$\begin{aligned}
\|\hat{z}\|_\triangle &= \left(|\hat{z}_1|^{\frac{2}{m_1}} + |\hat{z}_2|^{\frac{2}{m_2}}\right)^{\frac{1}{2}} \\
&\leq |\hat{z}_1|^{\frac{1}{m_1}} + |\hat{z}_2|^{\frac{1}{m_2}} \\
&\leq |\xi_1| + \left|z_2 - e_2^{\frac{m_2}{m_1}}\right|^{\frac{1}{m_2}} \\
&\leq |\xi_1| + c\big(|z_2|^{\frac{1}{m_2}} + |e_2|\big) \\
&\leq |\xi_1| + c\left(\left|\xi_2 + \beta_1^{\frac{1}{pm_2}}\xi_1\right| + |e_2|\right) \\
&\leq c\big(|\xi_1| + |\xi_2| + |e_2|\big). \qquad (6.6)
\end{aligned}$$

So, by Lemma 2.3,

$$|v(\hat{z})| \leq c\big(|\xi_1|^{\tau+m_2} + |\xi_2|^{\tau+m_2} + |e_2|^{\tau+m_2}\big). \qquad (6.7)$$

Using (3.16), we have

$$|z_2|^{\frac{q_2}{m_2}} \leq c\left|\xi_2 + \beta_1^{\frac{1}{pm_2}}\xi_1\right|^{q_2} \leq c\big(|\xi_1|^{q_2} + |\xi_2|^{q_2}\big). \qquad (6.8)$$

Finally, we estimate the last term on the right hand size of (6.5). By Lemma 2.5, rewrite $\hat{z}_2$ into e_2,

$$\begin{aligned}\left|z_2^{\frac{m_1}{m_2}} - \hat{z}_2^{\frac{m_1}{m_2}}\right| &\leq \frac{m_1}{m_2}|z_2 - \hat{z}_2|\left|z_2^{\frac{m_1}{m_2}-1} - \hat{z}_2^{\frac{m_1}{m_2}-1}\right| \\ &\leq c|e_2|^{\frac{m_2}{m_1}}\left(|z_2|^{\frac{m_1-m_2}{m_2}} + \left|z_2 - e_2^{\frac{m_2}{m_1}}\right|^{\frac{m_1-m_2}{m_2}}\right) \\ &\leq c|e_2|^{\frac{m_2}{m_1}}\left(|z_2|^{\frac{m_1-m_2}{m_2}} + |e_2|^{\frac{m_1-m_2}{m_1}}\right) \\ &\leq c|e_2|^{m_2}\left(|\xi_2 - \bar{\beta}_1\xi_1|^{m_1-m_2} + |e_2|^{m_1-m_2}\right) \\ &\leq c|e_2|^{m_2}\left(|\xi_1|^{m_1-m_2} + |\xi_2|^{m_1-m_2} + |e_2|^{m_1-m_2}\right). \qquad (6.9)\end{aligned}$$

Applying (6.7)-(6.9) into (6.5) yields

$$\begin{aligned}v(\hat{z})bz_2^{\frac{q_2}{m_2}}\left(z_2^{\frac{m_1}{m_2}} - \gamma_2\right) &\leq c\left(|\xi_1|^{\tau+m_2} + |\xi_2|^{\tau+m_2} + |e_2|^{\tau+m_2}\right)\left(|\xi_1|^{q_2} + |\xi_2|^{q_2}\right) \\ &\quad \times|e_2|^{m_2}\left(|\xi_1|^{m_1-m_2} + |\xi_2|^{m_1-m_2} + |e_2|^{m_1-m_2}\right) \\ &\leq c\left(|\xi_1|^{\tau+m_2} + |\xi_2|^{\tau+m_2} + |e_2|^{\tau+m_2}\right)\left(|\xi_1|^{m_1-\tau-m_2} + |\xi_2|^{m_1-\tau-m_2}\right) \\ &\quad \times|e_2|^{m_2}\left(|\xi_1|^{m_1-m_2} + |\xi_2|^{m_1-m_2} + |e_2|^{m_1-m_2}\right) \\ &\leq \frac{1}{4}(\xi_1^2 + \xi_2^2) + c_4e_2^2, \qquad (6.10)\end{aligned}$$

for a constant $c_4 \geq 0$. The last relation is obtained by applying Lemma 2.4 to each term in the last inequality above.

Proof of Proposition 3.3: We let $w(\cdot) = v(\cdot)^{\frac{m_1}{\tau+m_2}}$. By (2.6), we have

$$\begin{aligned}|v(\hat{z}) - v^*(z)| &= \left|w(\hat{z})^{\frac{\tau+m_2}{m_1}} - w^*(z)^{\frac{\tau+m_2}{m_1}}\right| \\ &\leq 2^{1-(\tau+m_2)}|w(\hat{z}) - w^*(z)|^{\frac{\tau+m_2}{m_1}}. \qquad (6.11)\end{aligned}$$

Now, because w is at least C^1. Let $\chi_2 = z_2 - \lambda e_2^{\frac{m_2}{m_1}}$. We expand this function as

$$\begin{aligned}|w(\hat{z}) - w^*(z)| &= \left|\int_0^1 \frac{\partial w(X)}{\partial X}\Big|_{X=\chi_2}\frac{\partial \chi_2}{\partial \lambda}d\lambda\right| \\ &\leq |e_2|^{\frac{m_2}{m_1}}\int_0^1\left|\frac{\partial w(X)}{\partial X}\right|\Big|_{X=\chi_2}d\lambda. \qquad (6.12)\end{aligned}$$

By homogeneity of $w(\cdot)$ whose degree is m_1, $\frac{\partial w(X)}{\partial X}\Big|_{X=\chi_2}$ is homogeneous of degree $m_1 - m_2$. Hence,

$$\begin{aligned} \frac{\partial w(X)}{\partial X}\Big|_{X=\chi_2} &\leq c\Big\|z_2 - \lambda e_2^{\frac{m_2}{m_1}}\Big\|_{\triangle}^{m_1-m_2} \\ &= c\Big[\Big(\big|z_2 - \lambda e_2^{\frac{m_2}{m_1}}\big|^{\frac{2}{m_2}}\Big)^{\frac{1}{2}}\Big]^{m_1-m_2} \\ &\leq c\big(|z_2| + |e_2|^{\frac{m_2}{m_1}}\big)^{\frac{m_1-m_2}{m_2}} \\ &\leq c\big(|\xi_1|^{m_2} + |\xi_2|^{m_2} + |e_2|^{\frac{m_2}{m_1}}\big)^{\frac{m_1-m_2}{m_2}} \\ &\leq c\big(|\xi_1|^{m_1-m_2} + |\xi_2|^{m_1-m_2} + |e_2|^{m_1-m_2}\big). \end{aligned} \tag{6.13}$$

From (A.11)-(A.13), we have

$$\begin{aligned} \xi_2^{2m_1-\tau-m_2}\big(v(\hat{z}) - v^*(z)\big) &\leq c\xi_2^{2m_1-\tau-m_2}|w(\hat{z}) - w^*(z)|^{\frac{\tau+m_2}{m_1}} \\ &\leq c\xi_2^{2m_1-\tau-m_2}\Big(|e_2|^{m_2}\big(|\xi_1|^{m_1-m_2} + |\xi_2|^{m_1-m_2} + |e_2|^{m_1-m_2}\big)\Big)^{\tau+m_2} \\ &\leq \frac{1}{4}\big(\xi_1^2 + \xi_2^2\big) + c_5 e_2^2, \end{aligned}$$

for a constant $c_5 \geq 0$.

Proof of Proposition 3.4: First, we will show that the vector field $\dot{\mathcal{Z}} = F(\mathcal{Z})$ in (3.28) is homogenous of degree τ. Consider a vector field with dilation weight is $\triangle = \Big(\underbrace{m_1,\ m_2}_{for\ z_1,\ z_2},\ \underbrace{m_1}_{for\ \eta_2}\Big)$,

$$\begin{aligned} F\big(\triangle_\varepsilon(\mathcal{Z})\big) &= F(\varepsilon^{m_1}z_1, \varepsilon^{m_2}z_2, \varepsilon^{m_1}\eta_2) \\ &= \big(\varepsilon^{pm_2}z_2^p, v(\varepsilon^{m_1}z_1, \varepsilon^{m_1}\eta_2), f_3(\varepsilon^{m_1}z_1, \varepsilon^{m_1}\eta_2)\big)^T \\ &= \big(\varepsilon^{\tau+m_1}z_2^p, -\beta_2(\varepsilon^{m_1}\eta_2 + \beta_1\varepsilon^{m_1}z_1)^{\tau+m_2}, -l_1(\varepsilon^{m_1}\eta_2 + l_1\varepsilon^{m_1}z_1)^{m_2p/m_1}\big)^T \\ &= \big(\varepsilon^{\tau+m_1}z_2^p, -\varepsilon^{\tau+m_1}\beta_2(\eta_2 + \beta_1 z_1)^{\tau+m_2}, -\varepsilon^{m_2p/m_1}l_1(\eta_2 + l_1 z_1)^{m_2p/m_1}\big)^T \\ &= \big(\varepsilon^{\tau+m_1}z_2^p, -\varepsilon^{\tau+m_1}\beta_2(\eta_2 + \beta_1 z_1)^{\tau+m_2}, -\varepsilon^{\tau+m_1}l_1(\eta_2 + l_1 z_1)^{m_2p/m_1}\big)^T \\ &= \big(\varepsilon^{\tau+m_1}z_2^p, \varepsilon^{\tau+m_1}v(\cdot), \varepsilon^{\tau+m_1}f_3(\cdot)\big)^T \\ &= \begin{bmatrix} \varepsilon^{\tau+m_1} & 0 & 0 \\ 0 & \varepsilon^{\tau+m_2} & 0 \\ 0 & 0 & \varepsilon^{\tau+m_1} \end{bmatrix} F(\mathcal{Z}). \end{aligned} \tag{6.14}$$

From, the definition of weighted homogeneity for a vector field in section 2. We can conclude that (3.28) is homogeneous of degree τ. Next, we want to show that the Lyapunov function of (3.15)-(3.18) is homogenous function of degree $2m_1 - \tau$. Consider,

$$
\begin{aligned}
V_2(\varepsilon^{m_1} z_1, \varepsilon^{m_2} z_2) &= V_1(\varepsilon^{m_1} z_1) + W_2(\varepsilon^{m_1} z_1, \varepsilon^{m_2} z_2) \\
&= \int_0^{\varepsilon^{m_1} z_1} s^{m_1-\tau} ds + \int_{z_2^*(\varepsilon^{m_1} z_1)}^{\varepsilon^{m_2} z_2} \left(s^{\frac{1}{m_2}} - \left(z_2^*(\varepsilon^{m_1} z_1) \right)^{\frac{1}{m_2}} \right)^{(2m_1-\tau-m_2)} ds \\
&= \int_0^{\varepsilon^{m_1} z_1} s^{m_1-\tau} ds \\
&\quad + \int_{-\beta_1 \varepsilon^{m_2} z_1^{m_2/m_1}}^{\varepsilon^{m_2} z_2} \left(s^{\frac{1}{m_2}} - \varepsilon \Big(-\beta_1 z_1^{\frac{m_2}{m_1}} \Big)^{\frac{1}{m_2}} \right)^{(2m_1-\tau-m_2)} ds. \qquad (6.15)
\end{aligned}
$$

By the change of variables, $y = \frac{s}{\varepsilon^{m_1}}, \ w = \frac{s}{\varepsilon^{m_2}}$. We have

$$
\begin{aligned}
V_2(\varepsilon^{m_1} z_1, \varepsilon^{m_2} z_2) &= \int_0^{z_1} (\varepsilon^{m_1-\tau} y^{m_1-\tau}) \varepsilon^{m_1} dy \\
&\quad + \int_{-\beta_1 z_1^{m_2/m_1}}^{z_2} \left(\varepsilon w^{\frac{1}{m_2}} - \varepsilon \Big(-\beta_1 z_1^{\frac{m_2}{m_1}} \Big)^{\frac{1}{m_2}} \right)^{(2m_1-\tau-m_2)} \varepsilon^{m_2} dw \\
&= \varepsilon^{2m_1-\tau} \int_0^{z_1} y^{m_1-\tau} dy \\
&\quad + \varepsilon^{2m_1-\tau} \int_{-\beta_1 z_1^{m_2/m_1}}^{z_2} \left(w^{\frac{1}{m_2}} - \Big(-\beta_1 z_1^{\frac{m_2}{m_1}} \Big)^{\frac{1}{m_2}} \right)^{(2m_1-\tau-m_2)} dw \\
&= \varepsilon^{2m_1-\tau} \int_0^{z_1} y^{m_1-\tau} dy \\
&\quad + \varepsilon^{2m_1-\tau} \int_{z_2^*}^{z_2} \left(w^{\frac{1}{m_2}} - z_2^{*\frac{1}{m_2}} \right)^{(2m_1-\tau-m_2)} dw \\
&= \varepsilon^{2m_1-\tau} \left(\int_0^{z_1} y^{m_1-\tau} dy + \int_{z_2^*}^{z_2} \left(w^{\frac{1}{m_2}} - z_2^{*\frac{1}{m_2}} \right)^{(2m_1-\tau-m_2)} dw \right) \\
&= \varepsilon^{2m_1-\tau} V_2(z_1, z_2). \qquad (6.16)
\end{aligned}
$$

and

$$
\begin{aligned}
U_2(\varepsilon^{m_1} z_1, \varepsilon^{m_2} z_2, \varepsilon^{m_1} \eta_2) &= \int_{(\varepsilon^{m_1}\eta_2 + l_1 \varepsilon^{m_1} z_1)^{(m_1-\tau)/m_1}}^{\varepsilon^{(m_1-\tau)} z_2^{(m_1-\tau)/m_2}} \left(s^{\frac{m_1}{m_1-\tau}} - \big(\varepsilon^{m_1} \eta_2 + l_1 \varepsilon^{m_1} z_1 \big) \right) ds \\
&= \int_{\varepsilon^{m_1-\tau} (\eta_2 + l_1 z_1)^{(m_1-\tau)/m_1}}^{\varepsilon^{(m_1-\tau)} z_2^{(m_1-\tau)/m_2}} \left(s^{\frac{m_1}{m_1-\tau}} - \varepsilon^{m_1} \big(\eta_2 + l_1 z_1 \big) \right) ds.
\end{aligned}
$$

Define the change of the variable, $x = \frac{s}{\varepsilon^{(m_1-\tau)}}$. We have,

$$\begin{aligned} U_2(\varepsilon^{m_1}z_1, \varepsilon^{m_2}z_2, \varepsilon^{m_1}\eta_2) &= \int_{(\eta_2+l_1z_1)^{(m_1-\tau)/m_1}}^{z_2^{(m_1-\tau)/m_2}} \Big(\varepsilon^{m_1}x^{\frac{m_1}{m_1-\tau}} - \varepsilon^{m_1}\big(\eta_2 + l_1z_1\big)\Big)\varepsilon^{(m_1-\tau)}dx \\ &= \varepsilon^{(2m_1-\tau)}\int_{(\eta_2+l_1z_1)^{(m_1-\tau)/m_1}}^{z_2^{(m_1-\tau)/m_2}} \Big(x^{\frac{m_1}{m_1-\tau}} - \big(\eta_2 + l_1z_1\big)\Big)dx \\ &= \varepsilon^{(2m_1-\tau)}\int_{\gamma_2^{(m_1-\tau)/m_1}}^{z_2^{(m_1-\tau)/m_2}} \Big(x^{\frac{m_1}{m_1-\tau}} - \gamma_2\Big)dx \\ &= \varepsilon^{(2m_1-\tau)}U_2(z_1, z_2, \eta_2). \end{aligned} \tag{6.17}$$

Since (6.16) and (6.17) are homogenous of degree $2m_1 - \tau$,

$$\begin{aligned} T(\varepsilon^{m_1}z_1, \varepsilon^{m_2}z_2, \varepsilon^{m_1}\eta_2) &= \varepsilon^{2m_1-\tau}V_2(z_1, z_2) + \varepsilon^{2m_1-\tau}U_2(z_1, z_2, \eta_2) \\ &= \varepsilon^{2m_1-\tau}T(z_1, z_2, \eta_2). \end{aligned} \tag{6.18}$$

Clearly, the Lyapunov function $T = V_2 + U_2$ of (3.15)-(3.18) is a homogenous function of degree $2m_1 - \tau$. By Lemma 2.2, we have the vector field $\frac{\partial T}{\partial \mathcal{Z}}$ is also homogeneous of degree $2m_1 - \tau$. Lastly, we use Lemma 2.1 with (6.14) and the previous result. It's obvious that $\dot{T} = \frac{\partial T}{\partial \mathcal{Z}}F(\mathcal{Z})$ is homogeneous of degree $2m_1$. This proof is complete.

VITA

Name	Mr. Teerapap Kuptarat
Date of Birth	November 3, 1987
Institutions Attended	Bachelor of Science (Mathematics), from Chaing Mai University in 2010, Chaing Mai, Thailand.