CHAPTER 2
PRELIMINARIES

In this chapter, we give some basic definitions, notations, lemmas and results which will be used in the later chapters.

Notations: The following notations will be used in this thesis:

$\mathbb{R}^n$ is the n dimensional Euclidean space,

$\mathbb{I}$ is the integer number set,

$\mathbb{R}_{odd} = \{b \in \mathbb{R} | b = \frac{2m+1}{2n+1},\ m,\ n \in \mathbb{I}\}$,

$\mathbb{R}_{even} = \{a \in \mathbb{R} | a = \frac{2m}{2n+1},\ m,\ n \in \mathbb{I}\}$,

$\|x\|$ is the Euclidean norm of vector x,

$\|x\|_{\triangle,p}$ is the homogeneous $p - norm$ of vector x,

$\mathbb{C}^1$ is the set of continuously differentiable function.

Definition 2.1 (Positive Definite Function) A function $f(x) \in \mathbb{R}$ is called positive definite if $f(0) = 0$ and $f(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. It is called positive semi-definite if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Definition 2.2 (Negative Definite Function) A function $f(x) \in \mathbb{R}$ is called negative definite if $f(0) = 0$ and $f(x) < 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. It is called negative semi-definite if $f(x) \leq 0$ for all $x \in \mathbb{R}^n$.

Stability and Lyapunov Theory of Autonomous Systems: [16]

Consider the autonomous system

$$\dot{x} = f(x) \tag{2.1}$$

where $f : D \to \mathbb{R}^n$ is a locally lipschitz mapping from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and $f(0) = 0, \forall x \in D$

Definition 2.3 The equilibrium point $x = 0$ of (2.1) is

- **Stable** if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0$$

- **Unstable** if not stable

- **Asymptotically stable** if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0$$

Theorem 2.1 (Lyapunov theory) Let $x = 0$ be an equilibrium point for (2.1). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

$$V(0) = 0 \text{ and } V(x) > 0, \ \forall x \neq 0$$
$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$
$$\dot{V}(x) < 0, \ \forall x \neq 0$$

then $x = 0$ is globally asymptotically stable.

Consider a nonautonomous system

$$\dot{x} = f(t, x), \qquad t \in \mathbb{R}, \ x \in \mathbb{R}^n \tag{2.2}$$

with $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ being a continuous function of (t, x) and $f(t, 0) = 0, \ \forall t \in \mathbb{R}$. Notice that the system (2.2) might not have a uniquen solution from any initial condition.

Definition 2.4 [17, 18] The trivial solution $x = 0$ of (2.2) is said to be globally strongly stable (GSS) if there are two functions $B : (0, \infty) \to (0, \infty)$ and $T : (0, \infty) \times (0, \infty) \to (0, \infty)$ with B being an increasing function and $lim_{s\to 0} B(s) = 0$, such that $\forall \alpha > 0$ and $\forall \epsilon > 0$, for every solution $x(t)$ of (**??**) defined on $[0, t)$, $0 \leq t < \infty$ with $\|x(0)\| \leq \alpha$, there is a solution $z(t)$ of (2.2) defined on $[0, \infty)$ satisfying

- $z(t) = x(t), \quad \forall t \in [0, t_1)$

- $\|z(t)\| \leq B(\alpha), \quad \forall t \geq 0$

- $\|z(t)\| < \epsilon, \qquad \forall t \geq T(\alpha, \epsilon)$.

Theorem 2.2 (Kurzweil, [18, p. 23]) Suppose there exist a C^1 function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, a continuous function $U_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, 3$, which are positive definite, such that

$$\begin{aligned} U_1(x) \leq V(t,x) \leq U_2(x), \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -U_3(x). \end{aligned} \tag{2.3}$$

Then, the trivial solution $x = 0$ of the system (2.2) is globally strongly stable.

State Feedback Stabilization Problems:

The global state feedback stabilization problem for the system

$$\dot{x} = f(x, u)$$

is the problem of designing a feedback control law

$$u = \gamma(x)$$

such that the origin $x = 0$ is a global asymptotically stable equilibrium point of the closed-loop system

$$\dot{x} = f(x, \gamma(x))$$

The feedback control law $u = \gamma(x)$ is usually called "static feedback" because it is a memory less function of x.

Output Feedback Stabilization Problems:

The global output feedback stabilization problem for the system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

is the problem of designing a static output feedback control law

$$u = \gamma(y)$$

or a dynamic output feedback control law

$$\begin{aligned} u &= \gamma(y, z) \\ \dot{z} &= g(y, z) \end{aligned}$$

such that the origin of the closed loop system is global asymptotically stable.

Weighted Homogeneity: (refer to [2], [11], [13], [14], [15] for details) For fixed coordinates $(x_1, \cdots, x_n)^T \in \mathbb{R}^n$ and real numbers $r_i > 0$, for $i = 1, \cdots, n$,

- the dilation $\triangle_\varepsilon(x)$ is defined by $\triangle_\varepsilon(x) = (\varepsilon^{r_1} x_1, \cdots, \varepsilon^{r_n} x_n)$, $\forall \varepsilon > 0$, with r_i being called as the weights of the coordinates (For simplicity of notation, we define dilation weight $\triangle = (r_1, \cdots, r_n)$).

- a function $V \in C(\mathbb{R}^n, \mathbb{R})$ is said to be homogeneous of degree τ if there is a real number $\tau \in \mathbb{R}$ such that $\forall x \in \mathbb{R}^n \setminus \{0\}, \varepsilon > 0, V(\triangle_\varepsilon(x)) = \varepsilon^\tau V(x_1, \cdots, x_n)$.

- a vector field $f \in C(\mathbb{R}^n, \mathbb{R}^n)$ is said to be homogeneous of degree τ if there is a real number $\tau \in \mathbb{R}$ such that for $i = 1, \cdots, n$ $\forall x \in \mathbb{R}^n \setminus \{0\}, \varepsilon > 0$, $f_i(\triangle_\varepsilon(x)) = \varepsilon^{\tau + r_i} f_i(x_1, \cdots, x_n)$.

- a homogeneous $p - norm$ is defined as $\|x\|_{\triangle,p} = (\sum_{i=1}^n |x_i|^{\frac{p}{r_i}})^{1/p}$, $\forall x \in \mathbb{R}^n$, for a constant $p \geq 1$. For the simplicity, in this thesis, we choose $p = 2$ and write $\|x\|_\triangle$ for $\|x\|_{\triangle,2}$.

Lemma 2.1: Given a dilation weight $\triangle = (r_1, \cdots, r_n)$, suppose V_1 and V_2 are homogenous functions of degree τ_1 and τ_2, respectively. Then $V_1 \cdot V_2$ is also homogeneous with respect to the same dilation weight $\triangle$. Thus, the new homogeneous degree of $V_1 \cdot V_2$ is $\tau_1 + \tau_2$.

Lemma 2.2: Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is a homogenous function of degree τ with respect to the dilation weight $\triangle$. Then the followings hold:

(1) $\partial V / \partial x_i$ is still homogeneous of degree $\tau - r_i$ with r_i being the homogeneous weights of x_i.

(2) There is a constant c such that
$V(x) \leq c \|x\|_\triangle^\tau$.

Moreover, if $V(x)$ is positive definite, $\underline{c}\|x\|_\triangle^\tau \leq V(x)$, for some a positive constant $\underline{c} > 0$.

Useful Inequalities

The next 3 lemmas are used as the implicit tools for adding a power integrator [9], [10], and proved therein.

Lemma 2.3: For $x,\ y \in \mathbb{R},\ p \geq 1$ is a constant, the following inequalities hold:

$$|x+y|^p \leq 2^{p-1}|x^p + y^p|, \tag{2.4}$$

$$(|x|+|y|)^{\frac{1}{p}} \leq |x|^{\frac{1}{p}} + |y|^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(|x|+|y|)^{\frac{1}{p}}. \tag{2.5}$$

If $p \in \mathbb{R}_{odd},\ p \geq 1$ then

$$\begin{aligned} |x-y|^p &\leq 2^{p-1}|x^p - y^p|, \\ |x^{\frac{1}{p}} - y^{\frac{1}{p}}| &\leq 2^{1-\frac{1}{p}}|x-y|^{\frac{1}{p}}. \end{aligned} \tag{2.6}$$

Lemma 2.4: Let c, d be positive constants. Given any positive number $\gamma > 0$, the following inequality holds:

$$|x|^c|y|^d \leq \frac{c}{c+d}\gamma|x|^{c+d} + \frac{d}{c+d}\gamma^{\frac{-c}{d}}|y|^{c+d}. \tag{2.7}$$

Lemma 2.5: Let $p \in \mathbb{R}_{odd},\ p \geq 1$ and $x,\ y$ be real-valued functions. Then, for some constant $c > 0$

$$|x^p - y^p| \leq p|x-y|(x^{p-1} + y^{p-1}) \tag{2.8}$$

$$\leq c|x-y||(x-y)^{p-1} + y^{p-1}| \tag{2.9}$$