CHAPTER 3

MAIN RESULTS

In this section, we will present the construction of the output feedback control law for the system (1.1). The control design consists of three steps. First, in Section 3.1, we assume that all of the states are available and then construct a Lyapunov function and a state feedback control law for (1.1). Followed by showing how to choose an observer for a nominal system of (1.1) based on the state feedback control law in Section 3.2. the so-called nominal system is the system (1.1) without the uncertain function $\phi_i(\cdot)$. Finally, we solve the problem of output feedback of (1.1) by introducing a change of coordinates to scale (1.1) into an appropriated form and applying the output feedback control law of the nominal system to the scaled system.

3.1 STABILIZATION BY HOMOGENEOUS STATE FEEDBACK

In this section, we will design a new method for a state feedback stabilizer for (1.1) under the following assumption:

Assumption 3.1: There is a negative constant τ, satisfying $\frac{-1}{p+1} < \tau \leq 0$ and a positive constant c such that

$$|\phi_1(x,t)| \leq c(|x_1|^{\frac{m_1+\tau}{m_1}}), \tag{3.1}$$

$$|\phi_2(x,t)| \leq c(|x_1|^{\frac{m_2+\tau}{m_1}} + |x_2|^{\frac{m_2+\tau}{m_2}}), \tag{3.2}$$

with

$$m_1 = 1, \quad pm_2 = m_1 + \tau, \quad m_2 + \tau > 0 \tag{3.3}$$

which m_1 and m_2 will always be $\mathbb{R}_{odd}$. For simplicity, we assume $\tau = \frac{-q}{d}$, with positive even integers q and positive odd integers d. Note that $m_1 > m_2 > m_2 + \tau > 0$.

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Lemma 3.1: By Assumption 3.1, there exists a homogeneous state feedback controller such that the nonlinear system (1.1) is globally asymptotically stable.

Proof. The proof itself is a 2 step process which relies on the simultaneous constructions of a C^1 Lyapunov function which is positive definite and proper.

Step 1. We define

$$V_1 = \int_0^{x_1} s^{m_1-\tau} ds.$$

The time derivative of V_1 along the trajectory of (1.1) is

$$\dot{V}_1 = x_1^{m_1-\tau}[x_2^p + \phi_1(x,t)]. \tag{3.4}$$

By Assumption 3.1,

$$\begin{aligned} \dot{V}_1 &\leq x_1^{m_1-\tau} x_2^{*p} + |x|_1^{m_1-\tau} c|x|_1^{\frac{\tau+m_1}{m_1}} + x_1^{m_1-\tau}[x_2^p - x_2^{*p}], \\ \dot{V}_1 &\leq x_1^{m_1-\tau} x_2^{*p} + c|x|_1^2 + x_1^{m_1-\tau}[x_2^p - x_2^{*p}] \end{aligned}$$

Then, the virtual controller x_2^{*p} defined by

$$x_2^{*p} = -(2+c)x_1^{pm_2/m_1} = -\beta_1 x_1^{(\tau+m_1)/m_1}$$

yields

$$\dot{V}_1 \leq -2x_1^2 + x_1^{m_1-\tau}[x_2^p - x_2^{*p}]. \tag{3.5}$$

Step 2. We define the following changes of coordinates:

$$\xi_1 = x_1, \qquad \xi_2 = x_2^{1/m_2} - x_2^{*1/m_2} \tag{3.6}$$

and the Lypunov function $V_2 : \mathbb{R}^2 \to \mathbb{R}$,

$$V_2(x_1, x_2) = V_1(x_1) + W_2(x_1, x_2) \text{ where } W_2 = \int_{x_2^*}^{x_2} \left(s^{\frac{1}{m_2}} - x_2^{*\frac{1}{m_2}}\right)^{(2m_1-\tau-m_2)} ds \tag{3.7}$$

which can be proven to be C^1 using a similar method as in [12]. The derivative of V_2 along the trajectory of (1.1) is

$$\begin{aligned} \dot{V}_2 &= \dot{V}_1 + \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \frac{\partial W_2}{\partial x_2}\dot{x}_2 \\ &= \dot{V}_1 + \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \xi_2^{(2m_1-\tau-m_2)}\dot{x}_2 \\ &\leq -2\xi_1^2 + \xi_1^{m_1-\tau}[x_2^p - x_2^{*p}] + \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \xi_2^{(2m_1-\tau-m_2)}\big(u + \phi_2(x,t)\big) \\ &= -2\xi_1^2 + \xi_1^{m_1-\tau}[x_2^p - x_2^{*p}] + \frac{\partial W_2}{\partial x_1}\dot{x}_1 + \xi_2^{(2m_1-\tau-m_2)}\phi_2(x,t) \\ &\quad + \xi_2^{(2m_1-\tau-m_2)} u \end{aligned} \tag{3.8}$$

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Next, we estimate the terms in the right hand side of (3.8). First, it follows from $pm_2 = 1 + \tau \leq 1$ and Lemma 2.3 in the Preliminaries that

$$\begin{aligned}(x_2^p - x_2^{*p}) &\leq \Big|\big(x_2^{\frac{1}{m_2}}\big)^{pm_2} - \big({x_2^*}^{\frac{1}{m_2}}\big)^{pm_2}\Big| \\ &\leq 2^{1-pm_2}\Big|x_2^{\frac{1}{m_2}} - {x_2^*}^{\frac{1}{m_2}}\Big|^{pm_2} \\ &= 2^{1-pm_2}|\xi_2|^{pm_2} \end{aligned} \tag{3.9}$$

and by Lemma 2.4, it can be seen that for a constant $c_1 > 0$,

$$\begin{aligned}\xi_1^{m_1-\tau}(x_2^p - x_2^{*p}) &\leq |\xi_1|^{(m_1-\tau)}2^{1-pm_2}|\xi_2|^{pm_2} \\ &= |\xi_1|^{(m_1-\tau)}2^{1-pm_2}|\xi_2|^{m_1+\tau} \\ &\leq \left(\frac{m_1-\tau}{2m_1}\right)\left(\frac{2m_1}{3(m_1-\tau)}\right)\xi_1^{2m_1} \\ &\quad +\left(\frac{m_1+\tau}{2m_1}\right)\left(\frac{2m_1}{3(m_1-\tau)}\right)^{-\left(\frac{m_1-\tau}{m_1+\tau}\right)}\left(2^{1-pm_2}\right)^{\frac{2m_1}{m_1+\tau}}\xi_2^{2m_1} \\ &\leq \frac{1}{3}\xi_1^2 + c_1\xi_2^2. \end{aligned} \tag{3.10}$$

Using Lemma 2.3 and the equations (3.6), (3.2) can be rewritten as

$$\begin{aligned}|\phi_2(x,t)| &\leq c\Big(|x_1|^{\frac{m_2+\tau}{m_1}} + |x_2|^{\frac{m_2+\tau}{m_2}}\Big) \\ &= c\Big(|x_1|^{\frac{m_2+\tau}{m_1}} + \big(|x_2|^{\frac{1}{m_2}}\big)^{m_2+\tau}\Big) \\ &= c\Big(|\xi_1|^{m_2+\tau} + \big|\xi_2 + {x_2^*}^{\frac{1}{m_2}}\big|^{m_2+\tau}\Big) \\ &\leq c\Big(|\xi_1|^{m_2+\tau} + \big|\xi_2 + \big((-\beta_1)^{\frac{1}{p}}\xi_1^{m_2}\big)^{\frac{1}{m_2}}\big|^{m_2+\tau}\Big) \\ &\leq c\Big(|\xi_1|^{m_2+\tau} + \big|\xi_2 - \beta_1^{\frac{1}{pm_2}}\xi_1\big|^{m_2+\tau}\Big) \\ &\leq c\Big(|\xi_1|^{m_2+\tau} + 2^{1-(m_2+\tau)}\big(|\xi_2|^{m_2+\tau} + |\beta_1^{\frac{1}{pm_2}}\xi_1|^{m_2+\tau}\big)\Big) \\ &\leq \check{c}_2\big(|\xi_1|^{m_2+\tau} + |\xi_2|^{m_2+\tau}\big) \end{aligned} \tag{3.11}$$

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for a constant $\check{c}_2 \geq 0$. From Lemma 2.4 and (3.11), we have

$$\begin{aligned}
\xi_2^{2m_1-\tau-m_2}\phi_2(\cdot) &\leq |\xi_2|^{2m_1-\tau-m_2}\check{c}_2\big(|\xi_1|^{m_2+\tau}+|\xi_2|^{m_2+\tau}\big) \\
&= \check{c}_2|\xi_2|^{2m_1-\tau-m_2}|\xi_1|^{m_2+\tau}+\check{c}_2|\xi_2|^{2m_1-\tau-m_2}|\xi_2|^{m_2+\tau} \\
&\leq \left(\frac{m_2+\tau}{2m_1}\right)\left(\frac{2m_1}{3(m_2+\tau)}\right)\xi_1^{2m_1} \\
&\quad +\left(\frac{2m_1-\tau-m_2}{2m_1}\right)\left(\frac{2m_1}{3(m_2+\tau)}\right)^{-\left(\frac{m_2+\tau}{2m_1-\tau-m_2}\right)}\check{c}_2^{\left(\frac{2m_1}{2m_1-\tau-m_2}\right)}\xi_2^{2m_1} \\
&\quad +\check{c}_2\xi_2^{2m_1} \\
&\leq \frac{1}{3}\xi_1^2+\left(\left(\frac{2m_1-\tau-m_2}{2m_1}\right)\left(\frac{2m_1}{3(m_2+\tau)}\right)^{-\left(\frac{m_2+\tau}{2m_1-\tau-m_2}\right)}\check{c}_2^{\left(\frac{2m_1}{2m_1-\tau-m_2}\right)}+\check{c}_2\right)\xi_2^2 \\
&\leq \frac{1}{3}\xi_1^2+c_2\xi_2^2 \qquad (3.12)
\end{aligned}$$

for a constant $c_2 > 0$. The third term in (3.8) can be estimated with the help of the following Proposition 3.1 whose proof is included in the Appendix.

Proposition 3.1: There is a constant $c_3 > 0$ such that

$$\frac{\partial W_2}{\partial x_1}\dot{x}_1 < \frac{1}{3}\xi_1^2 + c_3\xi_2^2. \qquad (3.13)$$

Substituting the estimates (3.10), (3.12) and (3.13) into (3.8), we arrive at

$$\dot{V}_2 \leq -\xi_1^2 + \bar{c}\xi_2^2 + \xi_2^{(2m_1-\tau-m_2)}u$$

for a constant $\bar{c} = c_1 + c_2 + c_3 > 0$. Choosing an intermediate controller

$$u = u^* = -(1+\bar{c})\xi_2^{\tau+m_2} = -\beta_2\xi_2^{\tau+m_2}, \quad \beta_2 > 0$$

yields

$$\dot{V}_2 \leq -(\xi_1^2+\xi_2^2) + \xi_2^{(2m_1-\tau-m_2)}(u-u^*). \qquad (3.14)$$

If the state x_2 are available for feedback, the control law can be implemented and u can be set to u^*. Then, the last term of $\dot{V}_2$ in (3.14) will be disappeared and we can conclude that $\dot{V}_2 < 0, \ \forall x \neq 0$. This implies that the system (1.1) can be globally asymptotically stabilized by a full-state feedback u^*. However, in our case, only x_1 is available, the control law u^* cannot be implemented. But, the inequality (3.14) still holds for any system of the form (1.1) and satisfies the growth condition in Assumption 3.1.

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3.2 STABILIZATION OF (1.1) BY OUTPUT FEEDBACK

In this section, we show that under Assumption 3.1, the problem of global output feedback stabilization for system (1.1) is solvable. We will first construct a homogeneous output feedback controller for the nominal chain of power integrator, i.e. $\phi_1(\cdot) = \phi_2(\cdot) = 0$:

$$\begin{aligned} \dot{z}_1 &= z_2^p \\ \dot{z}_2 &= v \\ y &= z_1, \end{aligned} \tag{3.15}$$

with p is positive odd integer number. Then, based on this output feedback controller, we will develop a scaled observer and controller to render the system (1.1) globally asymptotically stable under the growth condition (3.1)-(3.2).

3.2.1 Output Feedback Control of Nominal Nonlinear System

Theorem 3.1: Given a real number $\frac{-1}{p+1} < \tau \leq 0$, there is a homogeneous output feedback controller of degree τ rendering the nonlinear systems (3.15) is global asymptotically stable.

Proof. The construction of the homogeneous output feedback controller is accomplished in 3 steps. First, by Lemma 3.1, a homogeneous state feedback stabilizer is constructed. Then, a homogeneous observer is designed, and lastly, the unmeasurable states are replaced with the estimates. The closed-loop system can be proven globally asymptotically stable by an appropriate observer gain.

State Feedback Controller: For nonlinear systems (3.15), Assumption 3.1 is automatically satisfied since $\phi_1(\cdot)$, $\phi_2(\cdot)$ are trivial. Hence, by Lemma 3.1, there is a homogeneous (with respect to the weight (3.3)) state feedback controller that globally stabilizes (3.15). Therefore , there exist a Lyapunov function of the form

$$V_2(z_1, z_2) = \int_0^{z_1} s^{m_1-\tau} ds + \int_{z_2^*}^{z_2} \left(s^{\frac{1}{m_2}} - z_2^{*\frac{1}{m_2}} \right)^{(2m_1-\tau-m_2)} ds,$$

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a homogeneous control law

$$v^*(z) = -\beta_2 \xi_2^{\tau+m_2} \text{ with } \xi_1 = z_1,\ z_2^{*p} = -\beta_1 \xi_1^{\tau+m_1},\ \xi_2 = z_2^{1/m_2} - z_2^{*1/m_2}, \quad (3.16)$$

and constants $\beta_1,\ \beta_2 > 0$ that renders

$$\dot{V}_2 \leq -(\xi_1^2 + \xi_2^2) + \xi_2^{2m_1-\tau-m_2}\big(v - v^*(z)\big). \quad (3.17)$$

Homogeneous Observer Design: Next, similar to [2] and [12], a homogeneous observer is constructed as follows

$$\dot{\eta}_2 = -l_1 \hat{z}_2^p, \quad \hat{z}_2^p = [\eta_2 + l_1 \hat{z}_1]^{\frac{m_2 p}{m_1}}, \quad (3.18)$$

where $z_1 = \hat{z}_1$ and $l_1 > 0$ is the gains to be determined in a later step. Based on the estimated state $\hat{z}_2$, we design an output feedback controller

$$v(\hat{z}) = -\beta_2 \Big(\hat{z}_2^{\frac{1}{m_2}} + \beta_1^{\frac{1}{pm_2}} z_1\Big)^{\tau+m_2}. \quad (3.19)$$

We choose the Lyapunov function for the observer (3.18) as follows

$$U_2(z_1, z_2, \eta_2) = \int_{\gamma_2^{(m_1-\tau)/m_1}}^{z_2^{(m_1-\tau)/m_2}} \Big(s^{\frac{m_1}{m_1-\tau}} - \gamma_2\Big) ds \text{ where } \gamma_2 = \eta_2 + l_1 z_1 = \hat{z}_2^{\frac{m_1}{m_2}}.$$

It can be verified that U_2 is C^1. In addition, with a constant b, we have the following relationships

$$\begin{aligned}
\frac{\partial U_2}{\partial z_2} &= b z_2^{\frac{(m_1-\tau)}{m_2}-1}\Big(z_2^{\frac{m_1}{m_2}} - \gamma_2\Big), \\
\frac{\partial U_2}{\partial \eta_2} &= -\Big(z_2^{\frac{(m_1-\tau)}{m_2}} - \gamma_2^{\frac{(m_1-\tau)}{m_1}}\Big), \\
\frac{\partial U_2}{\partial z_1} &= -l_1\Big(z_2^{\frac{(m_1-\tau)}{m_2}} - \gamma_2^{\frac{(m_1-\tau)}{m_1}}\Big).
\end{aligned}$$

Hence, the time derivative of U_2 along the trajectories of (3.15)-(3.18) is

$$\dot{U}_2 = vbz_2^{\frac{(m_1-\tau)}{m_2}-1}\Big(z_2^{\frac{m_1}{m_2}} - \gamma_2\Big) - l_1\big(z_2^p - \hat{z}_2^p\big)\Big(z_2^{\frac{(m_1-\tau)}{m_2}} - \gamma_2^{\frac{(m_1-\tau)}{m_1}}\Big).$$

From the definition of γ, we can rearrange the terms in the above equation as follows.

$$\begin{aligned}
\dot{U}_2 &= vbz_2^{\frac{(m_1-\tau)}{m_2}-1}\Big(z_2^{\frac{m_1}{m_2}} - \gamma_2\Big) - l_1\big(z_2^p - \hat{z}_2^p\big)\Big(z_2^{\frac{(m_1-\tau)}{m_2}} - \gamma_2^{\frac{(m_1-\tau)}{m_1}}\Big) \\
&= vbz_2^{\frac{(m_1-\tau)}{m_2}-1}\Big(z_2^{\frac{m_1}{m_2}} - \gamma_2\Big) - l_1\big(z_2^p - \hat{z}_2^p\big)\Big(z_2^{\frac{(m_1-\tau)}{m_2}} - \big(\hat{z}_2^{\frac{m_1}{m_2}}\big)^{\frac{(m_1-\tau)}{m_1}}\Big) \\
&= vbz_2^{\frac{(m_1-\tau)}{m_2}-1}\Big(z_2^{\frac{m_1}{m_2}} - \gamma_2\Big) - l_1\big(z_2^p - \hat{z}_2^p\big)\Big(z_2^{\frac{(m_1-\tau)}{m_2}} - \hat{z}_2^{\frac{(m_1-\tau)}{m_2}}\Big) \quad (3.20)
\end{aligned}$$

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Let $e_2 = (z_2 - \hat{z}_2)^{m_1/m_2}$. We estimate the second terms in (3.20). By Lemma A.1, with constant $m > 0$

$$\begin{aligned} -l_1\left(z_2^p - \hat{z}_2^p\right)\left(z_2^{\frac{(m_1-\tau)}{m_2}} - \hat{z}_2^{\frac{(m_1-\tau)}{m_2}}\right) &\le -l_1 m e_2^{m_2 p}\left(z_2 - \hat{z}_2\right)^{\frac{(m_1-\tau)}{m_2}} \\ &= -l_1 m e_2^{\tau+m_1}\left(z_2 - \hat{z}_2\right)^{\frac{m_1(m_1-\tau)}{m_2 m_1}} \\ &= -l_1 m e_2^{\tau+m_1}(e_2)^{\frac{(m_1-\tau)}{m_1}} \\ &= -l_1 m e_2^2. \end{aligned} \tag{3.21}$$

The first terms in (3.20) can be estimated using the following Proposition 3.2 whose proofs are in the Appendix.

Proposition 3.2: For controller $v(\hat{z})$. there is a constant $c_4 \ge 0$ such that

$$v(\hat{z}) b z_2^{\frac{(m_1-\tau)}{m_2}-1}\left(z_2^{\frac{m_1}{m_2}} - \gamma_2\right) \le \frac{1}{4}(\xi_1^2 + \xi_2^2) + c_4 e_2^2. \tag{3.22}$$

With the help of the previous proposition and the estimates (3.21), the derivative of U_2 becomes

$$\dot{U}_2 \le \frac{1}{4}(\xi_1^2 + \xi_2^2) - (l_1 m - c_4) e_2^2. \tag{3.23}$$

Determination of Observer Gain l_1: To choose the gain l_1, we combine the Lyapunov functions of the nominal system (3.15) and the observer (3.18).

$$T = V_2 + U_2$$

whose derivative is the combination of (3.17) and (3.23). Due to the unmeasurable states, the controller $v = v(\hat{z})$ gives a redundant term in (3.17). To deal with this term, we use the following proposition.

Proposition 3.3: There is a constant $c_5 > 0$ such that

$$\xi_2^{2m_1-\tau-m_2}\left(v(\hat{z}) - v^*(z)\right) \le \frac{1}{4}(\xi_1^2 + \xi_2^2) + c_5 e_2^2. \tag{3.24}$$

Combining (3.17), (3.23) and (3.24) together yields

$$\dot{T} \le -\frac{1}{2}(\xi_1^2 + \xi_2^2) - (l_1 m - c_4 - c_5) e_2^2. \tag{3.25}$$

Clearly, by choosing $l_1 = \frac{1}{m}[\frac{1}{2} + c_4 + c_5]$, (3.25) becomes

$$\dot{T} \le -\frac{1}{2}(\xi_1^2 + \xi_2^2 + e_2^2). \tag{3.26}$$

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Note that from the construction of T, it is easy to verify that T is positive definite and proper with respect to

$$\mathcal{Z} = (z_1, z_2, \eta_2)^T. \tag{3.27}$$

In addition, the right hand side of (3.26) is negative definite with respect to $\mathcal{Z}$. Therefore, the closed-loop system (3.15)-(3.16)-(3.18) is globally asymptotically stable. Denoting $f_3 = \dot{\eta}_2$, it is straightforward to verify that the closed-loop system (3.15)-(3.16)-(3.18) can be rewritten in the following form

$$\dot{\mathcal{Z}} = F(\mathcal{Z}) = (z_2^p, v(z_1, \eta_2), f_3)^T \tag{3.28}$$

which is homogeneous. In fact, by choosing the dilation weight

$$\triangle = \Big(\underbrace{m_1,\ m_2}_{for z_1,\ z_2},\ \underbrace{m_1}_{for\ \eta_2}\Big), \tag{3.29}$$

It can be shown that (3.28) is homogeneous of degree τ. In addition, T is homogeneous of degree $2m_1 - \tau$ and the right hand side of (3.26) is homogeneous of degree $2m_1$. The proofs are shown in Proposition 3.4 of the Appendix.

Remark 3.1: The right hand side of (3.26) is negative definite and homogenous of degree $2m_1$. From Lemma 2.2, it can be shown that there is a constant $\underline{c}_1 > 0$ such that

$$\tfrac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}} F(\mathcal{Z}) \leq -\underline{c}_1 \|\mathcal{Z}\|_\triangle^{2m_1} \leq -\underline{c}_1 \|\mathcal{Z}\|_\triangle^2$$

where $\|\mathcal{Z}\|_\triangle = \sqrt{|z_1|^{2/m_1} + |z_2|^{2/m_2} + |\eta_2|^{2/m_1}}$.

3.2.2 Global Output Feedback Stabilization for System (1.1)

Utilizing of the homogeneous controller and observer established in the previous sections, we are ready to construct the output feedback for (1.1).

Theorem 3.2: Under Assumption 3.1, the system (1.1) can be globally stabilized by output feedback.

Proof: Under the new coordinates

$$z_1 = x_1,\ z_2 = \frac{x_2}{L^{\kappa_2}},\ v = \frac{u}{L^{\kappa_2+1}}\ \text{ with } \kappa_1 = 0 \text{ and } \kappa_2 = \frac{1}{p}, \tag{3.30}$$

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the system (1.1) can be rewritten as

$$\dot{z}_1 = Lz_2^p + \phi_1(\cdot),\ \ \dot{z}_2 = Lv + \phi_2(\cdot)/L^{\kappa_2} \tag{3.31}$$

with the scaling gain, $L \geq 1$. Next, we construct an observer with the scaling gain L.

$$\dot{\eta}_2 = -Ll_1\hat{z}_2^p,\quad \hat{z}_2^p = [\eta_2 + l_1\hat{z}_1]^{m_2p/m_1} \tag{3.32}$$

where $\hat{z}_1 = z_1$ and l_1 is the gain selected by (3.25) in Theorem 3.1. Using the same notations as (3.27) and (3.28), the closed-loop system (3.31)-(3.32)-(3.19) can be written as

$$\dot{\mathcal{Z}} = LF(\mathcal{Z}) + \Big(\phi_1(\cdot),\ \frac{\phi_2(\cdot)}{L^{\kappa_2}},\ 0\Big)^T. \tag{3.33}$$

Note that the $F(\mathcal{Z})$ in (3.33) has the exact same structure as (3.28) due to the use of same gains l_1 and $\beta_1,\ \beta_2$. Hence, adopting the same Lyapunov function $T(\mathcal{Z})$ used in preceding section, it can be concluded from Remark 3.1 that

$$\begin{aligned}\dot{T} &= L\frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}}F(\mathcal{Z}) + \frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}}\Big(\phi_1(\cdot), \frac{\phi_2(\cdot)}{L^{\kappa_2}}, 0\Big)^T \\ &\leq -L\underline{c}_1\|\mathcal{Z}\|_{\triangle}^2 + \frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}}\Big(\phi_1(\cdot), \frac{\phi_2(\cdot)}{L^{\kappa_2}}, 0\Big)^T. \end{aligned} \tag{3.34}$$

Using the change of coordinates (3.30) and the fact that $L \geq 1$, we deduce from Assumption 3.1 that for some constants v_1 and $v_2 > 0$,

$$\begin{aligned}\Big|\phi_1(x,t)\Big| &\leq c|z_1|^{\frac{m_1+\tau}{m_1}} \leq cL^{1-v_1}|z_1|^{\frac{m_1+\tau}{m_1}} \\ \Big|\frac{\phi_2(x,t)}{L^{\kappa_2}}\Big| &\leq \frac{c}{L^{\kappa_2}}\Big(|x_1|^{\frac{m_2+\tau}{m_1}} + |x_2|^{\frac{m_2+\tau}{m_2}}\Big) \\ &\leq cL^{1-v_2}\Big(|z_1|^{\frac{m_2+\tau}{m_1}} + |z_2|^{\frac{m_2+\tau}{m_2}}\Big). \end{aligned} \tag{3.35}$$

Recall that T is homogeneous of degree $2m_1 - \tau$. Therefore, for $i = 1, 2$, $\frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}_i}$ is homogeneous of degree $2m_1 - \tau - m_i$, $|z_1|^{\frac{m_1+\tau}{m_1}}$ is homogeneous of degree $m_1 + \tau$ and $\big(|z_1|^{\frac{m_2+\tau}{m_1}} + |z_2|^{\frac{m_2+\tau}{m_2}}\big)$ is homogeneous of degree $m_2 + \tau$. Then,

$$\Big|\frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}_1}\Big|\Big(|z_1|^{\frac{m_1+\tau}{m_1}}\Big) \text{ and } \Big|\frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}_2}\Big|\Big(|z_1|^{\frac{m_2+\tau}{m_1}} + |z_2|^{\frac{m_2+\tau}{m_2}}\Big) \tag{3.36}$$

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are homogeneous of degree $2m_1$. With (3.35) and (3.36) in mind, we can find a constant ρ_i such that

$$\begin{aligned}\frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}_1}\phi_1(\cdot) &\leq \rho_1 L^{1-v_1}\|\mathcal{Z}\|_{\triangle}^{2m_1} \leq \rho_1 L^{1-v_1}\|\mathcal{Z}\|_{\triangle}^{2} \\ \frac{\partial T(\mathcal{Z})}{\partial \mathcal{Z}_2}\frac{\phi_2(x,t)}{L^{\kappa_2}} &\leq \rho_2 L^{1-v_2}\|\mathcal{Z}\|_{\triangle}^{2m_1} \leq \rho_2 L^{1-v_2}\|\mathcal{Z}\|_{\triangle}^{2}. \end{aligned} \tag{3.37}$$

Substituting (3.37) into (3.34) yields

$$\dot{T}|_{\left((3.21)-(3.31)-(3.32)\right)} \leq -L(\underline{c}_1 - \rho_1 L^{-v_1} - \rho_2 L^{-v_2})\|\mathcal{Z}\|_{\triangle}^{2}. \tag{3.38}$$

Obviously, if L is large enough then the right hand side of (3.38) is negative definite. Clearly, the closed-loop system of (1.1) is globally asymptotically stable.

Note that when $\tau = 0$, Assumption 3.1 reduces to the bound described in [12] for planar systems, where $r_1 = 1,\ r_2 = 1/p$. Thus, the method presented here can be used to globally asymptotically stabilize any planar system studied in [12].

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