CHAPTER 3

MAIN RESULTS

In this chapter, we desire a switching law to stabilize a switched system with three subsystems. Two out of three subsystems used in this switched system cannot be stabilized. We then find the third subsystem and a new switching law to make the overall system asymptotically stabilizable.

We study the case in which the two subsystems have unstable foci and cannot be stabilized with $D > 0$ and $\rho > 1$ where D and ρ are defined in [7]. To stabilize the overall system, we introduce a third subsystem which is a stable system that has two complex eigenvalues with negative real parts. This subsystem will be used to stabilize the overall system according to the concept illustrated in figure 3.1.

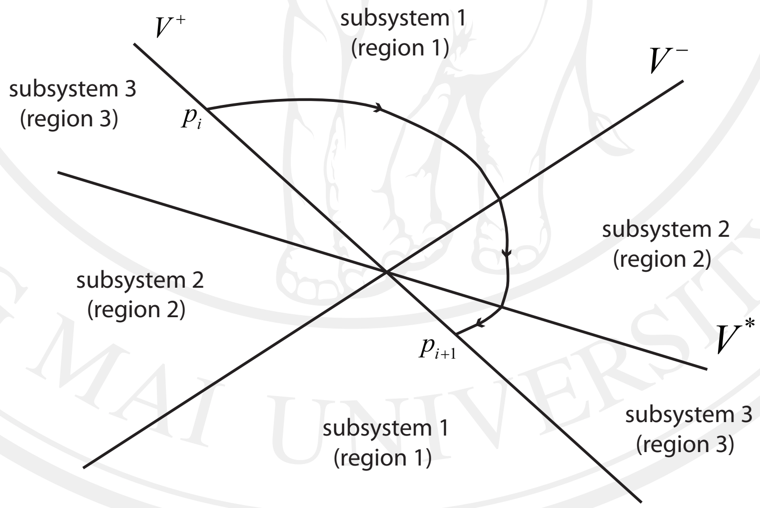

Figure 3.1: The idea of concept

According to figure 3.1, we divide the region for subsystem 2 by the line v^*, and we assign the regions between v^* and v^+ to our newly introduced subsystem 3. Let the initial condition starting at p_i, which is on the v^+ line. At this point, the switched system will use subsystem 1. The trajectory would traverse region 1 in the clockwise direction until intersects the v^- line, at where the switched system switched to use the subsystem 2. Then, we let the trajectory traversing region 2 in the clockwise direction until it intersects the v^* line, where the switched system switches to use the subsystem 3. The trajectory then traverses region 3 until it intersects the v^+ line again. We then define the point at which the trajectory intersects v^+ line to be p_{i+1}. From p_i to p_{i+1}, the trajectory has traversed the total angle of π. Clearly the overall switched system which consists of three subsystems is asymptotically stabilizable if $\|p_{i+1}\| < c\|p_i\|$ where $0 < c < 1$.

Consider the switched system

$$\dot{x} = A_{\sigma(t)}x(t)$$

$$\sigma(t) : [0, \infty) \to \{1, 2, 3\} \tag{3.1}$$

with

$$A_1 = \begin{pmatrix} \alpha_1 & \beta_1/E \\ -E\beta_1 & \alpha_1 \end{pmatrix}, A_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ -\beta_2 & \alpha_2 \end{pmatrix}, A_3 = \begin{pmatrix} \alpha_3 & \beta_3 \\ -\beta_3 & \alpha_3 \end{pmatrix}, \tag{3.2}$$

with $\lambda_i = \alpha_i \pm j\beta_i(\beta_i > 0)$ are the eigenvalues of A_i, $i = 1, 2, 3$. Assume that the system 1 and 2 cannot be stabilized but the system 3 is stable. That implies that $\alpha_1, \alpha_2 > 0$ and $\alpha_3 < 0$. For a general 2×2 nonsingular matrices, the matrices could be transformed into the eigenvalue form. So we let A_3 is the form as above for ease in the analysis.

According the Figure 3.1, the switching law would be : switching to subsystem 1 whenever the system trajectory enters the region 1. Switching to subsystem 2 whenever the system trajectory enters the region 2. And switching to subsystem 3 whenever the system trajectory enters the region 3.

By a switching law, we translate the switched system into a piecewise linear system.

$$\dot{x}(t) = \begin{cases} A_1 x(t), & x \in R1 \\ A_2 x(t), & x \in R2 \\ A_3 x(t), & x \in R3 \end{cases} \tag{3.3}$$

Lemma 1. *A switched system which consists of three subsystems is asymptotically stabilizable if* $\|p_{i+1}\| < k\|p_i\|$ *where* $0 < k < 1$.

Proof : For $0 < k < 1$, we assume $\|p_{i+1}\| \leq k\|p_i\|$.

Thus, we have

$$\begin{aligned}\|p_{i+1}\| &\leq k\|p_i\| \\ &\leq k^2\|p_{i-1}\| \\ &\vdots \\ &\leq k^n\|p_1\|.\end{aligned}$$

As $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|p_{n+1}\| \leq \lim_{n\to\infty} k^n\|p_1\| = 0.$$

The trajectory of system converges to the origin. Thus, the switched system is asymptotically stabilizable. □

Theorem 1. *Let $\alpha_i \pm j\beta_i$ be eigenvalues of A_i defined in* (3.1) *and (3.2). The autonomous switched system (3.3) consisting of three subsystems is asymptotically stabilizable if*

$$\frac{\alpha_3(\theta_2-\theta_3)}{\beta_3} < -\frac{\alpha_1\theta_1}{\beta_1} - \frac{\alpha_2(\theta_1-\theta_2)}{\beta_2} - \ln\sqrt{\cos^2(\theta_1)+E^2\sin^2(\theta_1)} - \ln c \qquad (3.4)$$

where $\theta_1+\theta_2+\theta_3=\pi$ and $c>1$.

Proof. Let $(\rho(t),\theta(t))$ be the solution of the switched system (3.3) in polar coordinate with initial condition $\rho(0)=1, \theta(0)=0$. The system will use the subsystem 1 from v^+ line to v^- line (see figure 3.1), for $t \geq 0$ and $t < t_1$. The system obeys the equation

$$x(t) = \rho_0 e^{\alpha_1 t}\begin{pmatrix}\cos(-\beta_1 t)\\ E\sin(-\beta_1 t)\end{pmatrix}. \qquad (3.5)$$

At $t=t_1$, we define $\theta_1=\beta_1 t_1$; i.e. $t_1=\theta_1/\beta_1$. From Lemma 2.5.3., we have

$$x_1(t_1) = e^{\alpha_1\theta_1/\beta_1}\begin{pmatrix}\cos(-\theta_1)\\ E\sin(-\theta_1)\end{pmatrix}. \qquad (3.6)$$

Therefore,

$$\rho_1(t_1) = e^{\alpha_1\theta_1/\beta_1}\sqrt{\cos^2(\theta_1)+E^2\sin^2(\theta_1)}. \qquad (3.7)$$

After instant t_1, the system will use the subsystem 2 from v^- line to v_* line (see figure 3.1), thus $t_2 > t_1$. The system obeys the equation

$$x(t) = \rho_1(t_1)e^{\alpha_2 t_2}\begin{pmatrix}\cos(-\beta_2 t_2 + \theta_1)\\ \sin(-\beta_2 t_2 + \theta_1)\end{pmatrix}. \tag{3.8}$$

At t_2 we define $\theta_2 = -\beta_2 t_2 + \theta_1$, then $t_2 = \dfrac{\theta_1 - \theta_2}{\beta_2}$.

Thus, the solution at $t = t_2$ is

$$x(t_2) = \rho_1(t_1)e^{\alpha_2\frac{(\theta_1-\theta_2)}{\beta_2}}\begin{pmatrix}\cos(-\theta_2)\\ \sin(-\theta_2)\end{pmatrix}. \tag{3.9}$$

Therefore,

$$\rho_2(t_2) = e^{\alpha_1\theta_1/\beta_1}e^{\alpha_2\frac{(\theta_1-\theta_2)}{\beta_2}}\sqrt{\cos^2(\theta_1) + E^2\sin^2\theta_1}. \tag{3.10}$$

After instant t_2, the system will use subsystem 3 at which the trajectory intersect v_+ line, for $t_3 > t_2$. The system obeys the equation

$$x(t) = \rho_2(t_2)e^{\alpha_3 t_3}\begin{pmatrix}\cos(-\beta_3 t_3 + \theta_2)\\ \sin(-\beta_3 t_3 + \theta_2)\end{pmatrix}. \tag{3.11}$$

At t_3 we define $\theta_3 = -\beta_3 t_3 + \theta_2$, thus $t_3 = \dfrac{\theta_2 - \theta_3}{\beta_3}$.

The solution at $t = t_3$ is

$$x(t_3) = \rho_2(t_2)e^{\alpha_3\frac{\theta_2-\theta_3}{\beta_3}}\begin{pmatrix}\cos(-\theta_3)\\ \sin(-\theta_3)\end{pmatrix}. \tag{3.12}$$

Therefore,

$$\rho_3(t_3) = e^{\alpha_1\theta_1/\beta_1}e^{\alpha_2\frac{(\theta_1-\theta_2)}{\beta_2}}e^{\alpha_3\frac{\theta_2-\theta_3}{\beta_3}}\sqrt{\cos^2\theta_1 + E^2\sin^2\theta_1}. \tag{3.13}$$

By lemma 3.0.1, the system is asymptotically stabilized if

$$e^{\alpha_1\theta_1/\beta_1}e^{\alpha_2\frac{(\theta_1-\theta_2)}{\beta_2}}e^{\alpha_3\frac{\theta_2-\theta_3}{\beta_3}}\sqrt{\cos^2\theta_1 + E^2\sin^2\theta_1} < \frac{1}{c} \tag{3.14}$$

$$\ln[e^{\alpha_1\theta_1/\beta_1}e^{\alpha_2\frac{(\theta_1-\theta_2)}{\beta_2}}e^{\alpha_3\frac{\theta_2-\theta_3}{\beta_3}}\sqrt{\cos^2\theta_1 + E^2\sin^2\theta_1}] < \ln c^{-1}$$

$$\frac{\alpha_1\theta_1}{\beta_1} + \alpha_2\frac{(\theta_1-\theta_2)}{\beta_2} + \alpha_3\frac{\theta_2-\theta_3}{\beta_3} + \ln\sqrt{\cos^2(\theta_1) + E^2\sin^2(\theta_1)} < -\ln c.$$

Therefore,

$$\alpha_3 \frac{\theta_2 - \theta_3}{\beta_3} < -\frac{\alpha_1 \theta_1}{\beta_1} - \alpha_2 \frac{(\theta_1 - \theta_2)}{\beta_2} - \ln \sqrt{\cos^2(\theta_1) + E^2 \sin^2(\theta_1)} - \ln c \qquad (3.15)$$

where $\theta_1 + \theta_2 + \theta_3 = \pi$ and $c > 1$.

In this chapter, we have a sufficient condition (3.15) to stabilize the overall switched system when two unstable foci subsystem cannot be stabilized the switched system.